

# Chapter 3

## Oscillations

### 1. Introduction

We begin by considering the oscillatory motion of a particle constrained to move in one dimension. We assume that position of stable equilibrium exists for the particle, and we designate this point as the origin. If the particle is displaced from the origin (in either direction), a certain force tends to restore the particle to its original position. An example is an atom in a long molecular chain. The restoring force is, in general, some complicated function of the displacement and perhaps of the particle's velocity or even of some higher time derivative of position coordinate. We consider here only cases in which the restoring force  $F$  is a function only of the displacement:  $F = F(x)$ .

We assume that the function  $F(x)$  that describes the restoring force possesses continuous derivatives of all orders so that the function can be expanded in a Taylor Series:

$$F(x) = F_0 + x \left( \frac{dF}{dx} \right)_0 + \frac{1}{2!} x^2 \left( \frac{d^2 F}{dx^2} \right) + \frac{1}{3!} x^3 \left( \frac{d^3 F}{dx^3} \right) + \dots \quad (3.1)$$

where  $F_0$ : the value of  $F(x)$  at the origin position ( $x=0$ ), it is equal to zero when we define

the origin as the equilibrium position. In addition, if the displacement ( $x$ ) is sufficiently small  $\Rightarrow x^2$  or higher order  $\Rightarrow$  Zero [can be neglected]

i.e.  $F(x) = -kx$  --- (3.2)

Where  $k = - (dF/dx)_0$ . The restoring force is always directed toward the equilibrium position (the origin), so the derivative  $(dF/dx)_0$  is negative, and  $k$  is positive. The restoring force is a linear force and it obeys Hook's law.

One of the classes of physical process that can be treated by applying Hook's law is that involving elastic deformations. As long as the displacements are small and the elastic limits are not exceeded, a linear restoring force can be used for problems of stretched springs, elastic springs, bending beams, and the like.

## 2. Simple Harmonic Oscillator

The equation of motion in this case can be obtained by substituting the Hook's law force in the Newtonian equation ( $F=ma$ ). Thus

$$-kx = m\ddot{x} \quad (3.3)$$

If we define  $\omega_0^2 = k/m$  (3.4)

Then, eq. 3.3 becomes

$$\ddot{X} + \omega_0^2 X = 0 \quad (3.5)$$

To solve this equation, we can express  $x(t)$  in either of the form:

$$x(t) = A \sin(\omega_0 t - \delta) \quad (3.6 a)$$

$$x(t) = A \cos(\omega_0 t - \phi) \quad (3.6 b)$$

Where the phases  $\delta$  and  $\phi$  differ by  $\frac{\pi}{2}$ . Equations 3.6 a and b exhibit the well-known sinusoidal behavior of the displacement of the simple harmonic oscillator.

We can obtain the relationship between the total energy of the oscillator and the amplitude of its motion as follows. Using eq 3.6 a for  $x(t)$ , we can find for the kinetic energy.

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \omega_0^2 A^2 \cos^2(\omega_0 t - \delta)$$

$$\therefore T = \frac{1}{2} k A^2 \cos^2(\omega_0 t - \delta) \quad \dots (3.7)$$

The incremental amount of work  $dW$  necessary to move the particle by an amount  $dx$  against the restoring force  $F$  is:

$$dW = -F dx = kx dx \quad \dots (3.8)$$

Integration from 0 to  $x$  and setting the work done on the particle equal to the potential energy

$$U = \int dW = \frac{1}{2} k x^2$$

$$\therefore U = \frac{1}{2} k x^2$$

$$\underline{\text{or}} \quad U = \frac{1}{2} k A^2 \sin^2(\omega_0 t - \delta) \quad \dots (3.9)$$

by combining the expressions for  $T$  and  $U$  [eqs. (3.7) and (3.9)]

$$E = T + U = \frac{1}{2} k A^2 [\cos^2(\omega_0 t - \delta) + \sin^2(\omega_0 t - \delta)]$$

$$\therefore E = T + U = \frac{1}{2} k A^2 \quad \dots (3.10)$$

Notice that the total energy is proportional to the square of the amplitude; this is a general result for linear system.

Notice also that  $E$  is independent of the time. Energy is conserved because we have been considering a system without frictional losses or other external forces.

The period  $T_0$  of the motion is defined:

$$\omega_0 T_0 = 2\pi \quad \dots (3.11)$$

$$\therefore T_0 = 2\pi \sqrt{\frac{m}{k}} \quad \dots (3.12)$$

It is clear that  $\omega_0$  represents the angular frequency of the motion, which is related to the frequency  $\nu_0$  by

$$\boxed{\omega_0 = 2\pi \nu_0 = \sqrt{\frac{k}{m}}} \quad \dots (3.13)$$

$$T_0 = \frac{1}{\nu_0} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

--- (3.14)

Notice that, the period of the simple harmonic oscillator is independent of the amplitude (or total energy).

### 3. The Simple pendulum

The small mass "m" swinging at the end of a tight, inextensible string of length  $l$ .

