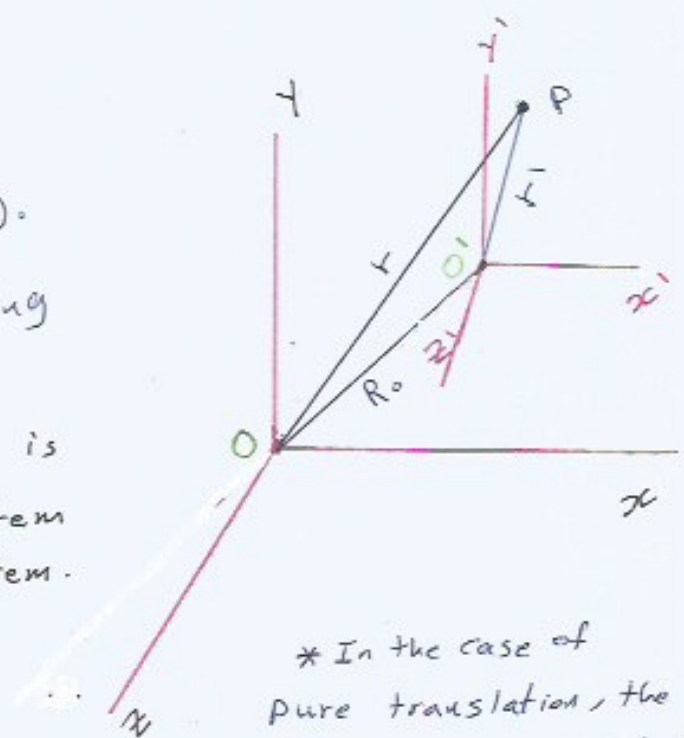# Chapter Five

Noninertial Reference Systems

1. Accelerated Coordinate Systems and Inertial Forces

In describing the motion of a particle, it is frequently convenient and sometimes necessary to employ a coordinate system that is not inertial. For example, a coordinate system fixed to the Earth is the most convenient one to describe the motion of a projectile, eventhough the Earth is accelerating and rotating.

* Consider $O\,xyz$ the primary coordinate axes (assumed fixed).

* Consider $O'x'y'z'$ the moving axes.

* The position of a partice P is denoted by r in the fixed system and by r' in the moving system.

* In the case of pure translation, the respective OX and $O'x'$ are remain parallel.

The displacement OO' of the moving origin is denoted by R_0. Thus, from the triangle OO'P we have

$$\vec{r} = \vec{R_0} + \vec{r^1}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{V} = \vec{V_0} + \vec{v^1}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = a = \vec{A_0} + \vec{a^1}$$

* Where, $\vec{V_0}$ and $\vec{A_0}$ are the velocity and acceleration of the moving system ($O' x'y'z'$).

* The $\vec{v^1}$ and $\vec{a^1}$ are the velocity and acceleration of the particle in the moving system ($O' x'y'z'$)

* The $\vec{V}$ and $\vec{a}$ are the velocity and acceleration of the particle in the non-moving system ($OXYZ$).

Now, if the moving system is not accelerated $A_0 = 0 \Rightarrow$

$$a = a^1$$

Newton's 2nd law

$$F = m\vec{a} = m\vec{a^1}$$

The moving system is also an inertial system (provided it is not rotating).

Now, if the system is accelerated $\Rightarrow$ Newton's 2nd law

$$\vec{F} = m\vec{a} = m(\vec{A_0} + \vec{a}')$$

$$\vec{F} = m\vec{A_0} + m\vec{a}'$$

$$\vec{F} - m\vec{A_0} = m\vec{a}'$$

In which $\vec{F}' = \vec{F} + (-m\vec{A_0})$ [Equation of motion]

* That is, an acceleration A_0 of the reference system can be taken into account by adding an inertial term $(-mA_0)$ to the force F and equating the result to the product of mass and acceleration in the moving system.

* $(-mA_0)$ is called (inertial force) or fictitious force). Such "forces" are not due to interaction with other bodies, rather, they stem from the acceleration of the reference system.

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Example 1 A block of wood rests on a rough horizontal table. If the table is accelerated in a horizontal direction, under what conditions will the block slip?

Answer:

Let μ_s be the coefficient of static friction between the block and the table top.

The condition of slipping is that the inertial force $(-m A_0)$ exceeds the frictional force, where A_0 is the acceleration of the table.

