Chapter 4

General Motion of Particle in Three Dimensions

1. Introduction: General principles

We now study the general case of motion of particle in three dimension. The Vector from the equation of motion Newton's Second law for such particle is:

$$F = \frac{dp}{dt} \quad \text{------------} \quad (4.1)$$

in which $p = mv$ is the linear momentum of the particle In Cartesian coordinates, we can write

$$\left.\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \\ F_z &= m\ddot{z} \end{aligned}\right] \quad \text{------------} \quad (4.2)$$

Equation 4.2 has no analytical solution, however, the simplest solution is assuming a Function (F) of spatial coordinates only

$$F = \vec{F}(r) \quad \text{-------} \quad (4.3)$$

1.1 The work principle

Work done on a particle causes it to gain or lose kinetic energy. Now, we will use equation (1) with taking the dot product with velocity (v), so we have:

$$\vec{F}\cdot\vec{v} = \frac{dP}{dt}\cdot\vec{v} = \frac{d(mv)}{dt}\cdot\vec{v} \quad \text{---- (4.4 A)}$$

$$\therefore \vec{F}\cdot\vec{v} = m\frac{d\vec{v}}{dt}\cdot\vec{v} \quad \text{---- (4.4 B)}$$

$$= 2mv\cdot\dot{v}$$

$$\boxed{\begin{aligned}\frac{d}{dt}(v\cdot v) &= \dot{v}\cdot v + v\cdot\dot{v}\\ &= 2v\cdot\dot{v}\end{aligned}}$$

So that,

$$\vec{F}\cdot\vec{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{dT}{dt} \quad \text{---- (4.5)}$$

Where T is the kinetic energy. However $v = \frac{d\vec{r}}{dt}$, so eq 4.5 becomes:

$$F\cdot\frac{d\vec{r}}{dt} = \frac{dT}{dt} \quad \text{---- (4.6)}$$

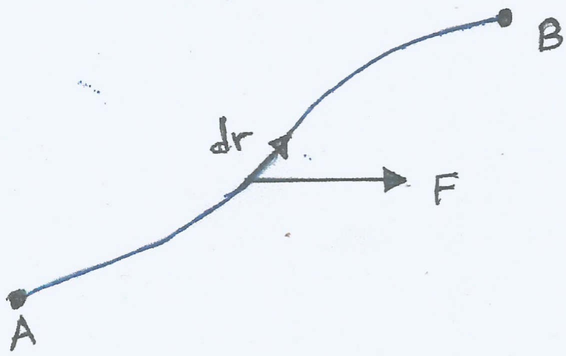


Fig 4.1 : The work done by a force F in the linear

$$\int \vec{F}\cdot d\vec{r} = \int dT \quad \text{------ } (4.7)$$

$$W = T_f - T_i \quad \text{------ } (4.8)$$

Equation (4.8) shows that the work done on a particle by net force acting on that particle to move it from one position to another is equal to the change in kinetic Energy of the particle at these position.

1.2 Conservative forces and force fields

If the force acting on a particle is conservative, it could be derived as the derivative of a Scalar potential energy function

$$F_x = -\frac{dV(x)}{dx} \quad \text{------ } (4.9)$$

This condition led us to the notion that the work done by such a force in moving a particle from point A to point B along x-axis is

$$W = \int F_x \, dx = -\int dV(x)$$

$$W = \int F_x \, dx = -\Delta V(x) = V(A) - V(B) \quad (4.10)$$

This means that we are no longer required a detailed knowledge of the motion of the particle from A to B to calculate the work done on it by a conservative force.

We needed to know only that it started at point A and ended up at point B. The work done depended only upon the potential energy function evaluated at the endpoints of the motion.

