Chapter 2

Newtonian Mechanics

1. Introduction

To describe the motion of bodies, we need certain fundamental concepts

Distance | Time | Mass

Distance + Time = Velocity + Acceleration

$$velocity = \frac{change\ in\ distance}{change\ in\ time}$$

$$Acceleration = \frac{change\ in\ velocity}{change\ in\ time}$$

$$Momentum\ unit = \frac{Mass \times distance}{time}$$

$$Energy\ unit = \frac{Mass \times distance}{time}$$

1. Newton's laws

I. A body remains at rest or in uniform motion unless acted upon by a force.

II. A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

III. If two bodies exert force on each other, these forces are equal in magnitude and opposite in direction.

The 1st law is meaningless without the concept of "Force".

Zero Force → Free Body(free particle) → Inertia

The 2nd law provides an explicit statement: Force is related to the time rate of changes of ***momentum***. Newton appropriately defined **momentum** to be the product of mass and velocity.

$$p = mv$$

Newton's second law can be expressed as :

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv)$$

The 3rd law states that

$$F_1 = -F_2$$

$$\frac{dp_1}{dt} = -\frac{dp}{dt}$$

$$m_1\left(\frac{dv_1}{dt}\right) = m_2\left(-\frac{dv_2}{dt}\right)$$

$$m_1 a_1 = m_2(-a_2)$$

$$\frac{m_2}{m_1} = -\frac{a_1}{a_2}$$

The negative sign indicates only that the two acceleration vectors are oppositely directed.

Another interpretation of Newton's third law is based on the concept of momentum

$$\frac{d}{dt}(p_1 + p_2) = 0 \quad \Rightarrow \quad p_1 + p_2 = constant$$

3. Frame of references

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Laws of motion have meaning when the motion of bodies have been measured relative to some reference frame.

