Chapter 6

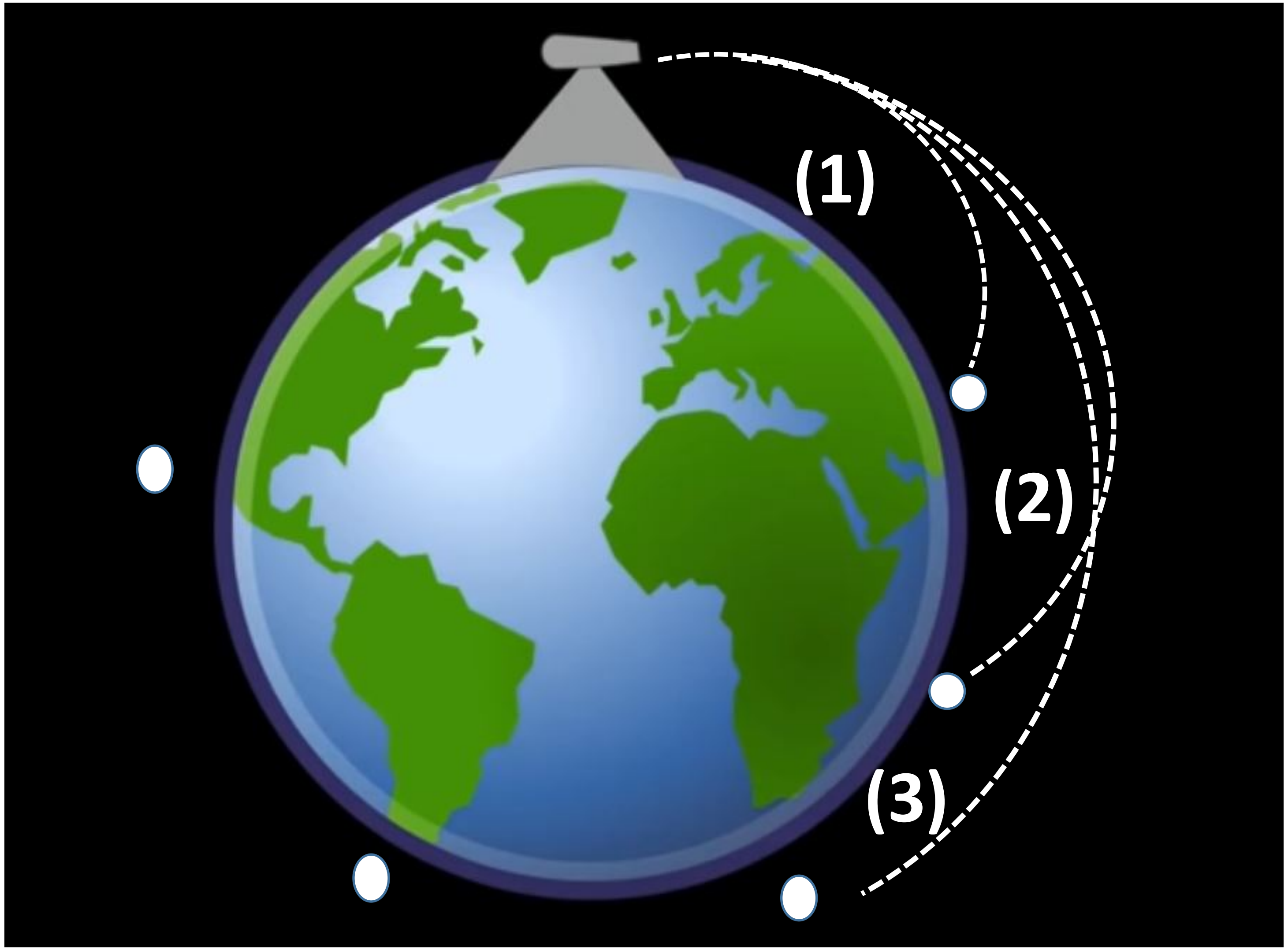
Gravitational and Central Force

6.1 Newton's Law of Universal Gravitation:

((*Every particle in the universe attracts every other particle with a force whose magnitude is proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between them. The direction of the force lies along the straight line connecting the two particles.*))

$$F_{ij} = G\frac{m_i m_j}{r_{ij}^2}\left(\frac{r_{ij}}{r_{ij}}\right)$$

$$G = (6.67259 \pm 0.00085 \times 10^{-11}\ \frac{Nm^2}{kg^2}$$



$$F_{ij} = G\frac{m_i m_j}{r_{ij}^2}\left(\frac{r_{ij}}{r_{ij}}\right)$$

- F_{ij} is the force on particle i of mass m_1 exerted by particle j of mass m_j.

