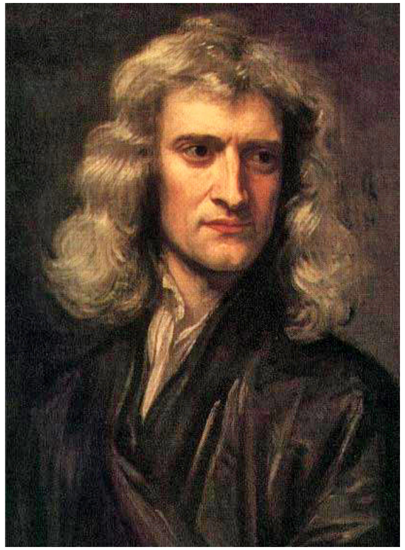
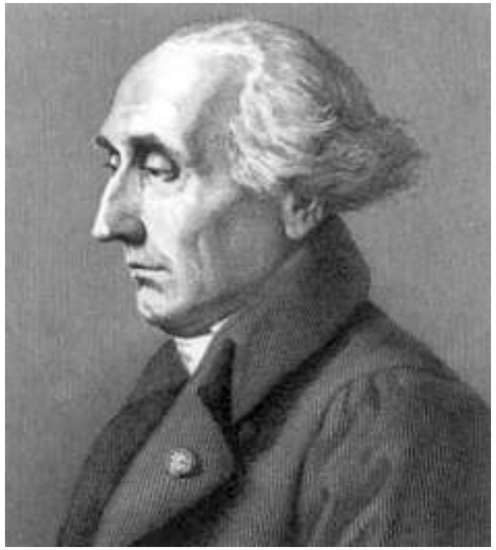
Chapter 7