Hence, the condition for slipping is

$$|-m A_0| > \mu_s mg$$

or $$\boxed{A_0 > \mu_s g}$$

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Example 2

A pendulum is suspended from the ceiling of a railroad car, as shown in the figure (Next page). Assume that the car is accelerating uniformly toward the right ($+x$ direction). A non inertial observer, the boy inside the car, sees the pendulum hanging at an angle θ, left of vertical. He believes it hangs this way because of the existence of an inertial force F_x', which acts on all objects in his accelerated frame of reference. An inertial observer, the girl outside the car, sees the same things. She knows, however, that there is no real force F_x' acting on the pendulum. She knows that it hangs this way because a net force in the horizontal direction is required to accelerate it at the rate A_o that she observes. Calculate the acceleration A_o of the car from the inertial observer's point of view. Show that, according to the noninertial observer, $F_x' = -m A_o$ is the force that causes the pendulum to hang at the angle θ.

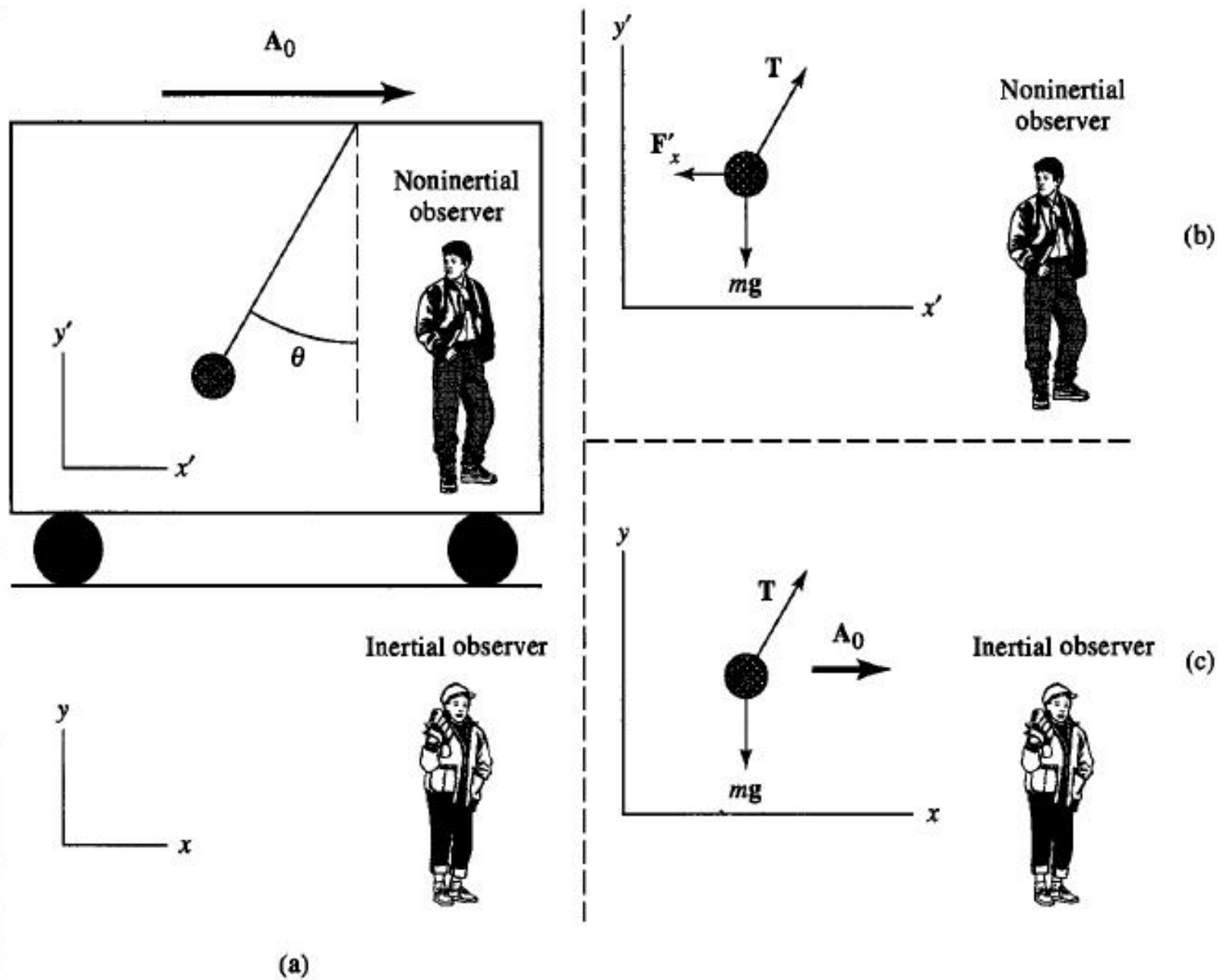Answer: The inertial observer writes down Newton's 2nd law for the hanging pendulum as