- The vector r_{ij} is the directed line segment running from particle i to particle j,

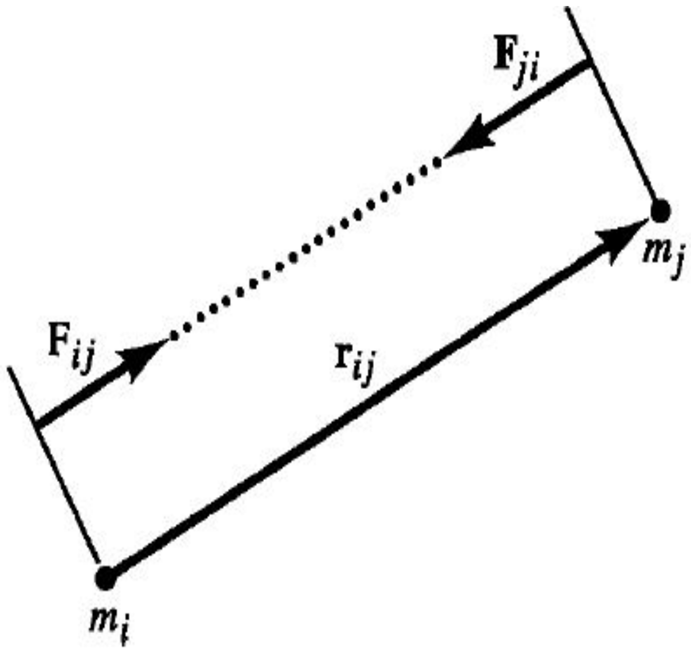


Figure 6.1.1 Action and reaction in Newton's law of gravity.

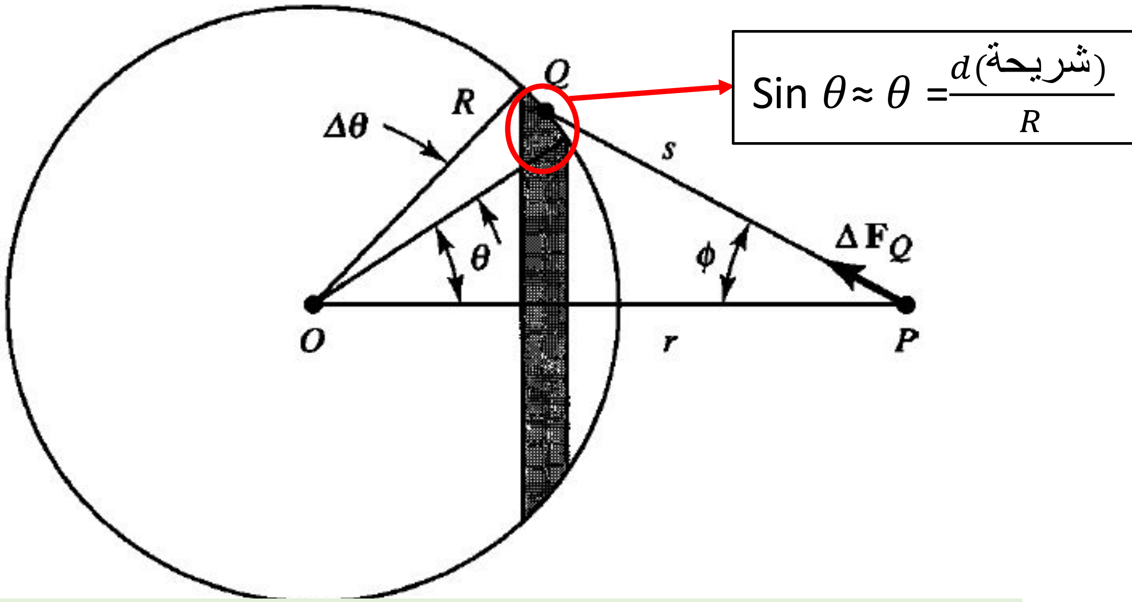
6.2 Gravitational Force between a Uniform Sphere and a Particle:

Consider first a thin uniform shell of mass M and radius R. Let r be the distance from the centre O to a test particle P of mass m (Fig. 6.2.1). We assume that r > R. We **shall divide the shell into circular rings** of width $R\,\Delta\,\theta$. Where, as shown in the figure, the angle