The motion is along circular arc. The restoring force along the path of motion is:

is:

$$F_s = -mg \sin \theta$$

$$m \ddot{s} = -mg \sin \theta$$

$$\ddot{m} \ddot{s} + mg \sin \theta = 0$$

$$s = l\theta \Rightarrow \ddot{s} = l\ddot{\theta}$$

Also, for small  $\theta$  [ $\sin \theta \approx \theta$ ]

$$\text{Thus, } \ddot{\theta} + \frac{g}{l} \theta = 0 \quad [\text{Simple pendulum}] \quad \text{--- (3.15)}$$

The differential equation is mathematically identical to that of the linear harmonic oscillator [ $\ddot{x} + \frac{k}{m}x = 0$ ]. We will replace  $(\frac{k}{m})$  by  $(\frac{g}{l})$  in the solution,

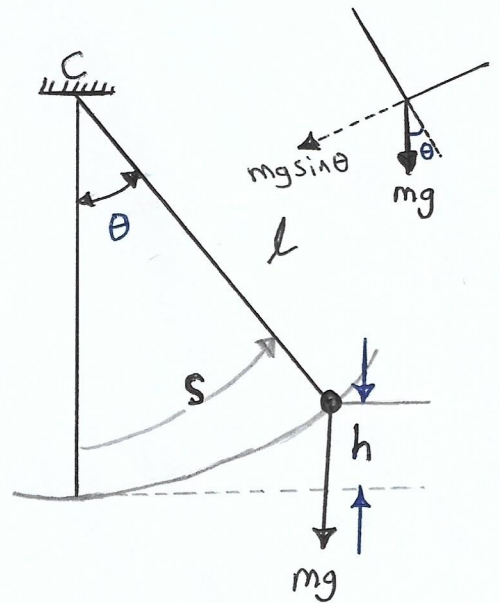


Fig. 1

$$\omega_0 = \sqrt{\frac{g}{l}}$$

--- (3.16)

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}}$$

--- (3.17)

#### 4. Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force  $F = -kx$ . Let calculate the work done by an external force  $F_{ext}$  in moving the particle from equilibrium position ( $x=0$ ) to some position  $x$ .

- Consider
- ① The particle moves slowly not gain any kinetic energy
  - ② Applied external force is greater in magnitude than the restoring force

$$\text{Hence, } F_{ext} = -F_x = kx$$

$$W = \int_0^x F_{ext} dx = \int_0^x kx dx$$

$$\therefore \boxed{W = \frac{1}{2} kx^2} \quad \text{--- (3.18)}$$

If the spring obeys Hooke's law, the work is stored as a potential energy  $W = V(x)$

$$W = V(x) = \frac{1}{2} k x^2$$

The "total energy" when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies.

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \quad \dots (3.19)$$

\* The kinetic energy is quadratic in the velocity variable

\* The potential energy is quadratic in the displacement variable.

\* The total energy is constant [no other force]

Solving for the velocity:

$$\dot{x} = \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2} \quad \dots (3.20)$$

We can integrate to give  $t$  as a function of  $x$ :

$$\frac{dx}{dt} = \pm \left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2}$$

$$\therefore dt = \frac{\pm dx}{\left( \frac{2E}{m} - \frac{kx^2}{m} \right)^{1/2}}$$

$$dt = \frac{\pm dx}{\left( \frac{2E}{m} \right)^{1/2} \left[ 1 - \frac{kx^2}{2E} \right]^{1/2}}$$

$$\text{let } x = \sqrt{\frac{2E}{K}} \cos \theta$$

$$\text{Thus, } x^2 = \frac{2E}{K} \cos^2 \theta \quad ; \quad dx = -\sqrt{\frac{2E}{K}} \sin \theta d\theta$$

$$\therefore dt = \frac{\pm (-\sqrt{\frac{2E}{K}} \sin \theta) d\theta}{\left(\frac{2E}{m}\right)^2 [1 - \cos^2 \theta]^{1/2}}$$

$$\therefore dt = \frac{\mp (\sqrt{\frac{2E}{K}} \sin \theta) d\theta}{\left(\frac{2E}{m}\right)^2 \sin \theta}$$

$$\therefore \int dt = \int \frac{\mp (\sqrt{\frac{2E}{K}}) d\theta}{\left(\frac{2E}{m}\right)^2}$$

$$t = \mp \frac{\sqrt{\frac{2E}{K}}}{\left(\frac{2E}{m}\right)^2} \theta + C$$

$$t = \mp \sqrt{\frac{2E}{K}} \left[\frac{m}{2E}\right]^2 \theta + C$$

$$= \mp \sqrt{\frac{2E}{K}} \left[\frac{m}{2E}\right]^2 \cos^{-1} \left[\frac{x}{\sqrt{\frac{2E}{K}}}\right] + C$$

$$= \mp \sqrt{\frac{2E}{K} \cdot \frac{m}{2E}} \cos^{-1} \left[\frac{x}{A}\right] + C$$

$$\therefore \boxed{t = \mp \sqrt{\frac{m}{K}} \cos^{-1} \left[\frac{x}{A}\right] + C} \quad \dots (3.21)$$

Where  $\boxed{A = \sqrt{\frac{2E}{K}}}$  A is the amplitude  $\dots (3.22)$

We can obtain the relation of A from the energy equation (3.19).