Because the work done was also equal to the change in kinetic energy of the particle, $\boxed{\Delta T = T(B) - T(A)}$ we were able to establish a general conservation of total energy principle, namely

$$\boxed{\underbrace{E_{total}}_{\text{constant}} = V(A) + T(A) = V(B) + T(B)} \quad 4.11$$

4.2 The potential Energy Function in three dimension motion: The Del operator

Assume we have F and V in three dimensions, so we can write

$$F(r) = -\frac{dV(r)}{dr} \quad \text{---- } 4.12$$

$$\left.\begin{aligned} F_x &= -\frac{\partial V(x)}{\partial x} \\ F_y &= -\frac{\partial V(y)}{\partial y} \\ F_z &= -\frac{\partial V(z)}{\partial z} \end{aligned}\right] \quad \text{----- } 4.13$$

So, Eq

$$F = -i\frac{\partial V(x)}{\partial x} - j\frac{\partial V(y)}{\partial y} + k\frac{\partial V(z)}{\partial z}$$

$$F = -\nabla V \quad \text{---}(4.14)$$

where ∇ is the "Del operator" and its given by :

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}} \quad \text{---}(4.15)$$

The expression ∇V is also called the gradient of V and is sometimes written as grad V.

Mathematically, the gradient of a function is a vector that represents the maximum spatial derivative of the function in direction and magnitude.

physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in the field created by other particles.

The negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction [see Fig 4.2]

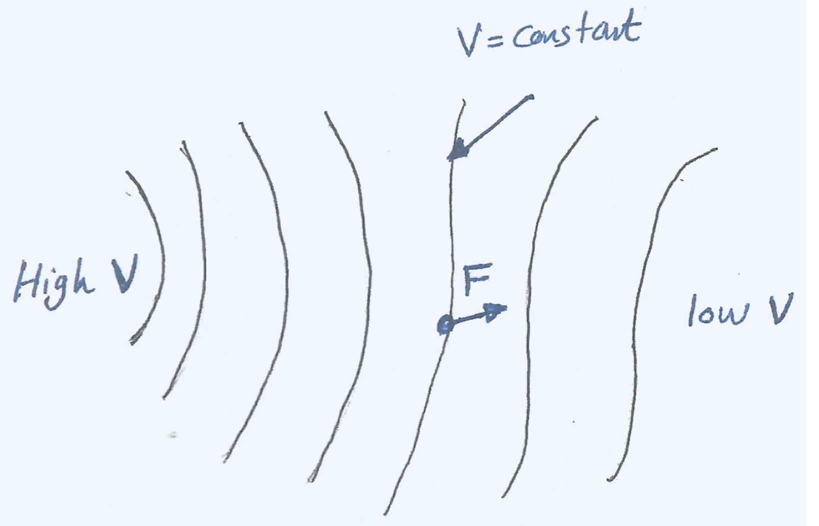


Fig. 4.2 A force field represented by equipotential contour curves.

The mathematical rule of Dell operator are listed below:

1. $\vec{\nabla} F$ [grade of a Function] is a vector.

2. $\vec{\nabla} \times F$ [curl of a Function] is a new vector

3. $\nabla \times (\nabla \times F) = 0$ [curl of any gradient $= 0$] (4-16)

Be aware that ∇F called a Gradient and it is a Numerical Function, however, $\nabla \times F$ called a Curl and gives us a Vector Function.

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \quad (4.1$$

$$\nabla \times F = i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + j\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = $$

(4.18)

Furthermore, if $\nabla \times F = 0$, then F can be derived from a scalar function $F = -\nabla V$, since $\nabla \times \nabla F = 0$ or the curl of any gradient is identically $= 0$

Now, we can generalize the conservation energy principle to three dimensions. The work done by a conservative force in moving a particle from point A to point B can be written as:

$$\int_A^B \vec{F} \cdot dr = -\int_A^B \nabla V(r) \cdot dr \qquad \text{as} \quad F = -\nabla V$$

(4-19)

$$= \int_A^B \left(i \frac{\partial V}{\partial x} + j \frac{\partial V}{\partial y} + k \frac{\partial V}{\partial z}\right) \cdot (i\,dx + j\,dy + k\,dz)$$