- The angle *POQ* is denoted by θ, *Q* being a point on the ring.
- Where S is the distance PQ (the distance from the particle P to the ring) as shown in above Figure.
- we can write the force between a shell and the particle as :

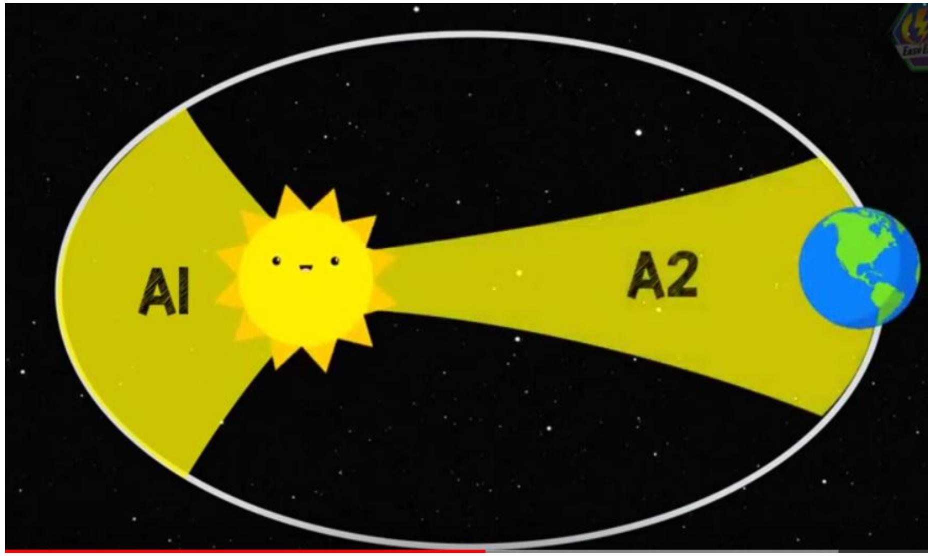
$$F = -G\,\frac{Mm}{r^2}\,e_r$$

e_r : is the radial vector from origin O.



- The gravitational force on a particle located inside a uniform spherical shell is zero.

6.3 Kepler's Laws of Planetary Motion:



I. Law of Ellipses (1609)

The orbit of each planet is an ellipse, with the Sun located at one of its foci (البؤرة)

II. Law of Equal Areas (1609)

A line drawn between the Sun and the planet sweeps out equal areas in equal times as the planet orbits the Sun.

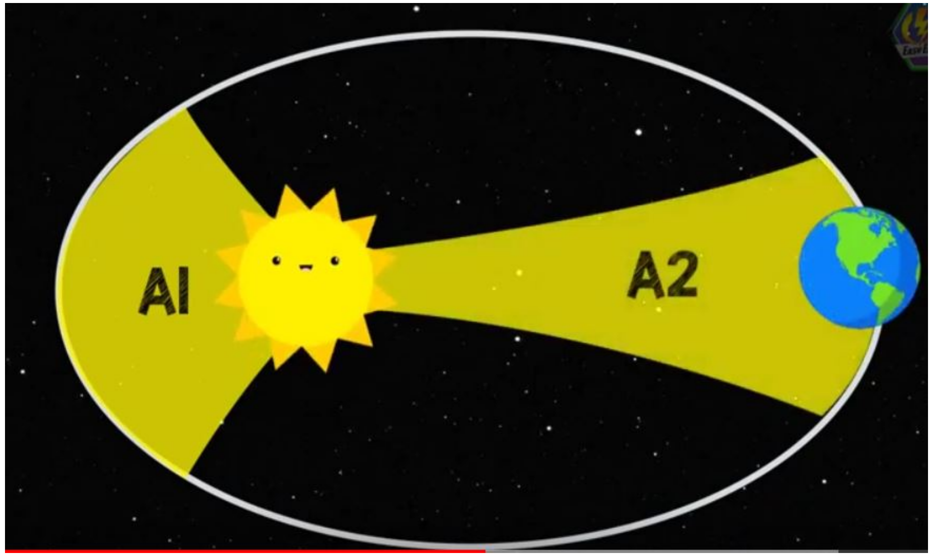
III. Harmonic Law (1618)

The square of the **sidereal period** **الفترة الفلكية** of a planet (the time it takes a planet to complete one revolution about the Sun relative to the stars) is directly proportional to the cube of the semi-major axis of the planet's orbit.

