Chapter Three- Exercises and Problems

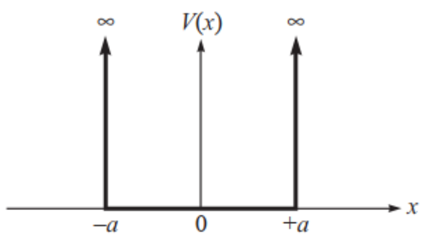
Problem (1): For a particle inside a one-dimensional infinite potential well of width $2a$. prove that the normalization constant equal to $\frac{1}{\sqrt{a}}$, for the wavefunction:

1- $\psi_n(x) = A\, sin(\frac{n\pi}{2a}x)$
2- $\psi_n(x) = B\, cos(\frac{n\pi}{2a}x)$

Problem (2): For a particle inside a one-dimensional infinite potential well of width $2a$. prove that the eigenfunctions are orthogonal.

1- $\psi_1(x) = \frac{1}{\sqrt{a}} cos\left(\frac{\pi}{2a}x\right)$ & $\psi_3(x) = \frac{1}{\sqrt{a}} cos\left(\frac{3\pi}{2a}x\right)$
2- $\psi_2(x) = \frac{1}{\sqrt{a}} sin\left(\frac{\pi}{a}x\right)$ & $\psi_3(x) = \frac{1}{\sqrt{a}} cos\left(\frac{3\pi}{2a}x\right)$
3- $\psi_1(x) = \frac{1}{\sqrt{a}} sin\left(\frac{\pi}{a}x\right)$ & $\psi_4(x) = \frac{1}{\sqrt{a}} sin\left(\frac{2\pi}{a}x\right)$

Problem (3): Calculate $\langle x\rangle, \langle p_x\rangle, \langle x^2\rangle$ *and* $\langle p_x^2\rangle$ for a particle in one dimensional box.



$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & |x| > a \end{cases}$$

Solution: We shall consider the even parity wave functions

$$\psi_n(x) = \frac{1}{\sqrt{a}} \cos kx, \quad k = \frac{n\pi}{2a}, \quad n = 1, 3, 5, \ldots$$

The same results are obtained if we consider the odd parity wave functions.

(a)
$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_n^* x\, \psi_n\, dx$$
$$= \frac{1}{a}\int_{-a}^{a} x \cos^2 kx\, dx$$
$$= 0, \text{ since the integrand is odd.}$$

This result is as expected; the probability density $\psi^*\psi$ is symmetric about $x = 0$, indicating that the particle spends as much time to the left of the centre as to the right.

(b)
$$\langle p\rangle = \int_{-\infty}^{\infty} \psi_n^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi_n\,dx$$
$$= -\frac{i\hbar}{a}\int_{-a}^{a} \cos kx\,(-k\sin kx)dx$$
$$= \frac{i\hbar k}{a}\int_{-a}^{a}\cos kx \sin kx\,dx$$
$= 0$, since the integrand is odd.

Again, the result is as expected. The particle moves back and forth, spending half its time moving towards the left and half its time moving towards the right. Thus the average momentum must be zero.

(c)
$$\langle x^2\rangle = \int_{-\infty}^{\infty}\psi_n^*\,x^2\,\psi_n\,dx$$
$$= \frac{1}{a}\int_{-a}^{a} x^2\cos^2 kx\,dx$$
$$= \frac{2}{2a}\int_0^a x^2(1+\cos 2kx)dx$$
$$= \frac{1}{a}\left[\frac{x^3}{3}\right]_0^a + \frac{1}{a}\int_0^a x^2\cos(2kx)dx$$
$$= \frac{a^2}{3} + \frac{1}{a}\int_0^a x^2\cos(2kx)\,dx$$

Integrating by parts
$$\int_0^a x^2\cos(2kx)\,dx = \frac{1}{2k}\left[x^2\sin 2kx\right]_0^a - \frac{1}{k}\int_0^a x\sin(2kx)\,dx$$

The first term vanishes since $2ka = n\pi$ and $\sin n\pi = 0$. The integral in the second term gives

$$\int_0^a x\sin(2kx)\,dx = -\frac{1}{2k}\left[x\cos 2kx\right]_0^a + \frac{1}{2k}\int_0^a \cos(2kx)\,dx$$
$$= \frac{a}{2k} + \frac{1}{4k^2}\left[\sin 2kx\right]_0^a = \frac{a}{2k}$$

The first term has been evaluated using $\cos n\pi = -1$ as n is odd. Thus, we obtain

$$\begin{aligned}\langle x^2 \rangle &= \frac{a^2}{3} + \left(\frac{1}{a}\right)\left(-\frac{1}{k}\right)\left(\frac{a}{2k}\right) \\ &= \frac{a^2}{3} - \frac{1}{2k^2} \\ &= \frac{a^2}{3} - \frac{1}{2}\left(\frac{2a}{n\pi}\right)^2 \\ &= \frac{a^2}{3} - \frac{2a^2}{n^2\pi^2} \\ &= a^2\left(\frac{1}{3} - \frac{2}{n^2\pi^2}\right)\end{aligned}$$

(d)

$$\begin{aligned}\langle p^2 \rangle &= \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n \, dx \\ &= \frac{1}{a}\int_{-a}^{a} \cos(kx)\left(-\hbar^2 \frac{d^2}{dx^2}\right) \cos\,(kx)\; dx \\ &= \frac{\hbar^2 k^2}{a}\int_{-a}^{a} \cos^2 kx\; dx \\ &= \frac{2\hbar^2 k^2}{2a}\int_0^a (1 + \cos(2kx))\; dx \\ &= \frac{2\hbar^2 k^2}{2a}\left[[x]_0^a + \int_0^a \cos\,(2kx)\, dx\right] \\ &= \hbar^2 k^2 = \frac{n^2\pi^2\hbar^2}{4a^2}\end{aligned}$$

Problem (4): Show that the uncertainty relation $\Delta x \Delta p \geq \frac{\hbar}{2}$ is satisfied in the case of a particle in a one-dimensional box.