Hamilton's Principle- Lagrangian and Hamiltonian Dynamics

Sir Isaac Newton

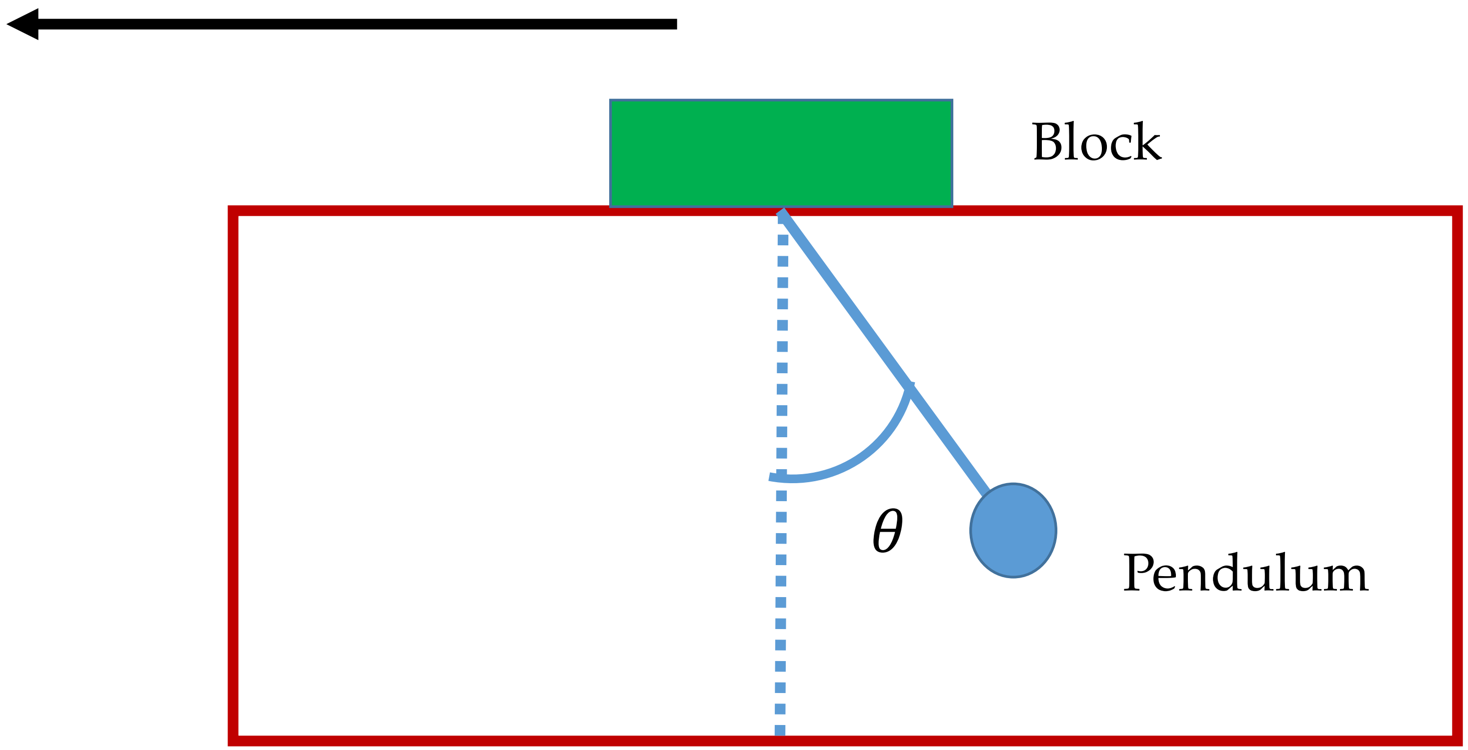




Joseph-Louis Lagrange

Experience has shown that a particle's motion in an internal reference frame is correctly described by the Newtonian equation (see Chapter 2) $F = \dot{p}$

This is a complicated system!



Lagrangian's equations (L)

$$L = \underbrace{T}_{\text{Kinetic energy}} - \underbrace{U}_{\text{Potential energy}}$$

Newtonian equations		**Lagrangian's equations**
Position Velocity Acceleration	Equation of motion	Energy