H.W

$$= \int_A^B dV(r) = \Delta V \Big]_A^B = V(B) - V(A) \qquad (4\text{-}20)$$

The last step illustrates the fact that $\nabla V \cdot dr$ is an exact differential equal to dV. The work done by any net force is always equal to the change in kinetic energy

$$\text{i.e } \int_A^B F \cdot dr = \Delta T = -\Delta V$$

$$\Delta(T+V) = 0 \qquad (4.21)$$

$$T(A) + V(A) = T(B) + V(B) = E = \text{Constant} \qquad (4.22)$$

We have arrived at our desired law of Conservation of total energy.

Note: if F is nonconservative force, it can be equal to $(-\Delta V)$

This leads, the work $(F.\, dr)$ is not an exact differential and can not be equaled to $(-\nabla V)$. So, the general form of work-energy theorem becomes:

$$\int_A^B F.\, dr = \Delta(T+V) = -\Delta E \qquad (4.23)$$

Example 1 :

Given the two-dimensional potential energy function

$$V(r) = V_0 - \frac{1}{2} k \delta^2 e^{-r^2/\delta^2}$$

Where $r = ix + jy$ and V_0, k and δ are constants, find the force function.

Solution:

We first write the potential energy function as a function of x and y.

$$V(x,y) = V_0 - \frac{1}{2} k \delta^2 e^{-(x^2+y^2)/\delta^2}$$

and then apply the gradient operator:

$$F = -\nabla V = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) V(x,y) \quad (4.24)$$

$$= -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)\left[V_0 - \frac{1}{2} k \delta^2 e^{-(x^2+y^2)/\delta^2}\right] \quad (4.25)$$

$$= -k r e^{-r^2/\delta^2} \quad (4.26)$$

Example 2

Suppose a particle of mass m is moving in the above force field, and at time $t=0$ the particle passes through the origin with speed v_0. What will the speed of the particle be at some small distance away from the origin given by $r = e_r \Delta$ where $\Delta \ll \delta$?

Solution

The force is Conservative, because a potential energy function exists.

The total energy $E = T + V = \text{constant}$

$$E = \frac{1}{2} m v^2 + V(r) = \frac{1}{2} m v_0^2 + V(0) \qquad (4\text{-}27)$$

$$\therefore \quad v^2 - v_0^2 = \frac{2}{m}\left[V(0) - V(r)\right] \qquad (4\text{-}28)$$

since $V(r) = V_0 - \frac{1}{2} k \delta^2 e^{-r^2/\delta^2}$ [Example 1]

$$\therefore \quad V(0) = V_0 - \frac{1}{2} k \delta^2 \qquad (4.29)$$

$$\therefore v^2 - v_0^2 = \frac{2}{m}\left[\left(V_0 - \frac{1}{2} k \delta^2\right) - \left(V_0 - \frac{1}{2} k \delta^2 e^{-r^2/\delta^2}\right)\right]$$

$\because r = e_r \Delta \Rightarrow r^2 = \Delta^2$

$$\therefore v^2 = v_0^2 + \frac{2}{m}\left[\left(\cancel{V_0} - \frac{1}{2} k \delta^2\right) - \left(\cancel{V_0} - \frac{1}{2} k \delta^2 e^{-\frac{\Delta^2}{\delta^2}}\right)\right]$$

(4·30)

Since $\Delta << \delta \Rightarrow e^{-\frac{\Delta^2}{\delta^2}} = 1 - \frac{\Delta^2}{\delta^2}$

Thus, $v^2 = v_0^2 + \frac{2}{m}\left[-\frac{1}{2}k\delta^2 + \frac{1}{2}k\delta^2\left(1 - \frac{\Delta^2}{\delta^2}\right)\right]$

(4-31)

$$= v_0^2 - \cancel{\frac{2}{m}} \cdot \frac{1}{\cancel{2}} k\delta^2\left[\cancel{1} - \cancel{1} + \frac{\Delta^2}{\delta^2}\right]$$

$$v^2 = v_0^2 - \frac{k\cancel{\delta^2}}{m}\left[\frac{\Delta^2}{\cancel{\delta^2}}\right]$$

$$v^2 = v_0^2 - \frac{k}{m}\Delta^2 \qquad (4\text{-}32)$$

The potential energy is a quadratic function of the displacement Δ from the origin for small displacement, so this solution reduces to the conservation of energy for the simple harmonic oscillator.