$$\sum F_i = ma$$

$$T \sin \theta = m A_o \qquad T \cos \theta - mg = 0$$

$$\therefore A_o = g \tan \theta$$

Figure 5.1.2 (a) Pendulum suspended in an accelerating railroad car as seen by (b) the noninertial observer and (c) the inertial observer.

She concludes that the suspended pendulum hangs at angle θ because the railroad car is accelerating in the horizontal direction and a horizontal force is needed to make it accelerate.

This force is the x-component of the tension in the spring. The acceleration of the car is proportional to the tangent of the angle of deflection. The pendulum, thus, serves as a linear accelerometer.

On the other hand, the noniertial observer unaware of the outside world [car with no window] observes that the pendulum just hangs there, tilted to the left of vertical. He concludes that

$$\sum F_i' = ma' = 0$$

$$T \sin \theta - F_{x'} = 0 \qquad\qquad T \cos \theta - mg = 0$$

$$\therefore F_{x'} = mg \tan \theta$$

All the forces acting on the pendulum are in balance, and the pendulum hangs left of vertical due to the force ($F_{x'} = -mA_0$).

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2. Rotating Coordinate Systems

* Here we will show that velocities, accelerations and force tranform between an inertial fram of reference and anoniertial one, that is rotating as well.

* We start with primed coordinate system rotating with respect to an unprimed, fixed, inertial one. The axes of the coordinate systems have a common origin.

* The rotation of the primed system takes place about some specific axis of rotation, whose direction is designated by unit vector ((n))
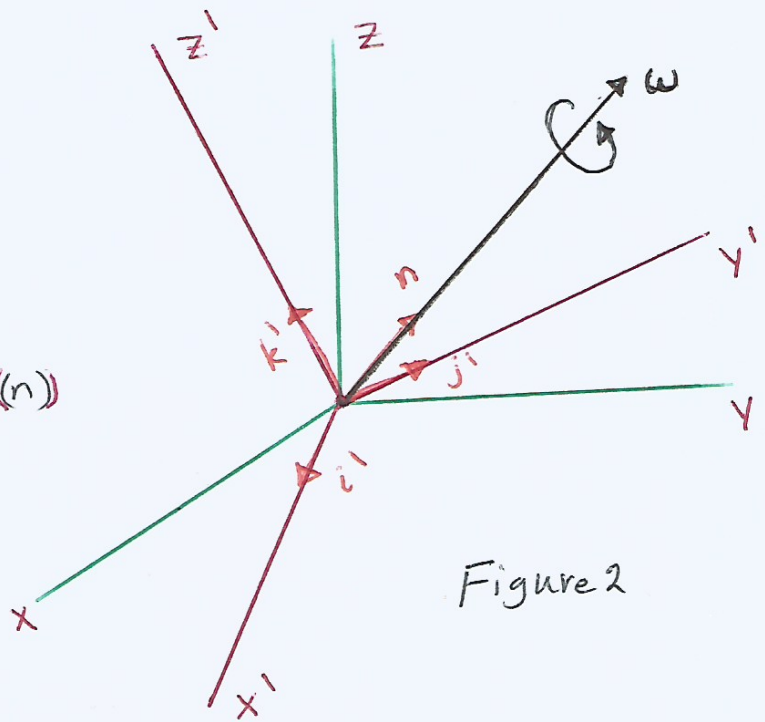

Figure 2

* The instantaneous angular speed of the rotating is designated by ((w))

* The product ((wn)) is the angular velocity of rotating system.