The Euler-Lagrange equations is given by

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1,2,3$$

2. Simple Harmonic Oscillator

تذكير من الفصل الثالث

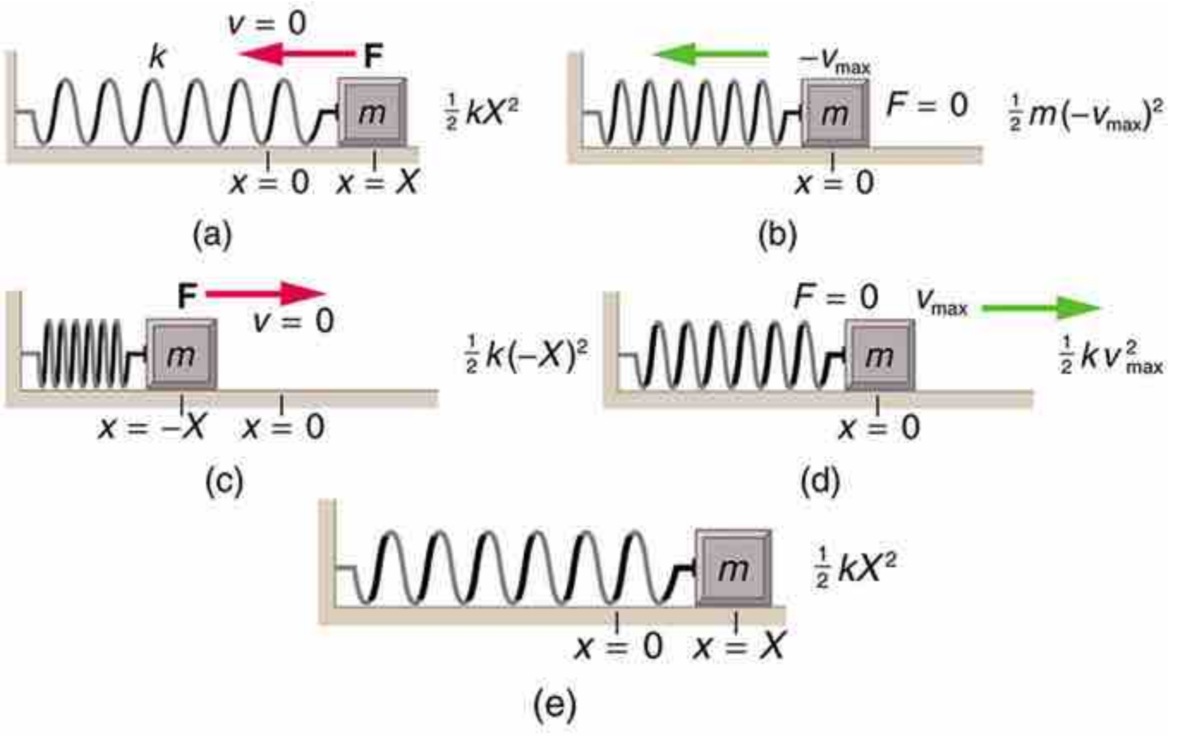
The equation of motion

Substitute the Hooke's Law in Newtonian equation

$$-kx = F$$

$$-kx = m\ddot{x}$$

Let $\omega_0^2 = {}^{k}/_{m}$



$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad \ddot{x} + \left(\frac{k}{m}\right)x = 0 \quad \Rightarrow \quad \boxed{\ddot{x} + \omega_0^2 x = 0}$$

Example 7.1: Use the Lagrange equation to obtain the equation of motion for one-dimensional harmonic oscillator.

Answer:

With the usual expressions for the kinetic and potential energies, we have

$$L = T - U$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\boxed{m\ddot{x} + kx = 0}$$

This is identical with the equation of motion obtained using Newtonian mechanics (See Chapter 3).

3. The Simple Pendulum

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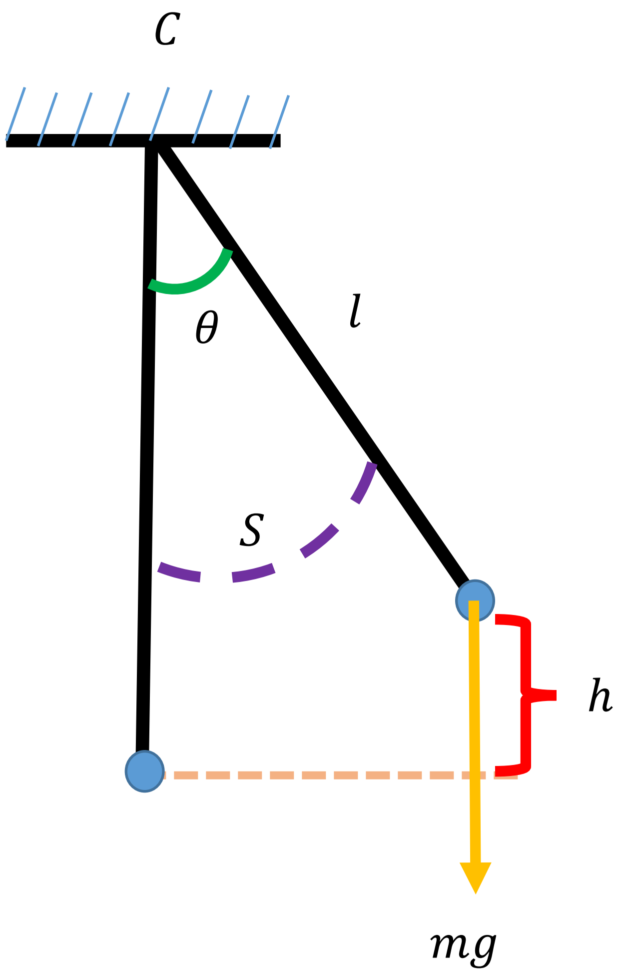
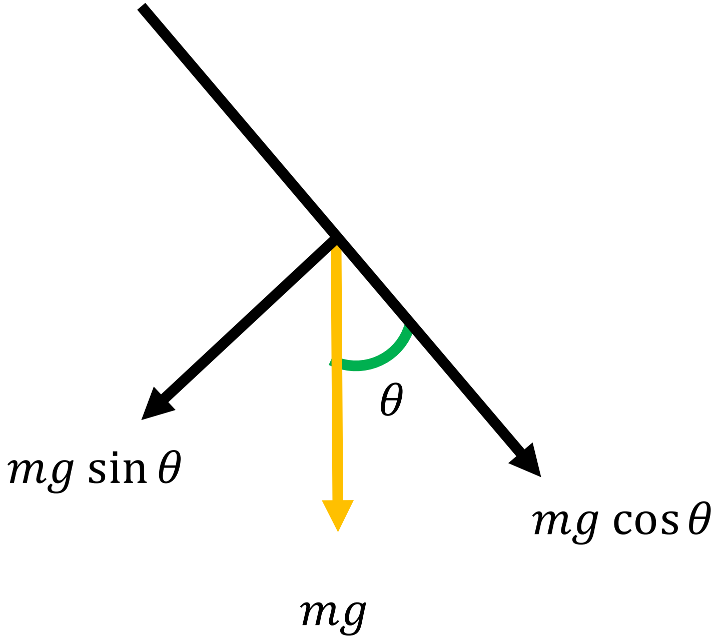