Example 3

Is the force field $F = i\,xy + j\,xz + k\,yz$ conservative?

Solution:

The curl of F is

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix}$$

$$= i\,(z - x) + j\,(0) + k\,(z - x)$$

$\nabla \times F \neq 0$, hence, the field is not conservative

Example 4

For what values of the constants a, b, and c is the force $F = i\,(ax + by^2) + j\,cxy$ conservative?

Solution

Talking the curl, we have

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax+by^2 & cxy & 0 \end{vmatrix} = k\,(c-2b)\,y$$

This shows that the force is Conservative, provided $c = 2b$. a value is immaterial

H.W (1)

show that the inverse-square law of force in three dimensions $F = (-k/r^2)e_r$ is conservative by the use of the Curl. Use spherical coordinates.

Hint: The Curl in spherical coordinates is:

$$\text{Curl } Q = \begin{vmatrix} e_r & r e_\theta & r\sin\theta\, e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ Q_r & r Q_\theta & r\sin\theta\, Q_\phi \end{vmatrix} \frac{1}{r^2 \sin\theta}$$

H.W (2) Find the force for each of the following potential energy Functions:

(a) $V = cxyz + C$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

(c) $V = c e^{-(\alpha x + \beta y + \gamma z)}$

(d) $V = c r^n$ in spherical coordinates

H.W (3) By Finding the curl, determine which of the following forces are conservative:

(a) $F = ix + jy + kz$

(b) $F = iy - jx + kz^2$

(c) $F = iy + jx + kz^3$

(d) $F = -k r^{-n} e_r$ in spherical coordinates

<u>H-W</u> (4) Find the value of the constant c such that each of the following force is conservative:

(a) $F = i\,xy + j\,cx^2 + k\,z^3$

(b) $F = i(z/y) + cj(xz/y^2) + k(x/y)$

3. The Harmonic Oscillator in Two and Three Dimensions:

Consider the motion of a particle subject to a linear restoring force that is always directed toward a fixed point, the origin of our coordinate system. Such a force can be represented by the expression

$$F = -kr \qquad (4.33)$$

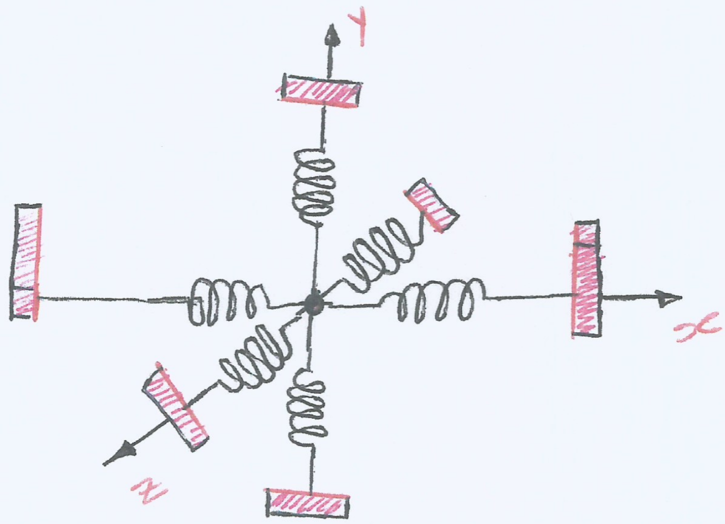


Fig 4.3: A model of a three dimensional harmonic oscillator.