□ The value of  $x$  must lie between  $\pm A$  in order



For  $\dot{x}$  to be real (see the figure below)  $[E = \frac{1}{2} k A^2]$

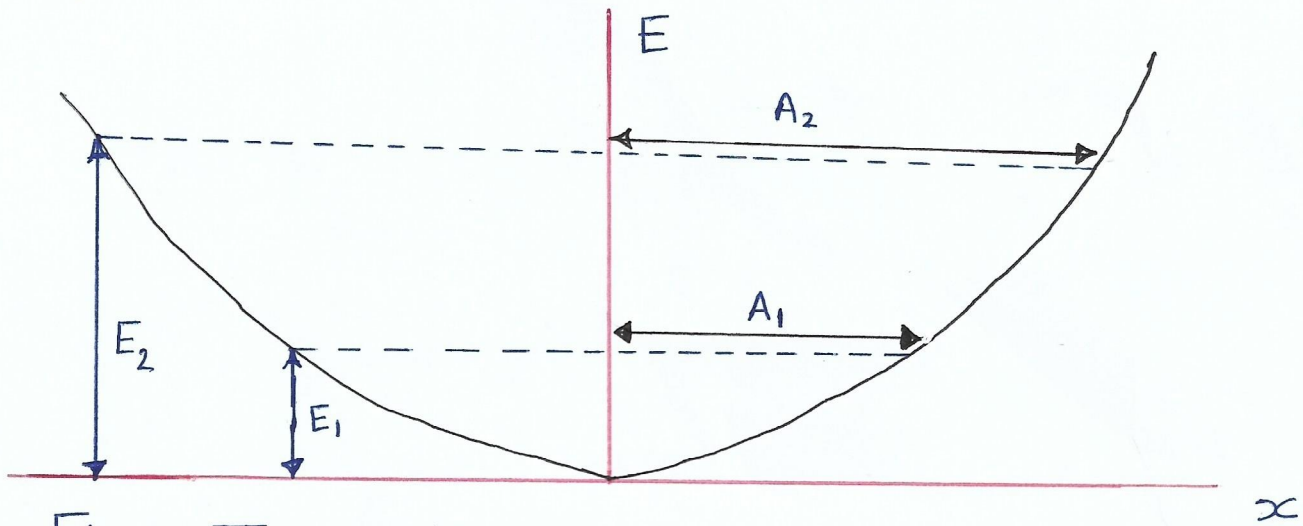


Fig. 2: The parabolic potential energy as a function of the harmonic oscillator.

[2] The maximum value of the speed  $[v_{max}]$  occurs at  $x = 0$

$$E = \frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2$$

**Example 1** Find the total energy of the simple pendulum?

The potential energy of the simple pendulum (Fig 1) is given by the expression

$$V = mgh$$

$h$  is the vertical distance

From Fig. 1, the displacement through an angle  $\theta$ , we see that  $h = l - l \cos \theta$

$$\therefore V(\theta) = mg(l - l \cos \theta)$$

Using  $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

Taking the first two terms, ... yield

$$V(\theta) = \frac{1}{2} mgl \theta^2$$

$$s = l\theta$$

$$V(s) = \frac{1}{2} \frac{mg}{l} s^2$$

The potential energy function is quadratic in the displacement variable; the total energy is given by:

$$E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \frac{mg}{l} s^2$$

### Example 2

Calculate the average kinetic, potential

and total energies of the harmonic oscillator.

$$\langle \text{Energy} \rangle = \frac{1}{T_0} \int_0^{T_0} \text{energy}(t) dt \quad \dots (3-23)$$

We use  $K$  for the kinetic energy and  $T_0$  for the period of the motion.

$$\begin{aligned} \langle K \rangle &= \frac{1}{T_0} \int_0^{T_0} K(t) dt \\ &= \frac{1}{T_0} \int_0^{T_0} \frac{1}{2} m \dot{x}^2 dt \end{aligned}$$

$$x = A \sin(\omega t - \delta)$$

[See 3-6.a]

$$x = w_0 A \cos(\omega_0 t - \delta)$$

let  $\delta = 0$  and  $u = \omega_0 t = \left(\frac{2\pi}{T_0}\right)t \Rightarrow dt = \frac{T_0}{2\pi} du$

$$\langle K \rangle = \frac{1}{T_0} \left[ \frac{1}{2} m w_0^2 A^2 \int_0^{T_0} \cos^2(\omega_0 t) dt \right]$$

$$\therefore \langle K \rangle = \frac{1}{T_0} \left[ \frac{1}{2} m w_0^2 A^2 \frac{T_0}{2\pi} \int_0^{2\pi} \cos^2 u du \right]$$

$$\langle K \rangle = \frac{1}{2\pi} \left[ \frac{1}{2} m w_0^2 A^2 \int_0^{2\pi} \cos^2 u du \right]$$

We can make use of the fact that:

$$\frac{1}{2\pi} \int_0^{2\pi} (\sin^2 u + \cos^2 u) du = \frac{1}{2\pi} \int_0^{2\pi} du = 1$$

Thus,  $\frac{1}{2\pi} \int_0^{2\pi} \cos^2 u du = \frac{1}{2}$

because the areas under the  $\cos^2$  and  $\sin^2$  terms throughout one cycle are identical.

$$\langle K \rangle = \frac{1}{4} m w_0^2 A^2$$

Average kinetic energy

The average potential energy proceeds along similar lines

$$V = \frac{1}{2} k x^2$$

$$= \frac{1}{2} k A^2 \sin^2 \omega_0 t$$

$$\langle V \rangle = \frac{1}{T_0} \int_0^{T_0} V(t) dt$$

$$= \frac{1}{T_0} \frac{1}{2} k A^2 \int_0^{T_0} \sin^2 \omega_0 t dt$$

$$= \frac{1}{2} k A^2 \frac{1}{2\pi} \int_0^{2\pi} \sin^2 u du$$

$$= \frac{1}{4} k A^2$$

□

$$\frac{k}{m} = \omega_0^2 \quad \text{or} \quad k = m\omega_0^2$$

$$\text{Thus, } \langle V \rangle = \frac{1}{4} k A^2 = \frac{1}{4} m \omega_0^2 A^2 = \langle K \rangle$$

$$\therefore \langle E \rangle = \langle K \rangle + \langle V \rangle$$

$$\langle E \rangle = \frac{1}{2} m \omega_0^2 A^2$$

$$\langle E \rangle = \frac{1}{2} k A^2 = E$$

The average kinetic energies and potential energies are equal. Therefore, the average energy of the oscillator is equal to its total instantaneous energy.

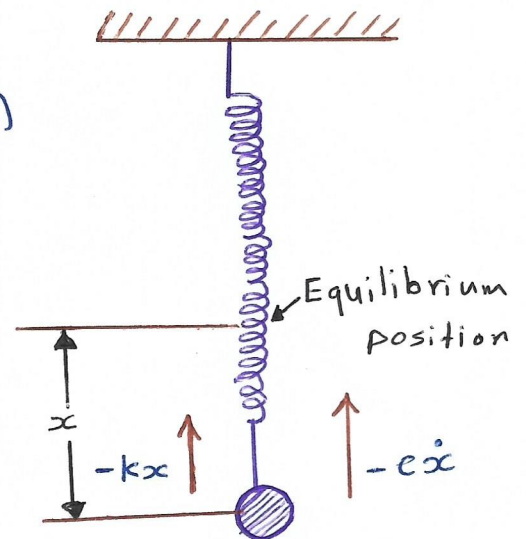
## 5. Damped Harmonic Motion

The foregoing analysis of harmonic oscillator is somewhat idealized in that we have failed to take into account frictional forces.