$$F_s = -mg\sin\theta \quad \Rightarrow \quad m\ddot{s} = -mg\sin\theta \quad \Rightarrow \quad \cancel{m}\ddot{s} + \cancel{m}g\sin\theta = 0$$

$$\ddot{s} + g\sin\theta = 0 \qquad s = l\theta \qquad \ddot{s} = l\ddot{\theta}$$

$$l\ddot{\theta} + g\sin\theta = 0 \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0} \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

$$\boxed{\tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}}$$

$$\boxed{\omega_0 = \sqrt{\frac{g}{l}}}$$

Simple pendulum

Example 7.2: Use the Lagrange equation to obtain the equation of motion of Simple pendulum.

Answer:

The kinetic and potential energies of the system are given by:

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgl(1 - \cos\theta)$$

$$L = T - U$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\boxed{L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)}$$

$$\boxed{\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0}$$

$$\frac{\partial L}{\partial \theta} = -mgl\ sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$-mgl\ sin\theta - ml^2\ddot{\theta} = 0$$

$$g\ sin\theta + l\ddot{\theta} = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{g}{l}\right) sin\theta = 0}$$

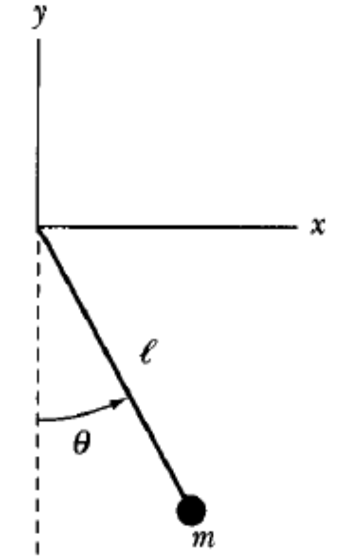
This is a remarkable result!

It has been obtained by calculating the kinetic and potential energies in terms of θ rather than x and then applying a set of operations designed for use with rectangular rather than angular coordinates.

Another important characteristic of the method used in two preceding simple examples is that nowhere in the calculation did there enter any statement **regarding force.**

Example 7.3. Use the (x, y) coordinate system to find the kinetic energy T, potential energy U, and the Lagrangian L for a simple pendulum (length l, mass bob m) moving in x,y plane .Determine the transformation equations from the (x, y) rectangular system to the coordinate θ. Find the equation of motion.

Answer:



The kinetic and potential energies and the Lagrangian become

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad\qquad U = mgy$$

$$\boxed{L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy}$$

Inspection reveals that the motion can be better described by using θ and $\dot{\theta}$. Let's transform x and y into the coordinate θ and then find L in terms of θ.

$$x = l\sin\theta \qquad\qquad y = -l\cos\theta$$

We now find for $\dot{x}$ and $\dot{y}$

$$\dot{x} = l\dot{\theta}\cos\theta \qquad\qquad \dot{y} = l\dot{\theta}\sin\theta$$

$$L = \frac{m}{2}\left(l^2\dot{\theta}^2 cos^2\theta + l^2\dot{\theta}^2 sin^2\theta\right) + mgl\cos\theta$$

$$L = \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$\ddot{\theta} + \left(\frac{g}{l}\right) sin\theta = 0$$