Solution: The uncertainty x is defined as, $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$, similarly, the uncertainty p_x is defined as, $(\Delta p_x)^2 = \langle {p_x}^2 \rangle - \langle p_x \rangle^2$,

Using the result of problem (1)

$$(\Delta x)^2 = a^2\left(\frac{1}{3} - \frac{2}{n^2\pi^2}\right)$$

$$(\Delta p)^2 = \frac{n^2\pi^2\hbar^2}{4a^2}$$

Therefore, $$(\Delta x)^2(\Delta p)^2 = n^2\pi^2\hbar^2\left(\frac{1}{12} - \frac{1}{2\pi^2 n^2}\right)$$

or $$\Delta x \ \Delta p = \hbar\left(\frac{\pi^2 n^2}{12} - \frac{1}{2}\right)^{1/2}$$

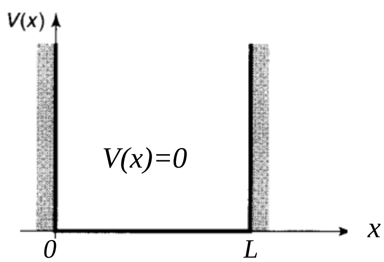
The smallest value of this uncertainty product is for the ground state ($n = 1$). We get on simplifying

$$(\Delta x \ \Delta p)_{n=1} = 0.567\hbar$$

This is in agreement with

$$\Delta x \Delta p \geq \hbar/2$$

Problem (5): Consider a particle of mass *m*, moving in a one-dimensional infinite square well of width *L*, such that the left corner of the well is at the origin. Obtain the energy eigenvalues and the corresponding normalized eigenfunctions of the particle.



$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & x < 0 \quad \text{and} \quad x > L \end{cases} \qquad \ldots(4.25)$$

Thus, we have to solve the Schrödinger equation, subject to the boundary conditions,

$$\psi(0) = \psi(L) = 0 \qquad \ldots(4.26)$$

The general solution is again given by (4.16), $\psi(x) = A \sin kx + B \cos kx$

The boundary condition at $x = 0$ requires that $B = 0$. The boundary condition at $x = L$ requires that,

$$kL = n\pi, \qquad n = 1, 2, 3\ldots$$

Thus the eigenvalues are

$$\boxed{E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1,2,3,\ldots} \qquad \ldots(4.27)$$

And the corresponding eigenfunctions are,

$$\frac{\pi^2\hbar^2}{2mL^2}$$

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots$$

The normalization condition requires, $A = \sqrt{\frac{2}{L}}$

Therefore, the normalized eigenfunctions are

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots} \qquad \ldots(4.28)$$

Note that the energies of the particle are the same as in Equation (4.23) because $L = 2a$.

Problem (6): Calculate the three lowest energy levels (in *eV*) for an electron inside a one-dimensional infinite potential well of width *2Å*. Also, determine the corresponding normalized eigenfunctions. $m = 9.1 \times 10^{-31}$ kg, $\hbar = 1.05 \times 10^{-34}$Js, 1 eV = 1.6×10^{-19} J.

Solution: **Energies**

If the width of the well is $2a$ from $-a < x < a$, then the energy of the nth level is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{8ma^2}$$

Here a = 1Å = 10^{-10}m. The three lowest levels correspond to n = 1, 2, 3.

$$E_1 = \frac{(3.14)^2 \times (1.05 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-10})^2}$$
$$= 14.93 \times 10^{-19} \text{ J}$$
$$= \boxed{9.3 \text{ eV}}$$
$$E_2 = 4E_1 = \boxed{37.2 \text{ eV}}$$
$$E_3 = 9E_1 = \boxed{83.7 \text{ eV}}$$

Eigenfunctions

$$\psi_1(x) = \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}$$
$$= \frac{1}{10^{-5}}\cos\left(\frac{\pi x}{2 \times 10^{-10}}\right)$$
$$\psi_2(x) = \frac{1}{10^{-5}}\sin\left(\frac{2\pi x}{2 \times 10^{-10}}\right)$$
$$= \frac{1}{10^{-5}}\sin\left(\frac{\pi x}{10^{-10}}\right)$$

Problem(7): Think of the nucleus as a cubical box of length $10^{-14} m$. Compute the minimum energy of a nucleon confined to the nucleus. Given: mass of a nucleon $1.6 \times 10^{-27} kg$.

Solution: The energy eigenvalue of a particle of mass *m* in a cubical box of length *a* is given by

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

In the ground state, $n_x, n_y, n_z = 1$,

$$E = \frac{3\pi^2\hbar^2}{2ma^2}$$

Therefore, the minimum energy of the nucleon is

$$E_{\min} = \frac{3\times(3.14)^2\times(1.054\times10^{-34})^2}{2\times1.6\times10^{-27}\times(10^{-14})^2}$$
$$= 9.75\times10^{-13}\ \text{J}$$
$$= \boxed{6.1\ \text{MeV}}$$

Problem: In the problem of cubical potential box with rigid walls, we have: $\ell^2 + m^2 + n^2 = 9$, Write down:

1- Schrödinger equation for the particle inside the box.

2- The possible values of: a- ℓ, m, n. b- $E_{\ell mn}$, c- $\psi_{\ell mn}$ d-degree of degeneracy.

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<u>Useful relations</u>

$$sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \cos(a-b)]$$

$$\cos a \cos b = \frac{-1}{2}[\cos(a+b) - \cos(a-b)]$$