chapter two: Exercises and examples (Part 1)

Physical Operator

Dynamical Variable	In Classical Mechanics	In Quantum Mechanics-3D	In Quantum Mechanics-1D
Position	$\vec{r}, \quad x, y, z$	$\hat{\boldsymbol{r}}$	$\hat{x}, \quad \hat{y}, \quad \hat{z}$
Momentum	$\vec{p} = m\vec{v}$, $p_x = m\dot{x}$	$\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}$	$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$
Kinetic energy	$T = \frac{p^2}{2m}$	$\hat{T} = \frac{\hat{\boldsymbol{p}}^2}{2m} = \frac{-\hbar^2}{2m}\boldsymbol{\nabla}^2$	$\hat{T}_x = \frac{\hat{p}_x^2}{2m} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
Total energy	$E = T + V$	$\hat{E} = i\hbar\frac{\partial}{\partial t}$	
Hamilton	$H = T + V$	$\hat{H} = \frac{-\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}, t)$	$\hat{H}_x = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x, t)$

$$\hat{A}\psi_n(x) = a_n\psi_n(x)$$

Make Measurement (→ $\hat{A}$) — on a system in State n (→ $\psi_n(x)$) — Result of Measurement (→ a_n)

a_n: eigen value of the operator $\hat{A}$ $\qquad$ $\Psi(x)$: eigen function of the operator $\hat{A}$

Eigen value equation

<u>Example 1:</u> By using the eigen value equation show that the function $\Psi(x) = e^{-4x}$ is an eigen function of the operator $\hat{A} = \frac{\partial}{\partial x}$

Solution:

$$\hat{A}\Psi(x) = \frac{\partial}{\partial x}(e^{-4x})$$

$$= -4e^{-4x}$$

$$\hat{A}\Psi(x) = a\Psi(x)$$

Then,

$a = -4$ $\quad$ is the eigen value to the operator $\hat{A} = \frac{\partial}{\partial x}$

$\Psi(x) = e^{-4x}$ $\quad$ is the eigen function to the operator $\hat{A}$

Example 2: By using the eigen value equation show that the function $\Psi(x) = \cos(4x)$ is an eigen function of the operator $-\frac{\partial^2}{\partial x^2}$

$$\hat{A}\Psi(x) = -\frac{\partial^2}{\partial x^2}\cos(4x)$$

$$\hat{A}\Psi(x) = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\cos(4x)\right) = 4\frac{\partial}{\partial x}\sin(4x) = 16\cos(4x)$$

$a = 16$ is the eigen value to the operator $\hat{A} = -\frac{\partial^2}{\partial x^2}$

$\Psi(x) = \cos(4x)$ is the eigen function to the operator $\frac{\partial^2}{\partial x^2}$

Example 3: Prove that, the function $\Psi(x) = Ae^{-\alpha x}$ is an eigen function of the operator $\hat{F}$

$$\hat{F} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{2\alpha}{x}$$

Where A, α are constant

Solution:

$$\hat{F}\psi = \frac{d^2}{dx^2}(Ae^{-\alpha x}) + \frac{2}{x}\frac{d}{dx}(Ae^{-\alpha x}) + \frac{2\alpha}{x}(Ae^{-\alpha x})$$

$$\hat{F}\psi = \alpha^2(Ae^{-\alpha x}) - \frac{2\alpha}{x}(Ae^{-\alpha x}) + \frac{2\alpha}{x}(Ae^{-\alpha x})$$

$$\hat{F}\psi = \alpha^2(Ae^{-\alpha x})$$

$\hat{F}\psi = \alpha^2\psi$ $\quad\therefore$ $Ae^{-\alpha x}$ is an eigen function of the operator $\hat{F}$

Example 4: Verify the following operator equation $\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right) = \left(\frac{d^2}{dy^2} - y^2 + 1\right)$

$$\left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right)\psi(y) = \left(\frac{d}{dy} - y\right)\left\{\left(\frac{d}{dy} + y\right)\psi(y)\right\}$$

$$= \left(\frac{d}{dy} - y\right)\left\{\frac{d\psi(y)}{dy} + y\psi(y)\right\}$$

$$= \frac{d}{dy}\left\{\frac{d\psi(y)}{dy} + y\psi(y)\right\} - y\left\{\frac{d\psi(y)}{dy} + y\psi(y)\right\}$$

$$= \left\{\frac{d^2\psi(y)}{dy^2} + \frac{d}{dy}\big(y\psi(y)\big)\right\} - \left\{y\frac{d\psi(y)}{dy} + y^2\psi(y)\right\}$$

$$= \frac{d^2\psi(y)}{dy^2} + y\frac{d\psi(y)}{dy} + \psi(y) - y\frac{d\psi(y)}{dy} - y^2\psi(y)$$

$$= \frac{d^2\psi(y)}{dy^2} + \psi(y) - y^2\psi(y)$$

$$= \left(\frac{d^2}{dy^2} + 1 - y^2\right)\psi(y)$$

$$\therefore \quad \left(\frac{d}{dy} - y\right)\left(\frac{d}{dy} + y\right) = \left(\frac{d^2}{dy^2} - y^2 + 1\right)$$

Normalization and expectation value

Example 5: Suppose that a particle moves in *x* direction for the interval ($0 \leq x \leq 0.3$), Calculate the normalization constant for the wave function $\psi(x) = Ae^{3ix}$