$$W = W\,n \qquad \text{---(5.9)}$$

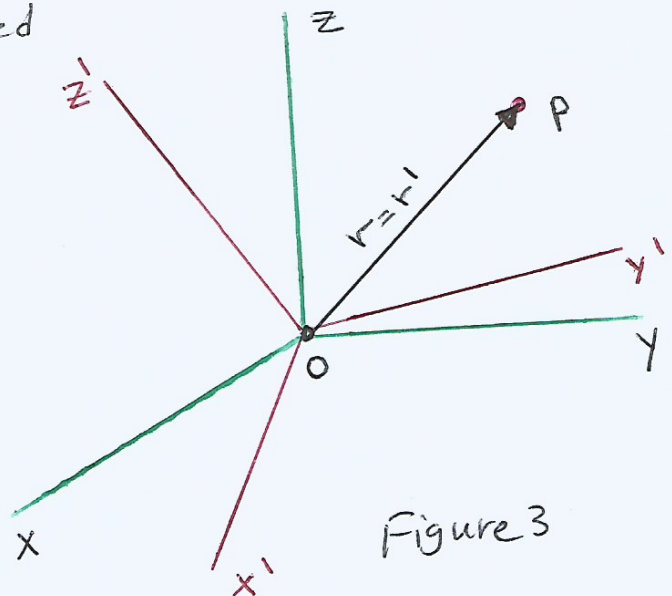Figure 3

The coordinate axes of the two systems have the same origin, i.e the position vectors are equal

$$r = r'$$ --- (5.10)

$$ix + jy + kz = i'x' + j'y' + k'z'$$ --- (5.11)

So, by deferentating both side with time

$$i \frac{dx}{dt} + j \frac{dy}{dt} + k \frac{dz}{dt} = i' \frac{dx'}{dt} + j' \frac{dy'}{dt} + k' \frac{dz'}{dt}$$

$$+ x' \frac{di'}{dt} + y' \frac{dj'}{dt} + z' \frac{dk'}{dt}$$ --(5.12)

The three terms [left hand side] $\Rightarrow$ Velocity v in the

fixed system.

The first thee terms [right hand side] $\Rightarrow$ velocity v'
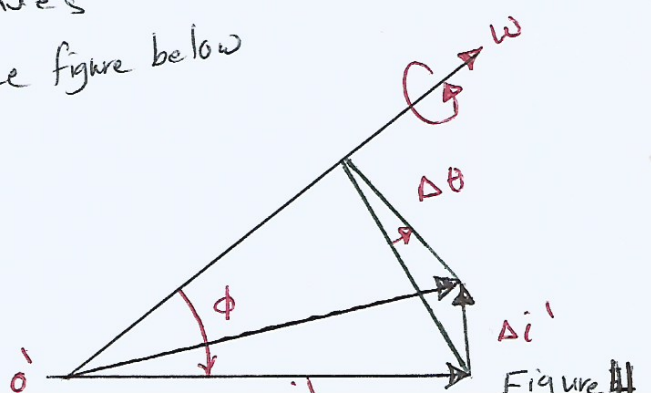
in the rotating system

Thus,

$$v = v' + x' \frac{di'}{dt} + y' \frac{dj'}{dt} + z' \frac{dk'}{dt}$$ --- (5.13)

The last three terms [right hand side] $\Rightarrow$ velocity

due to the rotating
of the primed coordinate

system.

To find the time derivatives

$$\left[\frac{di'}{dt}, \frac{dj'}{dt}, \text{ and } \frac{dk'}{dt} \right], \text{ see figure below}$$

Figure 11

$\Delta i'$ is the unit vector i' due to a small rotation $\Delta \theta$ about the axis of rotating.

From the figure, we see

$$|\Delta i'| \approx (|i|' \sin \phi) \Delta \theta \equiv \sin \phi \, \Delta \theta \qquad \text{--- (5.14)}$$

where ϕ is the angle between i' and w. Let Δt be the time interval for this change. Then, we can write:

$$\left| \frac{di'}{dt} \right| = \lim_{\Delta t \to 0} \left| \frac{\Delta i'}{\Delta t} \right| = \sin \phi \, \frac{d\theta}{dt}$$

$$= (\sin \phi) w \qquad \text{--(5.15)}$$

And, $\quad \dfrac{di'}{dt} = w \times i'$

$$\left. \begin{array}{l} \\ \dfrac{dk'}{dt} = w \times k' \end{array} \right\} \text{--(5.16)}$$