Accordingly, the differential equation of motion is simply expressed as

$$m \frac{d^2 r}{dt^2} = -kr \qquad (4.34)$$

This situation can be represented approximately by a particle attached to a set of elastic springs as shown in Fig 4.3. This is the three-dimensional generalization of the linear oscillator studied earlier. Eq. 4-34 is the differential equation of the linear isotropic oscillator.

The Two-Dimensional Isotropic oscillator:

In the case of motion in a single plane, Eq 4.33 is equivalent to two component equations

$$\left.\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \end{aligned}\right] \qquad (4\text{-}35)$$

These are separated and we can immediately write down the solutions in the form

$$\left.\begin{aligned} x &= A\cos(\omega t + \alpha) \\ y &= B\cos(\omega t + \beta) \end{aligned}\right] \qquad (4\text{-}36)$$

and

$$\omega = \sqrt{k/m} \qquad (3\text{-}37)$$

The constants of integration A, B, α and β are determined from the initial conditions in any given case. To find the equation of the path we eliminate the time t between the two equations. To do this, let us write the second equation in the form

$$y = B\cos(\omega t + \alpha + \Delta) \quad \text{as} \quad \beta = \alpha + \Delta$$

(4-38)

$$y = B\left[\cos(\omega t + \alpha)\cos\Delta - \sin(\omega t + \alpha)\sin\Delta\right]$$

$$\frac{y}{B} = \cos(\omega t + \alpha)\cos\Delta - \sin(\omega t + \alpha)\sin\Delta$$

(4-39)

But from eq (4-39), we have

$$x = A\cos(\omega t + \alpha) \Rightarrow \frac{x}{A} = \cos(\omega t + \alpha)$$

$$\text{and } \sin\theta = \left[1 - \cos^2\theta\right]^{1/2}$$

(4-40)

So that, equation (4-39) can be written as following:

$$\frac{y}{B} = \frac{x}{A}\left[\cos\Delta - \left[1 - \frac{x^2}{A^2}\right]^{1/2}\sin\Delta\right] \quad (4\text{-}41)$$

$$\frac{x}{A}\cos\Delta - \frac{y}{B} = \left[1 - \frac{x^2}{A^2}\right]^{1/2}\sin\Delta \quad (4\text{-}42)$$

By squaring the two sides and re-range the terms so equation (4-42) becomes:

$$\frac{x^2}{A^2}\cos^2\Delta + \frac{x^2}{A^2}\sin^2\Delta + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\Delta = \sin^2\Delta$$

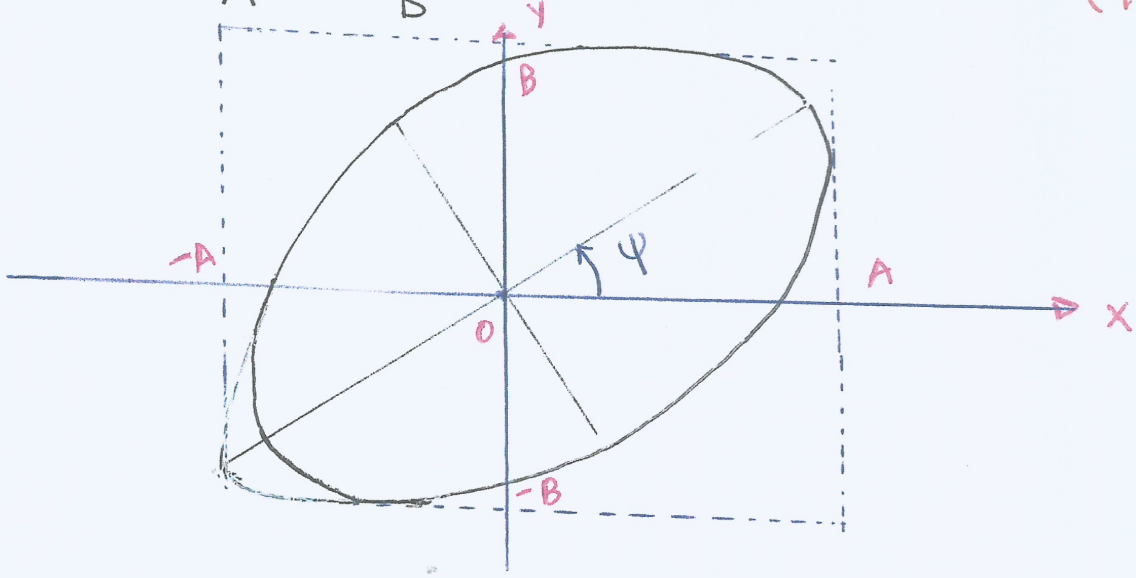
(4-43)