- \* There are always present in a mechanical system to some extent.
- \* There is always a certain amount of resistance in an electrical circuit.

Suppose there is an object of mass "m" that is supported by a light spring of stiffness k. Assume that there is a linear retarding force to the velocity.

$$m\ddot{x} = \underbrace{-kx}_{\text{restoring force}} - \underbrace{c\dot{x}}_{\text{retarding force}} \quad \dots (3.24)$$



Thus,  $m\ddot{x} + kx + c\dot{x} = 0$

$$\text{OR } \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \quad (3.25)$$

If we substitute the damping factor  $\delta$ , defined as

$$\delta = \frac{c}{2m} \quad \dots (3.26)$$

and  $\omega_0^2 (= k/m)$

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0 \quad \dots (3.27)$$

The general solution is:

$$x(t) = A_1 e^{-(\delta-q)t} + A_2 e^{-(\delta+q)t} \quad (3.28)$$

where  $q = \sqrt{\delta^2 - \omega_0^2}$

There are three possible scenarios:

- I.  $q$  real  $> 0$  "overdamping"
- II.  $q$  real  $= 0$  "critical damping"
- III.  $q$  imaginary  $< 0$  "underdamping"

### I. Overdamping

- $A_1$  and  $A_2$  are constants, determined by the initial conditions.

- The motion is an exponential decay with decay constants:

$(\delta - q)$  and  $(\delta + q)$

- The mass, given some initial displacement and released from a rest  $\Rightarrow$  returns slowly to equilibrium position and prevented from oscillating by the strong damping force.

## II. Critical damping

- $q=0 \Rightarrow x(t) = A_1 e^{-\delta t} + A_2 e^{-\delta t}$

let  $A = A_1 + A_2$

$\therefore x(t) = A e^{-\delta t}$

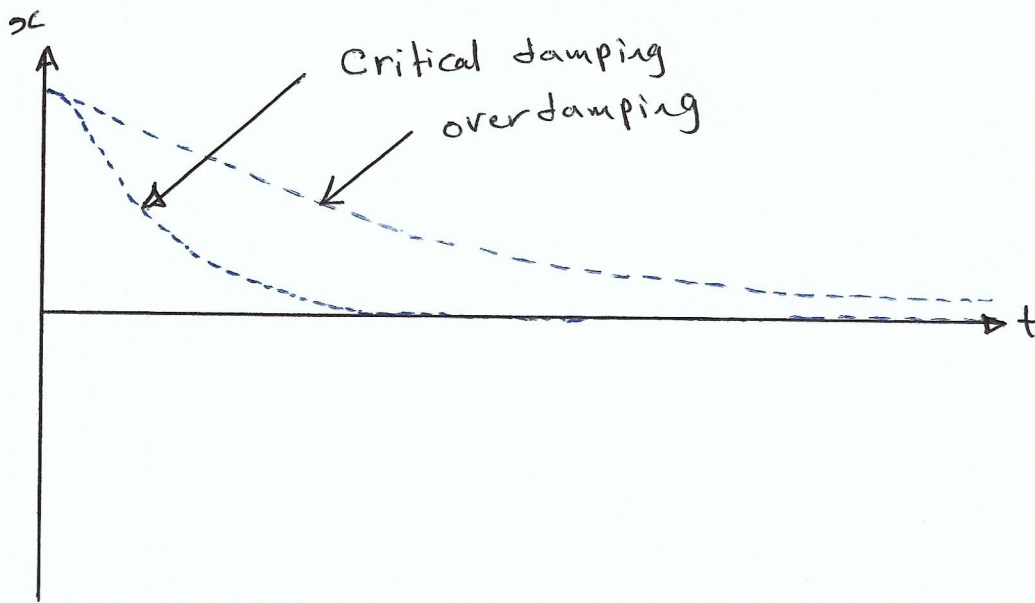
- The solution requires two different functions and independent constants to satisfy the boundary conditions specified by an initial position and velocity.

- The solution is  $x(t) = A t e^{-\delta t} + B e^{-\delta t}$  --- (3.29)

Note that, we have two different functions ( $t e^{-\delta t}$ ) and ( $e^{-\delta t}$ ). Also, we have two different constants  $A$  and  $B$

- If the mass ( $m$ ) displaced and released from a rest  $\Rightarrow$  the motion is nonoscillatory and returning to equilibrium.

• Critical damping is desirable in many systems such as the mechanical suspension systems of motor vehicles.



### III. Underdamping

If the damping factor ( $\gamma$ ) is small enough that:

$$\gamma^2 - \omega_0^2 < 0 \Rightarrow \gamma = \sqrt{\gamma^2 - \omega_0^2} = \text{imaginary}$$

i.e  $\gamma = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{-1} \sqrt{\omega_0^2 - \gamma^2} = i\omega_d$

where  $\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$

$\omega_0$  is the angular frequency of the undamped harmonic oscillator,  $\omega_d$  is the angular frequency of the underdamped harmonic oscillators.