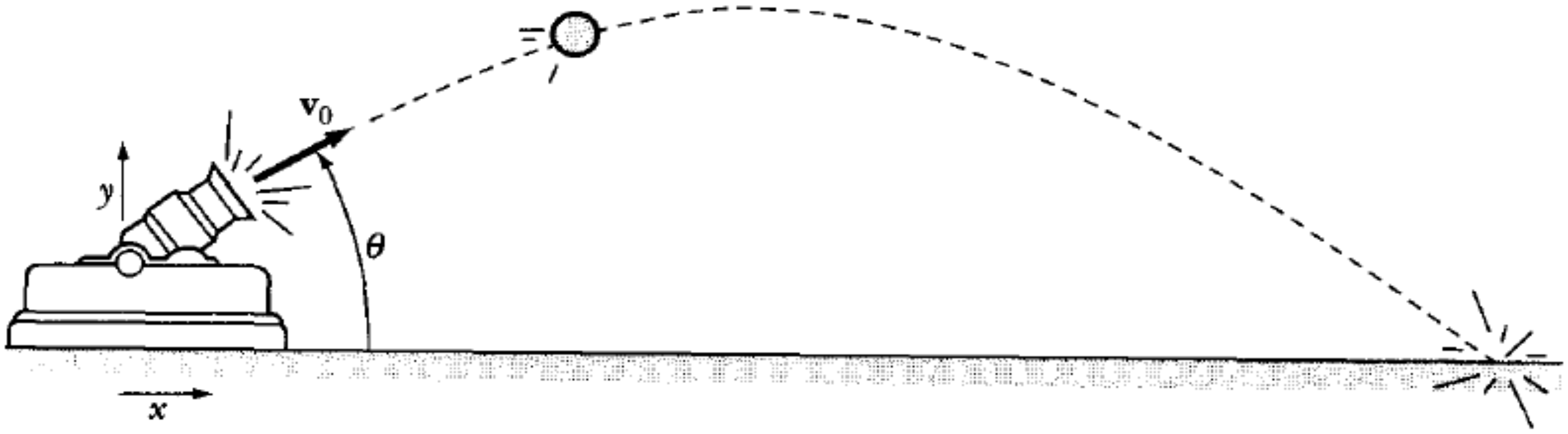
7.4 Lagrange's Equations of motion in Generalized coordinates.

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1,2, \dots \dots \dots .., s$$

It is important to realize that the validity of Lagrange's equation requires the following two conditions:

1. The force acting on the system (apart from any forces of constraint) must be derivable from the potential
2. The equations of constraint must be relations that connect the coordinates of the particles and may be functions of the time.

Example 7.4: Consider the case of projectile motion under gravity in two dimensions (as was discussed in Chapter 2). Find the equations of motion in both Cartesian and polar coordinates.



In Cartesian coordinate, we use xx (horizantoal) and y y (vertical).

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = mgy$$

Where $U = 0 \ at \ y = 0$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

x:	$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$ $$0 - \frac{d}{dt}\, m\dot{x} = 0$$ $$\ddot{x} = 0$$
y:	$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$ $$-mg - \frac{d}{dt}(m\dot{y}) = 0$$ $$\ddot{y} = -g$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

In polar coordinate, we use r (in radial direction) and θ (elevation angle from horizontal)

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(r\dot{\theta}\right)^2 \qquad U = mgr\sin\theta$$

Where $U = 0$ for $\theta = 0$

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

r:	$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$ $mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$ $r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$
θ:	$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$ $-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$ $-gr\cos\theta - 2r\dot{r}\theta - r^2\ddot{\theta} = 0$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

Canonical Equations of Motion-Hamiltonian

In the previous section, we found that if the potential energy of a system is velocity independent, then the linear momentum components in rectangular coordinates are given by

$$\boxed{p_i = \frac{\partial L}{\partial \dot{x}}}$$

By analogy, we extend this result to the case in which the Lagrangian is expressed in generalized coordinates and define the generalized momenta* according to