Similarly $\quad \dfrac{dj'}{dt} = w \times j'$

Thus,

$$x' \frac{di'}{dt} + y' \frac{dj'}{dt} + z' \frac{dk'}{dt} = x'(w \times i') + y'(w \times j') + z'(w \times k')$$

$$= w \times (i'x' + j'y' + k'z')$$

$$= w \times r' \qquad \text{--- (5.17)}$$

This is the velocity of P due to the rotation of the primed coordinate system. Accordingly, equation (5.13) can be shortened to read

$$v = v' + w \times r' \qquad \text{--- (5.18)}$$

More explicitly,

$$\left(\frac{dr}{dt}\right)_{fixed} = \left(\frac{dr'}{dt}\right)_{rot} + \omega \times r' = \left[\left(\frac{d}{dt}\right)_{rot} + \omega \times\right] r'$$

(5-1a)

A little reflections shows that the same applies to any vector Q, that is

$$\left(\frac{dQ}{dt}\right)_{fixed} = \left(\frac{dQ}{dt}\right)_{rot} + \omega \times Q \qquad (5-20)$$

let $[Q = v]$

$$\left(\frac{dv}{dt}\right)_{fixed} = \left(\frac{dv}{dt}\right)_{rot} + \omega \times v \qquad --- (5-21)$$

But $v = v' + \omega \times r'$ [equation 5.18]

$$\left(\frac{dv}{dt}\right)_{fixed} = \left(\frac{d}{dt}\right)_{rot} (v' + \omega \times r') + \omega \times (v' + \omega \times r') \qquad --- (5-22)$$

$$= \left(\frac{dv'}{dt}\right)_{rot} + \left[\frac{d(\omega \times r')}{dt}\right]_{rot} + \omega \times v' + \omega \times (\omega \times r') \qquad --- (5-23)$$

$$= \left(\frac{dv'}{dt}\right)_{rot} + \left(\frac{d\omega}{dt}\right)_{rot} \times r' + \omega \times \left(\frac{dr'}{dt}\right)_{rot} + \omega \times r'$$

$$+ \omega \times (\omega \times r') \qquad --- (5-24)$$

let $[Q = \omega]$

$$\left(\frac{d\omega}{dt}\right)_{fixed} = \left(\frac{d\omega}{dt}\right)_{rot} + \omega \times \omega \qquad --- (5-25)$$

But $\omega \times \omega = 0$

$\therefore \left(\dfrac{d\omega}{dt}\right)_{fixed} = \left(\dfrac{d\omega}{dt}\right)_{rot} = \dot{\omega}$ ---(5.26)

Because $v' = \left(\dfrac{dr'}{dt}\right)$ and $a' = \left(\dfrac{dv'}{dt}\right)_{rot}$ --- (5.27)

Thus,

$$\boxed{a = a' + \dot{\omega} \times r' + 2\omega \times v' + \omega \times (\omega \times r')}$$ --(5.28)

   Equation (5.28) given the <u>acceleration</u> in the <u>fixed</u> system in terms of the <u>position</u>, <u>velocity</u>, and <u>acceleration</u> of the <u>rotating</u> system.
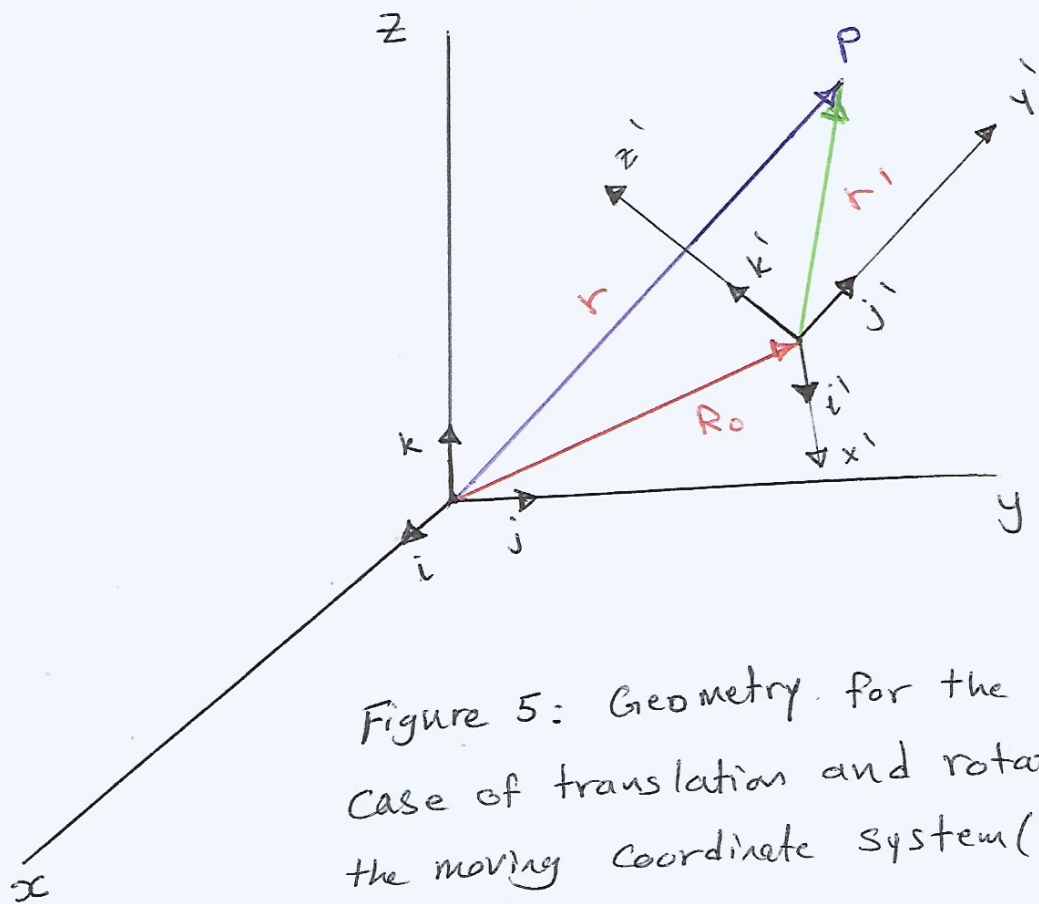