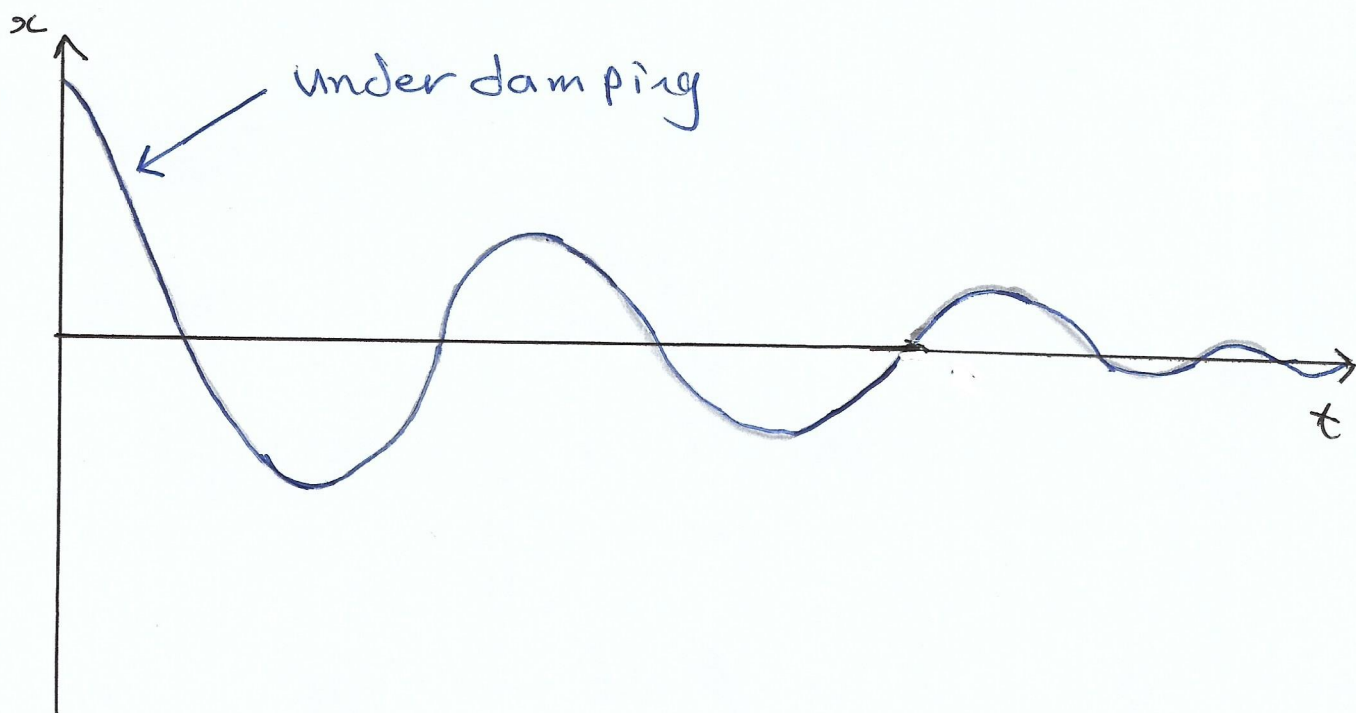
The general solution is

$$X(t) = e^{-\delta t} [A \sin(\omega_d t + \phi_0)] \quad \text{--- (3.30)}$$

From this equation, we note that the solution of an underdamped oscillator is nearly identical to that of the undamped oscillator.

There are two differences:

- 1] The presence of the real exponential factor  $[e^{-\delta t}]$  leads to a gradual death of the oscillations.
- 2] The underdamped oscillator's angular frequency is  $\omega_d$  not  $\omega_0$ , because the presence of the damping force.



The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} \quad \text{--- (3-31)}$$

## 6. Quality Factor $Q$

The rate of energy loss of a weakly damped harmonic oscillator is called the quality factor of the oscillator.

$$\frac{\Delta E}{E} = \frac{T_d}{\tau} \quad [\text{energy loss in one cycle}]$$

$T_d$ : period of underdamped oscillator

$t$ : the time of motion

$\tau$ : exponential time constant

$$Q = \frac{2\pi}{T_d/\tau} = \frac{2\pi\tau}{(2\pi/\omega_d)} = \omega_d\tau = \frac{\omega_d}{2\gamma} \quad \text{--- (3-32)}$$

where,

$$\tau = \frac{1}{2\gamma} \quad \text{(3-33)}$$

For weak damping, the period of oscillation  $T_d$  is much less than the time constant,  $\tau$

Table I: Some values of  $Q$  for different kind of oscillators

kind of oscillators	$Q$	
Earthquake	250 - 1400	→ Strong damping
piano string	3000	
Crystal in digital watch	$10^4$	
Microwave cavity	$10^4$	
Excited atom	$10^7$	→ Weak damping
Neutron star	$10^{12}$	
Excited $Fe^{57}$ nucleus	$3 \times 10^{12}$	→ Very weak damping

### Example 3

An automobile suspension system is

critically damped, and its period of free oscillation

with no damping is 1 s. If the system is initially

displaced by an amount  $x_0$  and released with zero

initial velocity, find the displacement at  $t = 1$  s.

Answer: critical damping  $\Leftrightarrow \gamma = 0$

at  $t = 0 \rightarrow x = x_0 \quad \dot{x}(0) = 0$

at  $t = 1 \rightarrow x(t) = ??$

$$x(t) = Ate^{-\delta t} + Be^{-\delta t}$$

For critical damping  
[see 3.29]

$$q = \sqrt{\delta^2 - \omega_0^2} \Rightarrow 0 = \sqrt{\delta^2 - \omega_0^2} \Rightarrow \delta = \omega_0$$