eliminate $(\sin^2 + \cos^2) = 1$ from the above equation, so we will get

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{xy}{AB}\cos\Delta = \sin^2\Delta \qquad (4.44)$$

Which is a quadratic equation in x and y from equation (4.44) if the phase $\Delta = \frac{\pi}{2}$, so

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \qquad (4.45)$$



Equation (4.45) is the equation of ellipse whose axis coincide with the coordinates axes.

If $\Delta = 0$, then equation (4.44) becomes:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB} = 0 \qquad (4.46)$$

and,

$$y = \frac{B}{A}x \qquad (4.47)$$

or if $\Delta = \pi$, $\quad y = -\frac{B}{A}x \qquad (4.48)$

In general case, it is possible to show that the axis of the elliptical path is inclined to the x-axis by the angle Ψ where

$$\tan 2\Psi = \frac{2AB\cos\Delta}{A^2 - B^2} \qquad (4\text{-}49)$$

The Three-Dimensional Isotropic Harmonic oscillator:

In the case of three-dimensional motion, the differential equation of motion is equivalent to the three equations:

$$\left.\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \\ m\ddot{z} &= -kz \end{aligned}\right\} \qquad (4\text{-}50)$$

Then, the suggested solution will be:

$$\left.\begin{aligned} x &= A\cos(\omega t + \alpha) \\ y &= B\cos(\omega t + \beta) \\ z &= C\cos(\omega t + \gamma) \end{aligned}\right\} \qquad (4\text{-}51)$$

As $\omega = \sqrt{\frac{k}{m}}$

Non-isotropic Oscillator in Three-Dimensional

The previous discussion considered the motion of the isotropic oscillator, wherein the restoring force is independent of the direction of the displacement. If the magnitudes of the components of the restoring force depend on the direction of the displacement, we have the case of the nonisotropic oscillator.

For a suitable choice of axes, the differential equations for the non-isotropic case can be written as:

$$\left.\begin{aligned} m\ddot{x} &= -k_1 x \\ m\ddot{y} &= -k_2 y \\ m\ddot{z} &= -k_3 z \end{aligned}\right] \quad (4.52)$$

Here we have a case of three different frequencies of oscillation, $\omega_1 = \sqrt{\frac{k_1}{m}}$, $\omega_2 = \sqrt{\frac{k_2}{m}}$, and $\omega_3 = \sqrt{\frac{k_3}{m}}$ and the motion is given by the solutions

$$\left.\begin{aligned} x &= A\cos(\omega_1 t + \alpha) \\ y &= B\cos(\omega_2 t + \beta) \\ z &= C\cos(\omega_3 t + \gamma) \end{aligned}\right] \quad 4.53$$

Energy Considerations

In the preceding chapter, we showed that the potential energy function of the one dimensional harmonic oscillator is quadratic in the displacement, $V(x) = \frac{1}{2}kx^2$. For the general three-dimensional case, it is easy to verify that

$$V(x,y,z) = \frac{1}{2}kx^2 + \frac{1}{2}ky^2 + \frac{1}{2}kz^2 \tag{4-54}$$

because $F_x = -\partial V/\partial x = -k_1 x$, and similarly for F_y and F_z. If $k_1 = k_2 = k_3 = k$, we have the isotropic case and

$$V(x,y,z) = \frac{1}{2}k(x^2+y^2+z^2) = \frac{1}{2}kr^2 \tag{4.55}$$

The total energy in the isotropic case is then given by the simple expression

$$\frac{1}{2}mv^2 + \frac{1}{2}kr^2 = E \tag{4-56}$$

which is similar to that of the one-dimensional case discussed in the previous chapter.