6.4 Kepler's Second Law: Equal Areas:

L = r x p

$$\frac{dL}{dt} = \frac{d(r \times p)}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$



- But $\frac{dr}{dt} = v$, so the first term in right became $v \times p = v \times mv = m(v \times v) = m(vv\, sin\theta) = 0 \quad as\ \theta = 0$

- And $\frac{dp}{dt} = F$ from 2nd law of **Newton,**

$$\frac{dL}{dt} = r \times F \qquad (6.4)$$

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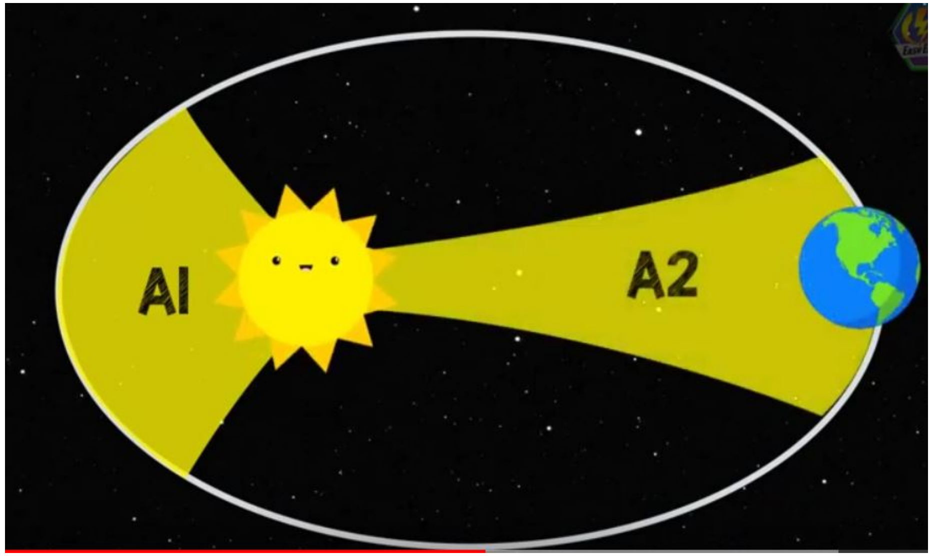


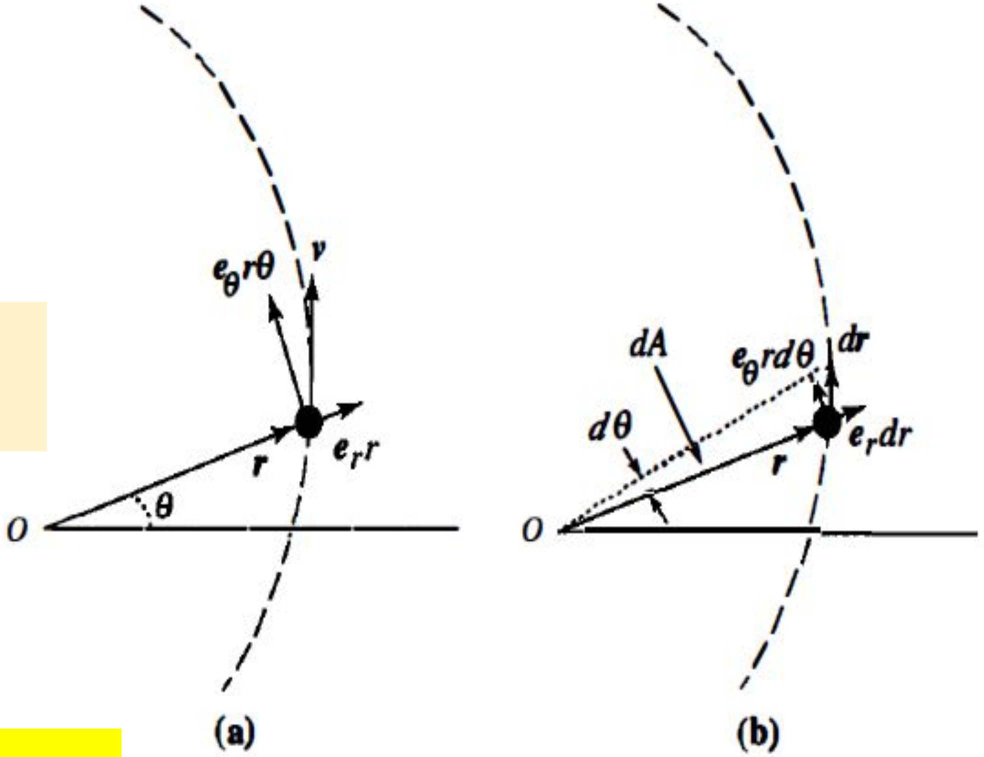
- $N = r \times F$: **moment of force**, or **torque**, on the particle about the origin of the coordinate system.