From 3.13  $\Rightarrow \omega_0 = \frac{2\pi}{T_0}$

$\therefore T_0 = 1 \text{ s} \Rightarrow \omega_0 = 2\pi$

$$\delta = 2\pi$$

Now,

$$x(t) = Ate^{-2\pi t} + Be^{-2\pi t}$$

At  $t=0$  and  $x = x_0$

$$x(0) = A(0)e^{-2\pi(0)} + Be^{-2\pi(0)}$$

$$x_0 = Be^0 \Rightarrow x_0 = B$$

Zero initial velocity [ $\dot{x}(0) = 0$ ]

$$\dot{x}(t) = -2\pi Ate^{-2\pi t} + Ae^{-2\pi t} - 2\pi Be^{-2\pi t}$$

$$\dot{x}(0) = 0 + Ae^0 - 2\pi Be^0$$

$$0 = A - 2\pi B \Rightarrow A = 2\pi B$$

$$A = 2\pi x_0$$

Now, the expression of displacement is:

$$x(t) = 2\pi x_0 t e^{-2\pi t} + x_0 e^{-2\pi t}$$

or

$$x(t) = (2\pi t + 1) x_0 e^{-2\pi t}$$

At  $t=1$  sec,  $x(1)$  will be

$$x(1) = (2\pi(1) + 1) x_0 e^{-2\pi(1)}$$

$$x(1) = (2\pi + 1) x_0 e^{-2\pi}$$

$$= (7.28) x_0 e^{-6.28}$$

$$\therefore x(1) = 0.0136 x_0$$

The system has practically returned to equilibrium.

### Example 4

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

$$\omega_d = \frac{1}{2} \omega_0 \Rightarrow \sqrt{\omega_0^2 - \gamma^2} = \frac{1}{2} \omega_0$$

$$\therefore \omega_0^2 - \gamma^2 = \frac{\omega_0^2}{4}$$

$$\gamma^2 = \omega_0^2 - \frac{\omega_0^2}{4} \Rightarrow \gamma^2 = \frac{3\omega_0^2}{4}$$

$$\therefore \gamma = \frac{\sqrt{3}\omega_0}{2}$$

at maxima oscillations

$$t = T_d$$

$$\therefore \gamma t = \gamma T_d = \frac{\sqrt{3}\omega_0}{2} \cdot \frac{2\pi}{\omega_d}$$

$$\therefore \gamma T_d = \sqrt{3}\pi \frac{\omega_0}{\omega_d} = \sqrt{3}\pi \frac{\omega_0}{\omega_0/2} \Rightarrow \gamma T_d = 2\sqrt{3}\pi$$

$$\gamma T_d = 10.88$$

The amplitude ratio is  $e^{-\gamma T_d} = e^{-10.88} = 0.00002$

## 7. Driven oscillations with Damping

The motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent is:

$$m\ddot{x} + c\dot{x} + kx = \underbrace{F}_{\text{driving force}}$$

$$\text{let } F = F_0 e^{i\omega t}$$

$$\therefore m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \quad \dots (3.34)$$

The solution is

$$x(t) = A e^{i(\omega t - \phi)} \quad \dots (3.35)$$

A : Amplitude

$\phi$  : phase difference

$$m \frac{d^2}{dt^2} A e^{i(\omega t - \phi)} + c \frac{d}{dt} A e^{i(\omega t - \phi)} + k A e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$-m\omega^2 A e^{i(\omega t - \phi)} + i c \omega A e^{i(\omega t - \phi)} + k A e^{i(\omega t - \phi)} = F_0 e^{i\omega t}$$

$$\text{Note that } e^{i(\omega t - \phi)} = e^{i\omega t} e^{-i\phi}$$

$$\therefore \cancel{-m\omega^2 A e^{i\omega t}} e^{-i\phi} + i c \omega A \cancel{e^{i\omega t}} e^{-i\phi} + k A \cancel{e^{i\omega t}} e^{-i\phi} = \cancel{F_0 e^{i\omega t}}$$

$$\left[ -m\omega^2 A + i c \omega A + k A \right] e^{-i\phi} = F_0$$

$$-m\omega^2 A + ic\omega A + kA = F_0 e^{i\phi}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$\underbrace{-m\omega^2 A + kA}_{\text{Real}} + \underbrace{ic\omega A}_{\text{Imaginary}} = \underbrace{F_0 \cos\phi}_{\text{Real}} + \underbrace{i F_0 \sin\phi}_{\text{Imaginary}}$$

Equating the real and imaginary parts yields the two equations:

$$\begin{aligned} A(k - m\omega^2) &= F_0 \cos\phi & \text{(a)} \\ c\omega A &= F_0 \sin\phi & \text{(b)} \end{aligned} \quad \dots (3.36)$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{c\omega A}{A(k - m\omega^2)} \quad \leftarrow \text{divide } \frac{\text{(b)}}{\text{(a)}}$$

$$\therefore \tan\phi = \frac{c\omega}{k - m\omega^2} \quad (3.37)$$

By squaring both sides of equations 3.36 a and b and adding and employing the identity  $\sin^2\phi + \cos^2\phi = 1$  we find after solving for A =

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{H.W} \quad (3.38)$$

We know that  $\omega_0^2 = \frac{k}{m}$  and  $\delta = \frac{c}{2m}$

$\downarrow$   $\downarrow$   
 $k = m\omega_0^2$   $c = 2m\delta$



Now, we can rewrite (3-37) and (3-38)

$$\tan \phi = \frac{2m\delta\omega}{m\omega_0^2 - m\omega^2} \Rightarrow \tan \phi = \frac{2\delta\omega}{\omega_0^2 - \omega^2} \quad (3-39)$$

$$A = \frac{F_0}{\sqrt{(m\omega_0^2 - m\omega^2)^2 + 4m^2\delta^2\omega^2}}$$

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + 4m^2\delta^2\omega^2}}$$

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}$$

$$\therefore A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}} \quad (3-40)$$

Amplitude resonance occurs at other  $\omega_r$  which is given as:

$$\omega_r^2 = \omega_0^2 - 2\delta^2 \quad (3-41)$$

$\omega_r$  approaches  $\omega_0$  as  $\delta$  goes to zero

$$\text{Also, } \omega_d = \sqrt{\omega_0^2 - \delta^2} \Rightarrow \omega_0^2 = \omega_d^2 + \delta^2$$

$$\therefore \omega_r^2 = \omega_d^2 + \gamma^2 - 2\gamma^2$$

$$\Rightarrow \boxed{\omega_r^2 = \omega_d^2 - \gamma^2} \quad (3.42)$$

$$\text{if } \gamma = 0 \Rightarrow \omega_r \approx \omega_d$$

Thus if the damping is weak, the resonant frequency  $\omega_r$ , the damped oscillator frequency  $\omega_d$ , and the natural frequency  $\omega_0$  are identical.