Figure 5: Geometry for the general case of translation and rotation of the moving coordinate system (primed system).

* The general case [system is undergoing both translation and rotation [see Figure 5], we must add the velocity of translation V_o to the right-hand side of Equation (5-18) and the acceleration (A_o) to the right-hand side of equation (5-28), this gives the general equations for transformation from a fixed system to a moving and rotating system.

$$U = v' + \omega \times r' + V_o \qquad \qquad \text{---(5.29)}$$

$$a = a' + \dot{\omega} \times r' + 2\omega \times v' + \omega \times (\omega \times r') + A_o$$
$$\text{--(5.30)}$$

* The term $2\omega \times v'$ is known as Coriolis acceleration. This term appears whenever a particle moves in a rotating coordinate system [except when the velocity v' is parallel to the axis of rotation].

* The term $\omega \times (\omega \times r')$ is known as Centripetal acceleration. This term is the result of the particle being carried around a circular path in the rotating system.
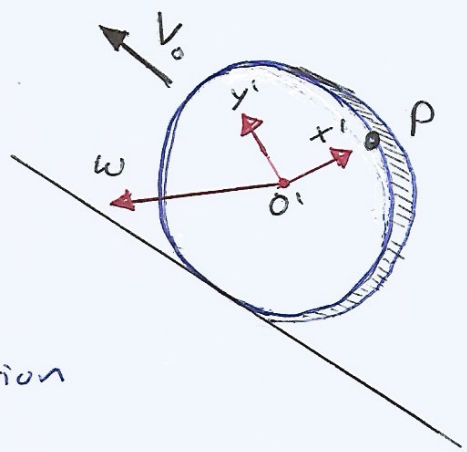
* The term $\dot{\omega} \times r'$ is called the transverse acceleration because it is perpendicular to the position vector r'. It appears as a result of any angular acceleration of the rotating system.

Example 3 | A wheel of radius b rolls along the ground with constant forward speed V_o. Find the acceleration relative to the ground, of any point on the rim.

Solution:

Let us choose a coordinate system fixed to the rotating wheel, and let moving origin be at the center with the x'-axis passing through the point in question as shown the figure.