Example 5

A particle of mass m moves in two dimensions under the following potential energy

$$V(r) = \frac{1}{2}k(x^2+4y^2)$$

Find the resulting motion, given the initial condition at $t=0$: $x=a$, $\dot{x}=0$, $\dot{y}=v_0$

Solution

This in nonisotropic oscillator potential. The force function is

$$F = -\nabla V = -\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}\right)\left[\frac{1}{2}k(x^2+4y^2)\right]$$

$$= -ikx - j4ky$$

The component differential equations of motion are then

$$m\ddot{x}+kx=0 \qquad m\ddot{y}+4ky=0$$

The x-motion has angular Frequency $\omega=(k/m)^{1/2}$, while the y-motion has angular frequency just twice that, namely, $\omega_y=(4k/m)^{1/2}=2\omega$

We shall write the general solution in the form

$$x = A_1\cos\omega t + B_1\sin\omega t$$
$$y = A_2\cos 2\omega t + B_2\sin 2\omega t$$

Thus, at $t=0$, we see that the above equations for the components of position and velocity reduce to

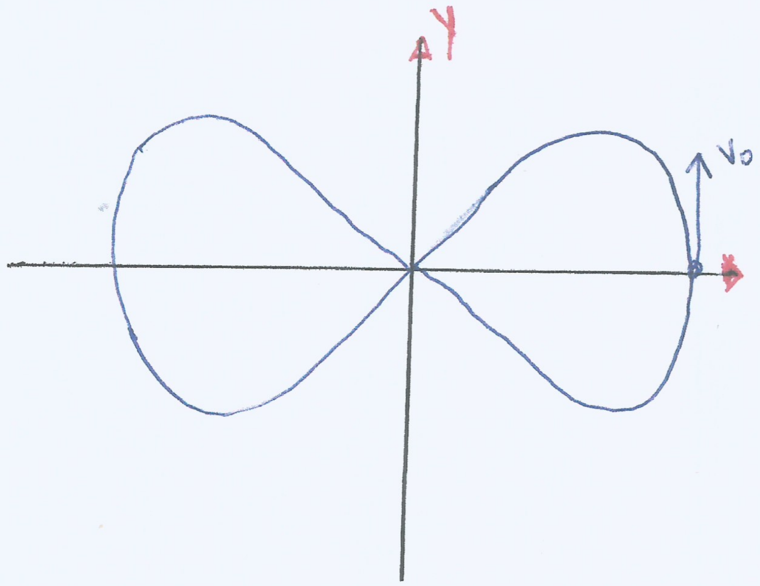
$$\boxed{a = A_1} \quad \boxed{0 = A_2} \quad \boxed{0 = B_1 w} \quad \boxed{V_0 = 2B_2 w}$$

These equations give directly the values of the amplitude coefficients, $\boxed{A_1 = a}$, $\boxed{A_2 = B_1 = 0}$ and $\boxed{B_2 = V_0/2w}$ so the final equations for the motion are

$$x = a \cos wt$$

$$y = \frac{V_0}{2w} \sin 2wt$$

The path is a Lissajous figure having the shape of a figure-eight as shown in Figure



H.W 5 A particle of mass m moving in three dimensions under the potential energy function $V(x,y,z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.

(a) what will its speed be if and when it passes through the point $(1,1,1)$?

(b) If the point $(1,1,1)$ is a turning point in the motion $(v=0)$, what is v_0 ?

(c) what are the component differential equations of motion of the particle ?

[Note: you do not have to solve the differential equations of motion in this problem]