At the extreme of strong damping, no amplitude resonance occurs if  $\gamma > \frac{\omega_0}{\sqrt{2}}$  because of a monotonically decreasing function of  $\omega$ . To see this, consider the limiting case  $\gamma^2 = \frac{\omega_0^2}{2}$ , so eq (3.41)  $\Rightarrow \omega_r = 0$

Now, substitute in (3.40)  $\Rightarrow$

$$\boxed{A(\omega) = \frac{F_0/m}{\sqrt{\omega_0^4 + \omega^4}}} \quad (3.43)$$

## Amplitude of oscillation at resonance peak:

The steady-state amplitude at the resonant frequency which we call  $A_{max}$  is obtained from eqs (3.40) and (3.41)

$$\omega = \omega_r$$

$$\therefore A_{max} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_r^2)^2 + 4\delta^2\omega_r^2}}$$

$$\omega_r^2 = \omega_0^2 - 2\delta^2$$

$$A_{max} = \frac{F_0/m}{\sqrt{(\cancel{\omega_0^2} - \cancel{\omega_0^2} + 2\delta^2)^2 + 4\delta^2\omega_0^2 - 8\delta^4}}$$

$$A_{max} = \frac{F_0/m}{\sqrt{4\delta^4 + 4\delta^2\omega_0^2 - 8\delta^4}}$$

$$A_{max} = \frac{F_0/m}{\sqrt{4\delta^2\omega_0^2 - 4\delta^4}}$$

$$A_{max} = \frac{F_0/m}{\sqrt{4\delta^2(\omega_0^2 - \delta^2)}}$$

$$A_{max} = \frac{F_0/m}{2\delta\sqrt{\omega_0^2 - \delta^2}}$$

(3.44)

For weak damping  $\Leftrightarrow \delta \approx$  very small

$$A_{\max} = \frac{F_0/m}{2\delta \sqrt{\omega_0^2 - \delta^2}}$$

$\delta^2$  neglect

$$\therefore A_{\max} = \frac{F_0/m}{2\delta\omega_0} \quad (3.45)$$

Since  $C = 2m\delta$

$$\therefore A_{\max} = \frac{F_0}{c\omega_0} \quad (3.46)$$

Sharpness of the resonance : Quality factor

In the case of weak damping  $\delta \ll \omega_0$

$$\text{eq (3.40)} \Rightarrow \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \\ \approx 2\omega_0(\omega_0 - \omega)$$

$$4\delta^2\omega^2 \approx 4\delta^2\omega_0^2$$

$$\therefore A(\omega) = \frac{F_0/m}{\sqrt{(2\omega_0(\omega_0 - \omega))^2 + 4\delta^2\omega_0^2}} \\ = \frac{F_0/m}{\sqrt{4\omega_0^2(\omega_0 - \omega)^2 + 4\delta^2\omega_0^2}} = \frac{F_0/m}{2\omega_0 \sqrt{(\omega_0 - \omega)^2 + \delta^2}}$$

From (3-45), we have  $\frac{F_0/m}{2\omega_0} = \delta A_{\max}$

$$\therefore A(\omega) = \frac{A_{\max} \delta}{\sqrt{(\omega_0 - \omega)^2 + \delta^2}} \quad \dots (3-47)$$

IF  $|\omega_0 - \omega| = \delta$  or  $\omega = \omega_0 \pm \delta$

$$\Downarrow \\ A(\omega) = \frac{1}{\sqrt{2}} A_{\max}$$

$$A^2(\omega) = \frac{1}{2} A_{\max}^2 \quad \dots (3-48)$$

$\delta$  is measuring the width of the resonance curve.

The quality factor is defined as  $\frac{\omega_d}{2\delta}$  [see eq 3-32]

In the case of weak damping ( $\omega_d \approx \omega_0$ ), one can write

$$Q = \frac{\omega_0}{2\delta} \Rightarrow 2\delta = \frac{\omega_0}{Q}$$

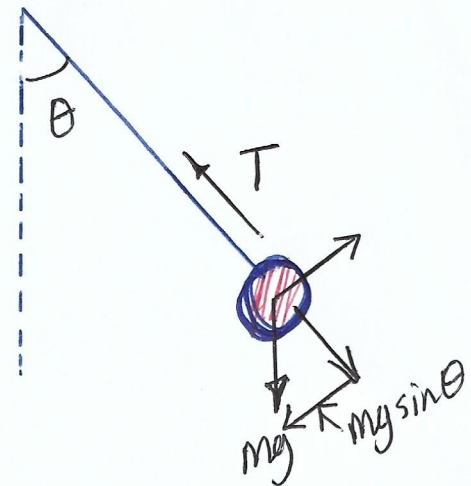
So, the total width at the half-energy point is

$$\Delta\omega = 2\delta \approx \frac{\omega_0}{Q} \quad \dots (3-49)$$

and  $\omega = 2\pi f \Rightarrow \frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q} \quad \dots 3-50$

**Example 5** Consider a pendulum of length  $l$  and a bob of mass  $m$  at its end moving through oil with  $\theta$  decreasing. The massive bob undergoes small oscillations, but the oil retards the bob's motion with a resistive force proportional to speed with  $F_{res} = 2m\sqrt{\frac{g}{l}}(l\dot{\theta})$ . The bob is initially pulled back at  $t=0$  with  $\theta = \alpha$  and  $\dot{\theta} = 0$ . Find the angular displacement  $\theta$  and velocity  $\dot{\theta}$  as a function of time.