We have, $r' = i'b$ $a' = \ddot{r}' = 0$ $v' = \dot{r}' = 0$

The angular velocity vector is given by

$$\omega = k'\omega = k'\frac{V_o}{b} \qquad \begin{array}{l} \dot{\omega} = 0 \\ A_o = 0 \end{array} \left[\begin{array}{c} rotating \\ only \end{array}\right]$$
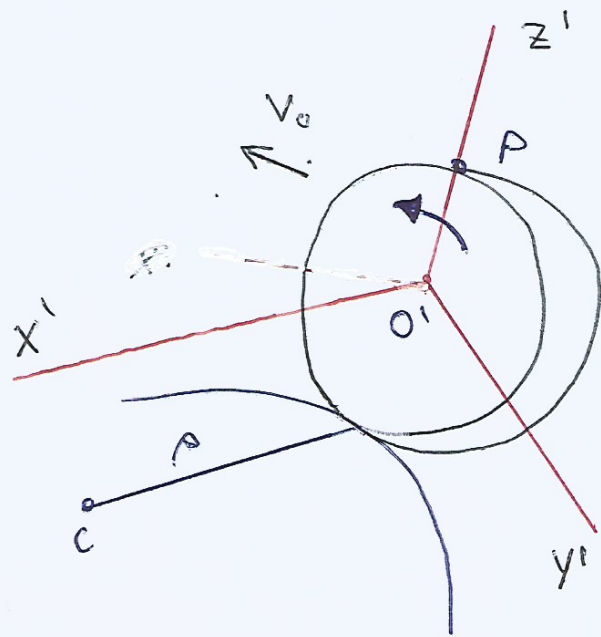
for the choice of coordinates shown: therefore all terms in the expression for acceleration vanish except the centripetal term

$$a = \omega \times (\omega \times r') = k'\omega \times (k'\omega \times i'b)$$

$$= \frac{V_o^2}{b} k' \times (k' \times i')$$

$$= \frac{V_o^2}{b} k' \times j'$$

$$\therefore \vec{a} = \frac{V_o^2}{b}(-\vec{i}') \quad [\text{directed toward the center of the}$$
$$\text{rolling wheel }]$$

Example 4 | A bicycle travels with constant speed around a track of radius ρ. what is the acceler-ation of the highest point on one of its wheels? Let V_o denote the speed of the bicycle and b the radius of the wheel.

We choose a coordinate system with origin at the center of the wheel and with the $\vec{x}'$-axis horizontal pointing toward the center of curvature C of the track.

~~The rotating~~
Rather than having the moving coordinate system rotate with the wheel, we choose a system in which the z'-axis remains vertical. Thus, $O'x'y'z'$ system rotates with angular velocity ω, which can be expressed as

$$\omega = k' \frac{V_o}{b}$$

And, the acceleration of the moving origin A_o is given by

$$A_o = i' \frac{V_o^2}{\rho}$$

Each point on the wheel is moving in a circle of radius b with respect to the moving origin $\longrightarrow$ the acceleration in the $o'\,x'y'z'$ system of any point on the wheel is directed toward o' and has magnitude $\frac{V_o^2}{b}$. !

Thus, in the moving system, we have:

$$a' = \ddot{r}' = -k' \frac{V_o^2}{b} \implies v' = -j' V_o$$

* The angular velocity ω is constant $[\dot{\omega} = 0]$

* The transverse acceleration is zero $[\dot{\omega} \times r' = 0]$

* The centripetal acceleration is also zero because

$$\omega \times (\omega \times r') = \frac{V_o^2}{\rho} k' \times (k' \times b k') = 0$$

* The Coriolis acceleration is

$$2\,\omega \times v' = 2\left(\frac{V_o}{\rho} k'\right) \times (-j' V_o) = 2\frac{V_o^2}{\rho} i'$$

* Now, the net acceleration, relative to the ground, of the highest point on the wheel is:

$$\vec{a} = \vec{a'} + \dot{\vec{\omega}} \times r' + 2\omega \times v' + \vec{\omega} \times (\omega \times r') + A_0$$

$$= a' + 0 + 2\omega \times v' + 0 + A_0$$

$$= -k \frac{V_o^2}{b} + 2 \frac{V_o^2}{\rho} i' + \frac{V_o^2}{\rho} i'$$

$$\boxed{a = 3 \frac{V_o^2}{\rho} i' - k \frac{V_o^2}{b}}$$

3. Dynamics of a Particle in a Rotating Coordinate System

The fundamental equation of motion of a particle in an inertial frame of reference is

$$F = ma$$

where F is the vector sum of all real, physical forces acting on the particle. In view of Equation 5-30, we can write the equation of motion in noninertial frame of reference as:

$$ma' = F - mA_0 - 2m(\vec{\omega} \times v') - m\dot{\vec{\omega}} \times r' - m\omega \times (\omega \times r')$$