- If r and F are collinear, this cross product vanishes and so does L(i.e. dL/dt=0), so , **the angular momentum L, in such cases, is a constant of the motion.**

Angular Momentum and Areal Velocity of a Particle Moving in a Central Field

$\mathbf{L}$ any particle moving in a central field of force =conserved

- we first calculate the magnitude of the angular momentum of a particle moving in a central field.
- We use polar coordinates to describe the motion
- The velocity of the particle is



$$v = e_r\dot{r} + e_\theta r\dot{\theta}$$

In the Polar coordinates(see Chapter 1)

And we have :

$$L = r \times p$$

So, the magnitude will be: $L = |r \times mv|$ → $L = |re_r \times m(e_r\dot{r} + e_\theta r\dot{\theta})|$

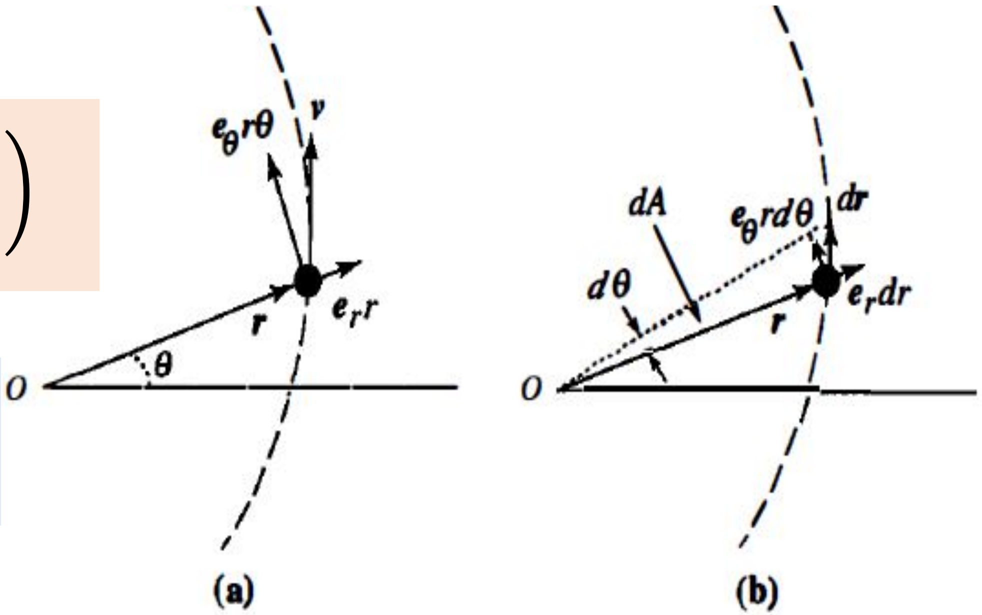
→ $L = mr^2\dot{\theta}$ =constant $\boldsymbol{as\ e_r \times e_r = 0\ and\ e_r \times e_\theta = 1}$

Now, we calculate the "areal velocity," $\dot{A}$, of the particle. Figure 6.4.l(b) shows the triangular area, dA, swept out by the radius vector r as a planet moves a vector distance dr in a time dt along its trajectory relative to the origin of the central field

$$A = \frac{1}{2}|r \times dr| = \frac{1}{2}|re_r \times (e_r dr + e_\theta r d\theta)| = \frac{1}{2} r\,(rd\theta))$$

$$\frac{dA}{dt} = \dot{A} = \frac{1}{2}\ r^2\dot{\theta} = \frac{L}{2m}$$

$$\frac{dA}{dt} = \dot{A} = \frac{L}{2m} = constant$$



Thus, the areal velocity, A, of a particle moving in a central field is directly proportional to its angular momentum and, therefore, is also a constant of the motion, exactly as Kepler discovered for planets moving in the central gravitational field of the Sun.

Example (1)

Let a particle be subject to an attractive central force of the from (r) , where r is the distance between the particle and the centre of the force. Find $f\ (r)$ if all circular orbits are to have identical areal velocities, $\dot{A}$.