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$$

Using the definition of the generalized momenta for the Hamiltonian may be written as

$$\boxed{H = \sum p_j \, \dot{q}_j - L}$$

$$\dot{q}_j = \dot{x} = v$$

$$p_j = p = linear\ momentum$$

$$H = \sum \underbrace{p_j \ \dot{x}_j}_{sine \ pv \ =} - L$$

$$As \ T = \left(\frac{1}{2} m v^2\right)$$

$sine \ \ pv =$ $(mv) \ v = mv^2$ $\Rightarrow$ $2 \left(\frac{1}{2} m v^2\right)$ $\Rightarrow$ $2 \, T$

$H = 2T \ - T + U$ $\Rightarrow$ $H = T + U$

$L = T - U$

$Hamiltonian$

$Lagrangian$

Example 7.5: Find the equations of motion for a system of particle moving in a potential region where $U = cx$ using Hamiltonian method.

If $U = cx$

Particle gain height

$$H = T + U$$

$$H = \frac{1}{2} m\dot{x}^2 + cx$$

$$sine \;\; p = m\dot{x}$$

$$p^2 = m^2\dot{x}^2$$

$$H = \frac{m^2\dot{x}^2}{2m} + cx$$

$$H = \frac{p^2}{2m} + cx$$

$$\frac{\partial H}{\partial x} = c$$

$$\frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\frac{\partial H}{\partial p} = \frac{m\dot{x}}{m} = \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}$$

$$\boxed{m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right)+\frac{\partial H}{\partial x}=0}$$

$$\boxed{m\ddot{x}+c=0}$$

OR

$$\boxed{\ddot{x}=-\frac{c}{m}}$$

$Equation\ of\ motion$

Example 7.6: Find the equations of motion for a system of simple Oscillator using the Hamiltonian method.