Transvers force $F'_{trans} = -m\dot{\omega} \times r'$

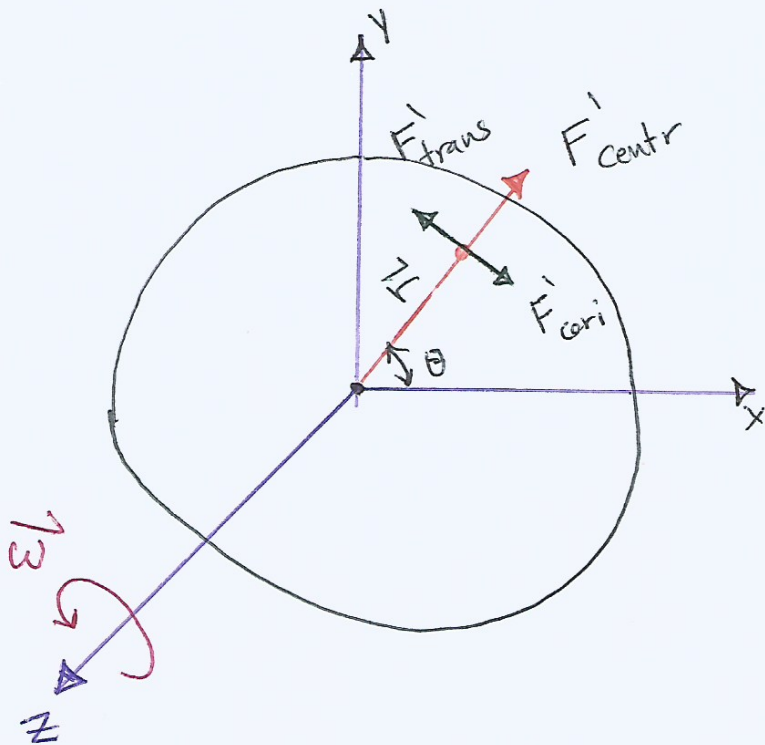Coriolis force $F'_{cor} = -2m(\vec{\omega} \times v')$

Centrifugal force $\qquad F_{centr}' = -m\vec{\omega} \times (\vec{\omega} \times r')$

If we write the force acting on the particle as:

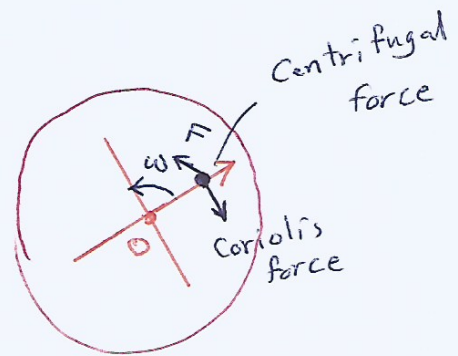$$F' = F_{phy} - m\vec{A_o} + F_{trans}' + F_{cor}' + F_{centr}'$$

where, $F' = ma'$

Example 5 A bug crawls outward with a constant speed v' along the spoke of a wheel that is rotating with constant angular velocity ω about a vertical axis. Find all the apparent forces acting on the bug

First, let us choose a coordinate system fixed on the wheel, and let the x'-axis point along the spoke in question

$$\vec{r} = \hat{i}\,x' = \hat{i}\,v't$$

$$\dot{\vec{r}}' = \hat{i}\,\dot{x}' = \hat{i}\,v'$$

$$\ddot{\vec{r}}' = 0$$

Centrifugal force

Coriolis force

Choose z'-axis to be vertical

$$\omega = \hat{k}'\,\omega$$

Thus, $\quad m\ddot{\vec{r}}' = F - mA_o - 2m\omega \times \dot{\vec{r}}' - m\dot{\omega} \times \vec{r}' - m\omega \times (\omega \times \vec{r}')$

$$-2m\omega \times \dot{\vec{r}}' = -2m\omega v'\,(\hat{k}' \times \hat{i}') = -2m\omega v'\hat{j}'$$

(Coriolis force)

$$-m\dot{\omega} \times \vec{r}' = 0 \qquad\qquad (\omega = \text{constant})$$

(transverse force)

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$$- m \omega \times (\omega \times r') = - m \omega^2 \left[k' \times (k' \times i' x') \right]$$

(centrifugal force) $\quad = - m \omega^2 \left(k' \times j' x' \right)$

$$= m \omega^2 x' i'$$

Thus,

$$0 = F - 2 m \omega v' j' + m \omega^2 x i'$$

Where F is the real force extented on the bug by the spoke. The forces are shown in the figure (last page).