$$\text{Force} = m(l\ddot{\theta}) = \text{Restoring force} + \text{Resistive force}$$



$$\therefore \cancel{m}l\ddot{\theta} = -\cancel{m}g\sin\theta - 2\cancel{m}\sqrt{\frac{g}{l}}l\dot{\theta}$$

$$\ddot{\theta} + 2\sqrt{\frac{g}{l}}\dot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\omega_0^2 = \frac{g}{l}, \quad \beta^2 = \frac{g}{l} \quad \Rightarrow \quad \omega_0^2 = \beta^2$$

The pendulum is critically damped

$$\theta(t) = (A + Bt)e^{-\beta t}$$

$$\theta(t=0) = \alpha = A$$

$$\dot{\theta}(t) = B e^{-\beta t} - \beta (A + Bt) e^{-\beta t}$$

$$\dot{\theta}(t=0) = 0 = B - \beta A$$

$$B = \beta A = \beta \alpha$$

$$\theta(t) = \alpha \left( 1 + \sqrt{\frac{g}{l}} t \right) e^{-\sqrt{g/l} t}$$

$$\dot{\theta}(t) = -\frac{\alpha g}{l} t e^{-\sqrt{g/l} t}$$

### Example 6

A simple harmonic oscillator consists of 100g mass attached to a spring whose force constant is  $10^4$  dyne/cm. The mass is displaced 3 cm and released from rest. Calculate

(a) The natural frequency  $\nu_0$ .

(b) The total energy.

(c) The maximum speed.

$$(a) \nu_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{10^4 \frac{g}{\text{sec}^2} \cdot \frac{\text{cm}}{\text{cm}}}{10^2 g}}$$

$$\nu_0 = \frac{10}{2\pi} \text{ sec}^{-1}$$

$$\nu_0 \approx 1.6 \text{ Hz}$$

$$(b) E = \frac{1}{2} k A^2 = \frac{1}{2} 10^4 \cdot 3^2$$

$$\Rightarrow E = 4.5 \times 10^4 \text{ erg}$$

(c) The maximum velocity is attained when the total energy of oscillator is equal to the kinetic energy. Therefore

$$\frac{1}{2} m v_{\max}^2 = 4.5 \times 10^4 \text{ erg}$$

$$v_{\max} = \sqrt{\frac{2 \times 4.5 \times 10^4}{100}}$$

$$\therefore v_{\max} = 30 \text{ cm/sec}$$



**Example 7** A damped harmonic oscillator with  $m = 10 \text{ kg}$ ,  $k = 250 \text{ N/m}$  and  $c = 60 \text{ kg/s}$  is subject to a driving force given by  $F_0 \cos \omega t$  where  $F_0 = 48 \text{ N}$ .

(a) what value of  $\omega$  results in steady-state oscillation with maximum amplitude under this condition.

(b) what is the maximum amplitude?

(c) what is the phase shift?

Solution

$$\gamma = \frac{c}{2m} = \frac{60}{2 \times 10} = 3 \text{ sec}^{-1} \Rightarrow \gamma^2 = 9 \text{ sec}^{-2}$$

$$\omega_0^2 = \frac{k}{m} = \frac{250}{10} = 25 \text{ sec}^{-2} \Rightarrow \omega_0 = 5 \text{ sec}^{-1}$$

(a)  $\omega_d^2 = \omega_0^2 - \gamma^2$

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$\therefore \omega_d^2 = 25 - 9 = 16 \text{ sec}^{-2} \Rightarrow \omega_d = 4 \text{ sec}^{-1}$$

$$\omega_r^2 = 25 - 18 = 7 \text{ sec}^{-2} \Rightarrow \omega_r = \sqrt{7} \text{ sec}^{-1}$$

(b)  $A_{\max} = \frac{F_0}{2m\gamma\omega_d} = \frac{F_0}{c\omega_d} = \frac{48}{(60)(4)} = 0.2 \text{ m}$

(c)  $\tan \phi = \frac{2\gamma\omega_r}{\omega_0^2 - \omega_r^2} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$

**Example 8** The frequency  $f_d$  of a damped harmonic oscillator is 100 Hz and the ratio of the amplitude of two successive maxima is one half

(a) what is the undamped frequency of this oscillator?

(b) what is the resonant frequency?

The ratio of the amplitude is  $e^{-\delta T_d} = \frac{1}{2}$

$$\Rightarrow -\delta T_d = \ln \frac{1}{2} \Rightarrow \delta T_d = \ln 2 \Rightarrow \delta = \frac{1}{T_d} \ln 2$$

$$\delta = f_d \ln 2 = 69.315 \text{ sec}^{-1}$$

$$\delta^2 = 4804.57 \text{ sec}^{-2}$$

(a)  $\omega_d = 2\pi f_d = 2(3.14)(100) = 628 \text{ sec}^{-1}$

$$\omega_d^2 = 394384 \text{ sec}^{-2}$$

$$\omega_d^2 = \omega_0^2 - \delta^2 \Rightarrow \omega_0^2 = \omega_d^2 + \delta^2$$

$$\Rightarrow \omega_0 = \sqrt{\omega_d^2 + \delta^2}$$

$$\therefore \omega_0 = 631.81 \text{ sec}^{-1}$$

But  $f_0 = \frac{\omega_0}{2\pi} = 100.6 \text{ sec}^{-1}$

$$\textcircled{b} \quad \omega_r^2 = \omega_d^2 - \gamma^2 \implies \omega_r = \sqrt{\omega_d^2 - \gamma^2}$$

$$\therefore \omega_r = 624.16 \text{ sec}^{-1}$$

$$\implies f_r = \frac{\omega_r}{2\pi} = 99.4 \text{ sec}^{-1}$$