Solution:

Because the orbits are circular, the acceleration,r, has no transverse component and is entirely in the radial direction. In polar coordinates, the acceleration is given by:

$$a = \ddot{r} - r\dot{\theta}^2$$

Thus,

$$ma_r = -mr\dot{\theta}^2 = f(r) \qquad \times(\frac{\boldsymbol{r^3}}{\boldsymbol{r^3}})$$

Because the orbits are circular, the acceleration, $i.e. \ddot{r} = 0$,

6.5 Kepler's First Law: Equal Areas:

To prove Kepler's first law, we develop a general differential equation for the orbit of a particle in any central, isotropic field of force. Then we solve the orbital equation for the specific case of an inverse-square law of force.

The equation of motion in polar coordinates is $m\ddot{r} = f(r)e_r$

Where $f(r)$ is the central, isotropic force that acts on the particle of mass m.

acceleration vector in polar coordinates

$$a = \ddot{r} = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta$$

So,

$$m(\ddot{r} - r\dot{\theta}^2)e_r = f(r)$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta = 0$$

No component toward θ direction

$$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

Or

$$r^2\dot{\theta} = constant = l$$

Where l is the angular momentum per unit mass:

$$l = \frac{L}{m} = |r \times v|$$

Given a certain radial force function f (r), we could, in theory, solve the pair of differential equations (Equations 6.10a and b) to obtain r and θ as functions of t. Often one is interested only in the path in space (the orbit) without regard to the time t. **To find the equation of the orbit, we use the variable u defined by**

$$r = \frac{1}{u} \text{ or } u = \frac{1}{r}$$

$$\text{And } l = r^2\dot{\theta} = \frac{1}{u^2}\dot{\theta}$$

$$dr = \dot{r} = \frac{-1}{u^2}\dot{u} = \frac{-1}{u^2}\frac{du}{d\theta}\frac{d\theta}{du} = \frac{-1}{u^2}\dot{\theta}\frac{du}{d\theta} = -l\frac{du}{d\theta}$$

$$dr = \dot{r} = \frac{-1}{u^2}\dot{u} = \frac{-1}{u^2}\ \frac{du}{d\theta}\frac{d\theta}{du} = \frac{-1}{u^2}\ \ \dot{\theta}\frac{du}{d\theta} = -\ l\ \frac{du}{d\theta}$$

As we employed the fact $l = \dot{\theta}u^2$ So the above equation can be written as:

$$\dot{r} = -\ l\ \frac{du}{d\theta} \xrightarrow{\text{H.W}} \ddot{r} = -\ l^2u^2\ \frac{d^2u}{d\theta^2}$$

Substituting the values found for r, $\dot{\theta}$, and $\ddot{r}$ into Equation 6.10a, we obtain

$$a = \ddot{r} = \left(\ddot{r} - r\dot{\theta}^2\right)e_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)e_\theta$$

$$m\left(\ddot{r} - r\dot{\theta}^2\right)e_r = f(r) \qquad \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)e_\theta = 0$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f(u^{-1})$$

Differential equation of the orbit of a particle moving under a central force.

Example (2):

A particle in a central field moves in the spiral orbit $r = c\theta^2$ Determine the force function.

Solution:

We have $u = \frac{1}{r} = \frac{1}{c\theta^2}$ and $\theta = \frac{1}{\sqrt{cu}}$ $\Rightarrow$ $\frac{du}{d\theta} = -\frac{2}{c}\frac{1}{\theta^3}$

$$\frac{d^2u}{d\theta^2} = -\frac{6}{c}\frac{1}{\theta^4} = 6\,cu^2$$

Now, eq. 6.17 will applied

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f(u^{-1})$$

$$6\,cu^2 + u = -\frac{1}{ml^2u^2}\,f(u^{-1})$$

$$f(u^{-1}) = -ml^2(6cu^2 + u^3)$$

$$f(r) = -ml^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right) \qquad \text{as } u = 1/r$$

Thus, the force is a combination of an inverse cube and inverse-fourth power law

$$f(r) = -\frac{mr^4}{r^3}\dot{\theta}^2$$

$$f(r) = -\frac{L^2}{mr^3}$$ as $L = mr^2\dot{\theta}$

OR

$$f(r) = -\frac{4m\dot{A}^2}{r^3}$$ as $\dot{A} = \frac{L}{2m}$

As we

latter

Or

Where