The normalization condition is,

$$\int \psi^*(x)\psi(x)dx = 1$$

$$\int_0^{0.3} A^* e^{-3ix} A e^{3ix}\, dx = 1$$

$$|A|^2 \int_0^{0.3} dx = 1$$

$$|A|^2(0.3) = 1$$

$$|A| = \frac{1}{\sqrt{0.3}}$$

$$\psi(x) = \frac{1}{\sqrt{0.3}} e^{3ix}$$

Example 5: If $\psi(x,t) = A(ax - x^2)$ is a normalized wave function in the period $0 \leq x \leq a$ find:

1- The normalization constant
2- The expectation value of the position in x-direction $\langle x \rangle$

Solution

$$\int |\psi(x)|^2 dx = 1$$

$$\int_0^a |A|^2 (ax - x^2)^2 dx = 1$$

$$|A|^2 \int_0^a (a^2x^2 - 2ax^3 + x^4) dx = 1$$

$$|A|^2 \left[a^2 \frac{x^3}{3} - 2a \frac{x^4}{4} + \frac{x^5}{5} \right]_0^a = 1$$

$$|A|^2 \left[\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5} \right] = 1$$

$$|A|^2 \frac{a^5}{30} = 1 \qquad\qquad A = \sqrt{\frac{30}{a^5}}$$

$$\psi(x) = \sqrt{\frac{a^5}{30}} \, (ax - x^2)$$

(2)

$$\langle x \rangle = \int_0^a \psi^* \, \hat{x} \, \psi \, dx = \int_0^a \left(\sqrt{\frac{a^5}{30}} \, (ax - x^2) \right)^* x \left(\sqrt{\frac{a^5}{30}} \, (ax - x^2) \right) dx$$

$$= \frac{a^5}{30} \int_0^a (ax - x^2)^2 \, dx$$

$$= \frac{30}{a^5}\int_0^a x\,(ax - x^2)^2\,dx = \frac{30}{a^5}\int_0^a (\,a^2x^3 - 2ax^4 + x^5)dx$$

$$= \frac{30}{a^5}\left[a^2\frac{x^4}{4} - 2a\frac{x^5}{5} + \frac{x^6}{6}\right]_0^a = \frac{30}{a^5}\left[\frac{a^6}{4} - 2\frac{a^6}{5} + \frac{a^6}{6}\right] = \frac{30}{a^5}\,\frac{a^6}{60} = \frac{a}{2}$$

$$\langle x\rangle = \frac{a}{2}$$

<u>Example 6:</u> Normalize the wave function $\boldsymbol{\psi(x,t) = Ae^{-\frac{(x-x_o)^2}{2a^2}}e^{-ip_ox/\hbar}}$

$$\psi(x) = A\,e^{-\frac{(x-x_\circ)^2}{2a^2}}\,e^{i\,p_\circ x/\hbar}\quad,\quad \psi^*(x) = A\,e^{-\frac{(x-x_\circ)^2}{2a^2}}\,e^{-i\,p_\circ x/\hbar}$$

$$|\psi(x)|^2 = AA^*\,e^{-\frac{(x-x_\circ)^2}{a^2}}$$

$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1$$

$$A\,A^*\int_{-\infty}^{\infty}e^{-\frac{(x-x_\circ)}{a^2}}\,dx = 1 \quad \text{let}\ \ (x - x_\circ) = z \Rightarrow dx = dz$$

$$A\,A^*\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dz = 1$$

$$AA^*\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}dz = 2\int_0^{\infty}e^{-\frac{z^2}{a^2}}dz = a\sqrt{\pi}$$

Usefull integral $\quad \displaystyle\int_0^{\infty}x^n\,e^{-\alpha x^2}dx = \frac{1}{2\,\alpha^{\frac{n+1}{2}}}\,\Gamma\left(\frac{n+1}{2}\right) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$A\,A^*\,a\sqrt{\pi} = 1$$

$$A\,A^* = \frac{1}{a\sqrt{\pi}}$$

If A is real then $A = A^* = \left(\frac{1}{a\sqrt{\pi}}\right)^{\frac{1}{2}}$

Example 7: Suppose that a particle moves in x direction for the period $(0 \le x \le 0.5)$, in the wave function $\Psi(x) = Ae^{ix^2}$ find:

1- normalization constant.
2- The expectation value of $\langle x \rangle$, $\langle p_x \rangle$, $\langle T_x \rangle$

$$\int \psi^*(x)\psi(x)dx = 1$$

$$\int_0^{0.5} A^* e^{-ix^2} A e^{ix^2} dx = 1$$

$$|A|^2 \int_0^{0.5} dx = 1$$

$$A = \sqrt{2}$$

$$< x >= \int \psi^* x\, \psi dx = \int_0^{0.5} \sqrt{2}\, e^{-ix^2} x\, \sqrt{2} e^{ix^2} dx = 2\int_0^{0.5} x dx = \frac{1}{4}$$

$$< p_x >= \int \psi^* \hat{p}_x\, \psi dx = \int_0^{0.5} \sqrt{2}\, e^{-ix^2} \left(-i\hbar \frac{\partial}{\partial x}\right) \sqrt{2} e^{ix^2} dx$$

$$\langle p_x \rangle = -i2\hbar \int_0^{0.5} e^{-ix^2}(2ix)e^{ix^2} dx = 4\hbar \int_0^{0.5} x\, dx$$

$$\langle p_x \rangle = \frac{\hbar}{2}$$

H.W. Find $\langle T_x \rangle$

H.W. : Suppose that a particle moves in the interval $(0 \le x \le 2)$, if the probability density is given by, $|\Psi|^2 = \frac{15}{16}(x^2 - \frac{x^4}{4})$, Find the expectation value of the position $\langle x \rangle$.