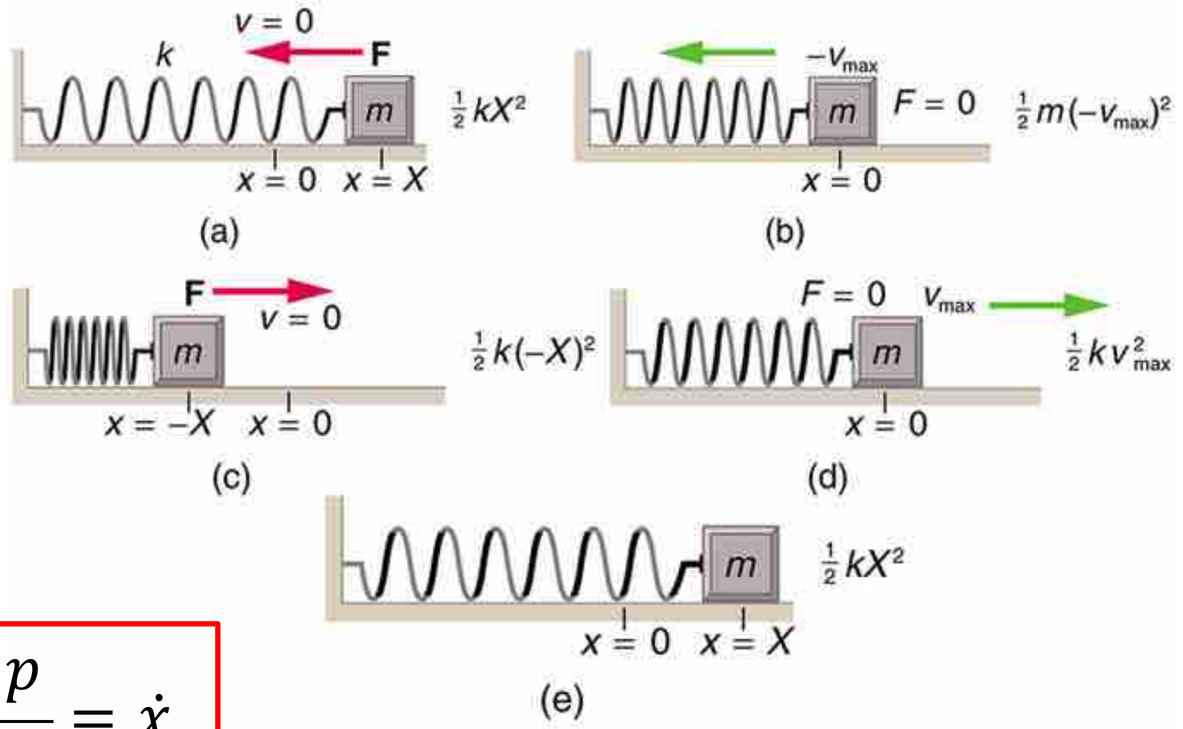
$$\boxed{H = T + U}$$

$$\boxed{H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2}$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2}$$

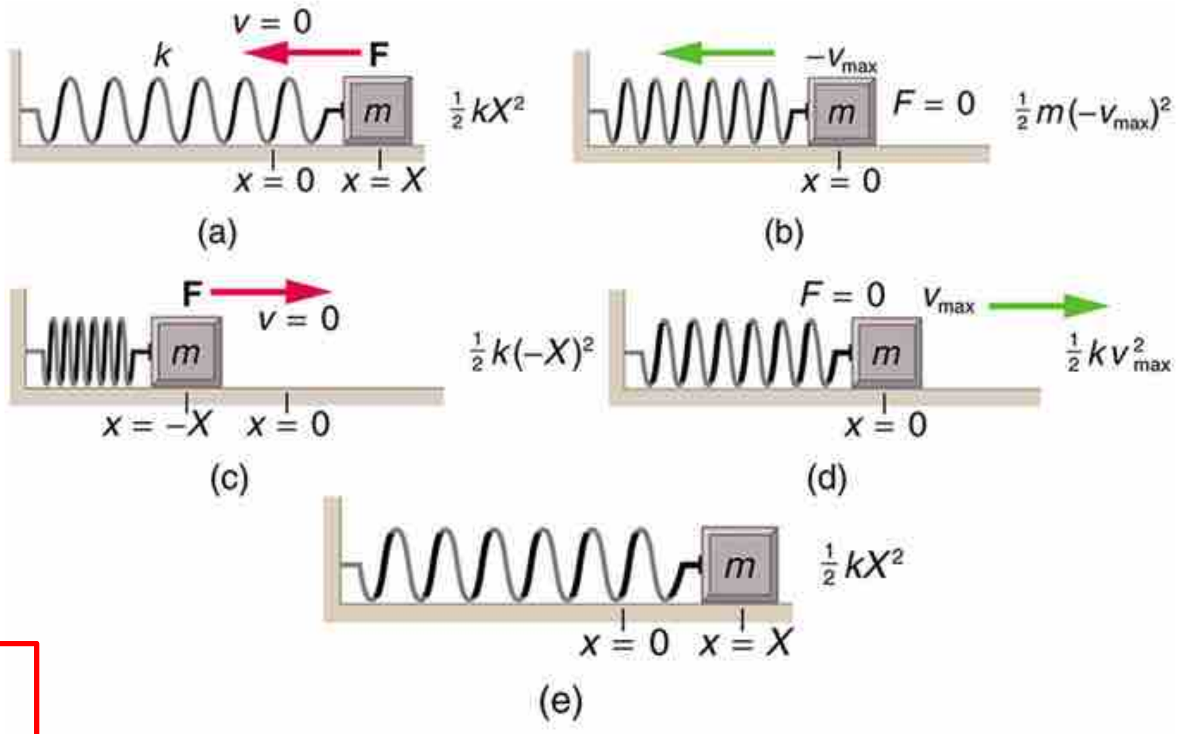
$$\boxed{\frac{\partial H}{\partial x} = kx}$$

$$\boxed{\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}}$$



$$\frac{\partial H}{\partial x} = kx$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$



$$\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}$$

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial x} = 0$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

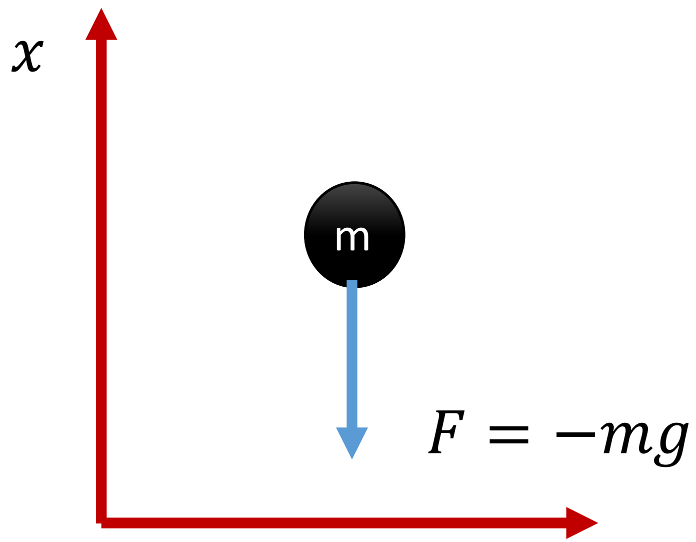
$$\text{As } \omega^2 = \frac{k}{m}$$

Example 7.7: Find the equations of motion for a system of free fall particle using the Hamiltonian method.

$$\boxed{H = T + U} \Rightarrow \boxed{H = \frac{1}{2}mv^2 + mgx}$$

$$\boxed{H = \frac{1}{2}m\dot{x}^2 + mgx} \Rightarrow \boxed{p = mv = m\dot{x}}$$

$$\boxed{p^2 = m^2\dot{x}^2} \Rightarrow \boxed{H = \frac{p^2}{2m} + mgx}$$



$$\boxed{\frac{\partial H}{\partial x} = mg} \qquad \boxed{\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) = \ddot{x}}$$

$$\boxed{m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right)+\frac{\partial H}{\partial x}=0}$$

$$\boxed{m\ddot{x}+mg=0}$$

OR

$$\boxed{\ddot{x}=-g}$$

$Equation\ of\ motion$

In general coordinate

Now for a system expressed in the generalized coordinates q_i and $\dot{q_i}$

q is a general coordinate $(x, y, z, \theta, , , \dots)$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j$$

Canonical Equations

$$\frac{\partial H}{\partial p} = \dot{x}$$

$$\frac{\partial H}{\partial x} = -\dot{p}$$

X-direction

Example 7.8: Obtain Hamilton's equations of motion for one-dimensional harmonic oscillator and use them to find the differential equation.

$$T = \frac{P^2}{2m} \qquad U = \frac{1}{2}kx^2$$

$$\boxed{\frac{\partial H}{\partial p_j} = \dot{q}_j} \qquad \boxed{\frac{\partial H}{\partial q_j} = -\dot{p}_j}$$

Canonical Equations

$$H = T + U$$

One-dimension

$$H = \frac{P^2}{2m} + \frac{1}{2}kx^2$$

$$\boxed{\frac{\partial H}{\partial x} = -\dot{p}} \qquad \boxed{\frac{\partial H}{\partial p} = \dot{x}}$$

Left-hand side

Left-hand side

$$\frac{\partial H}{\partial x} = kx \qquad \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$kx = -\dot{p}$$

$$\frac{p}{m} = \dot{x}$$

$$kx = -\frac{\partial p}{\partial t}$$

$$p = m\dot{x}$$

$$kx = -\frac{\partial (p)}{\partial t}$$

$$kx = -\frac{\partial (m\dot{x})}{\partial t}$$

$$kx = -m\ddot{x}$$

$$\boxed{kx + m\ddot{x} = 0}$$

Lagrangian VS Hamiltonian

$L = T - V$

$H = T + V$

$L(q, \dot{q})$ OR $L(x, \dot{x})$

$H(q, p)$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{p}$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \frac{\partial H}{\partial q_j} = -\dot{p}_j$$

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial q} = 0$$

Equation of Motion