1 Inertial frame of references

Frame of reference is at **rest** or in **constant speed**

Newton's law does hold

Demonstration

Part 1: free fall in a fixed car

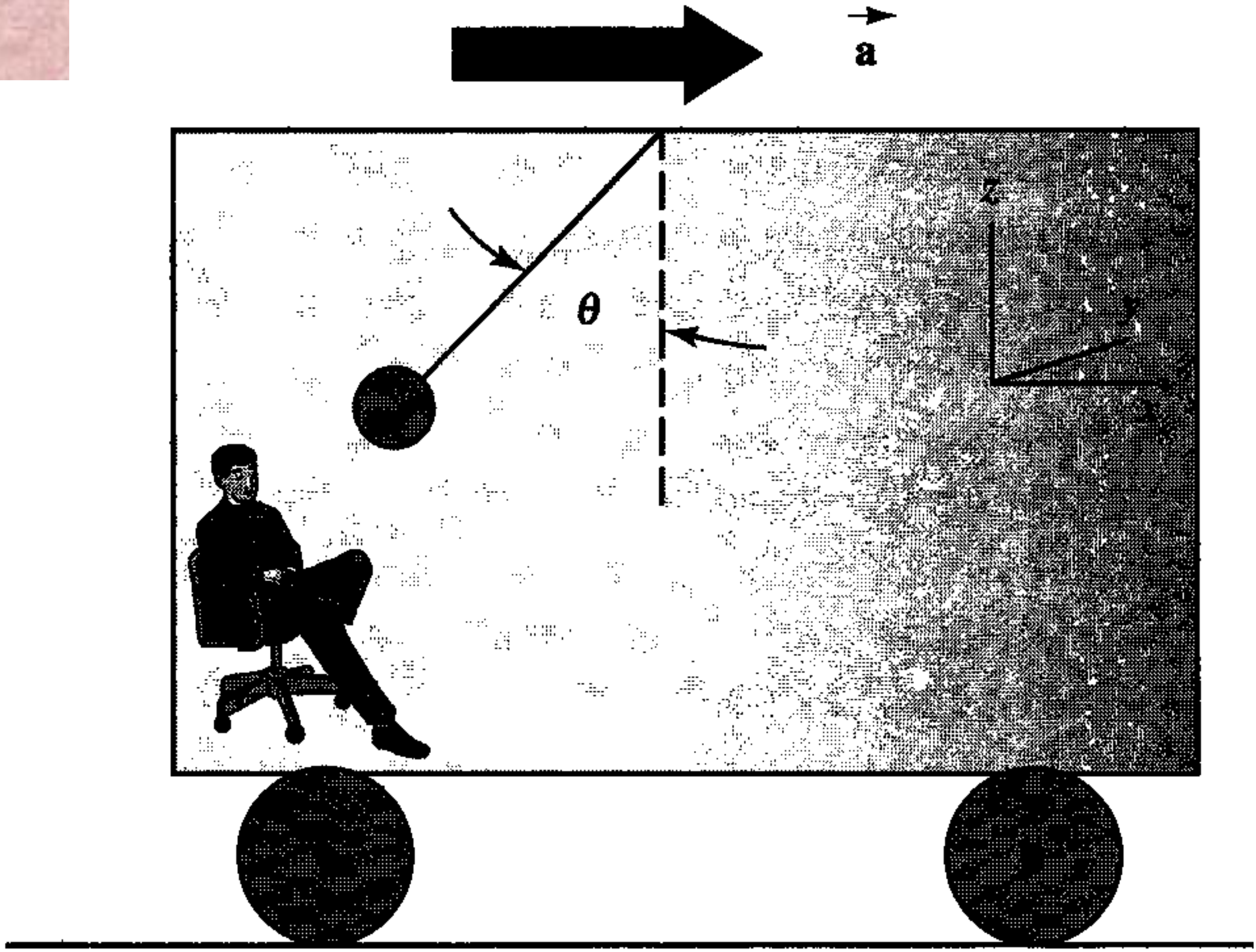
2

Non-Inertial frame of references

Frame of reference is accelerated

Newton's law does not hold

Fictitious force



Demonstration

Part 4: Accelerated frame of reference

A coordinate system attached to the earth's surface is ***approximately inertial***

The theory of relativity resulted from the failure of attempts to find an absolute frame of reference in which all of the fundamental laws of physics, not just Newton's first law of motion, were supposed to be valid. This led Einstein to the conclusion that the failure to find an absolute frame was because of the simple reason that none exists.

4. The equation of motion for a particle

$$F = \frac{dP}{dt}$$

$$F = \frac{d}{dt}(mv) \quad = m\frac{dv}{dt} \quad = m\ddot{r}$$

$F(r, v, t)$ The force F may be a combination of position, velocity, and time.

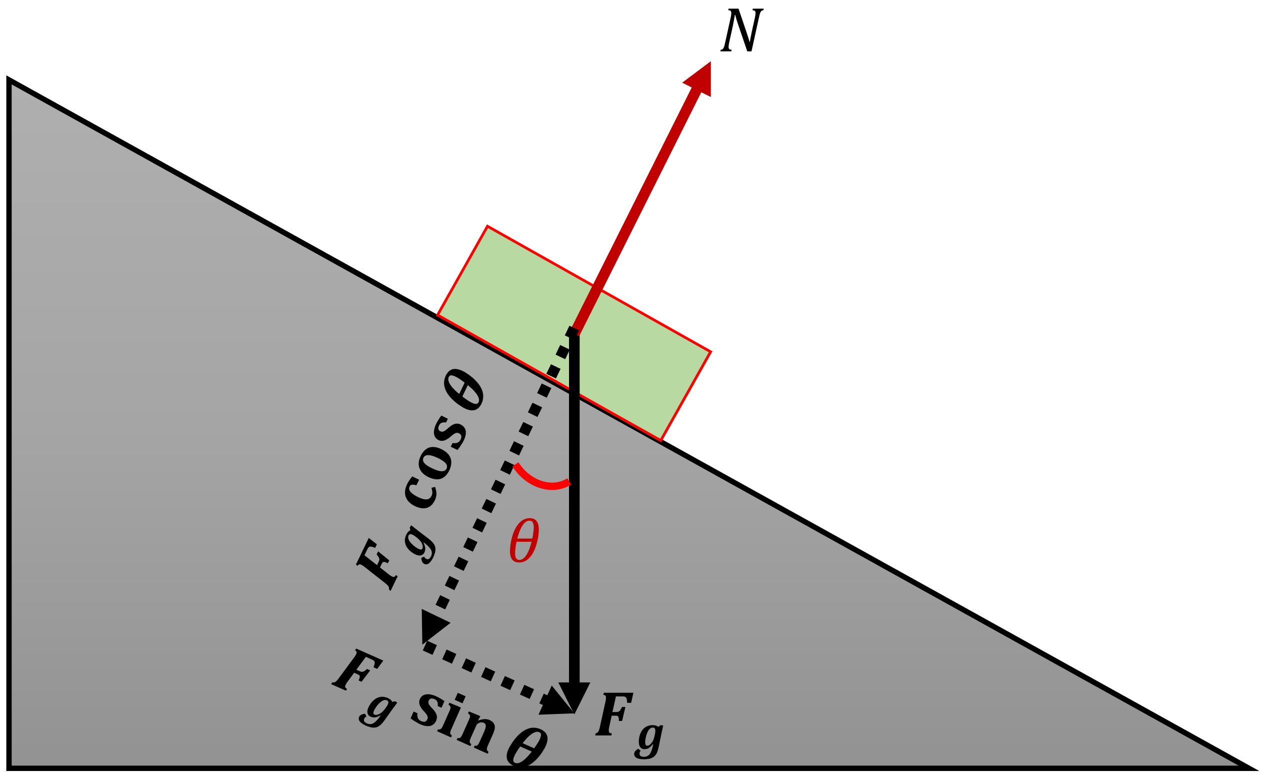
Example 1

If a block slides without friction down a fixed, inclined plane, with $\theta = 30^o$. What is the block's acceleration? Find the velocity of the block after it moves from rest to a distance x_0 down the plane?

Two forces act on the block:

1- The gravitational force F_g.

2- The plane's normal force N



The total force is constant

$$F_{net} = F_g + N$$

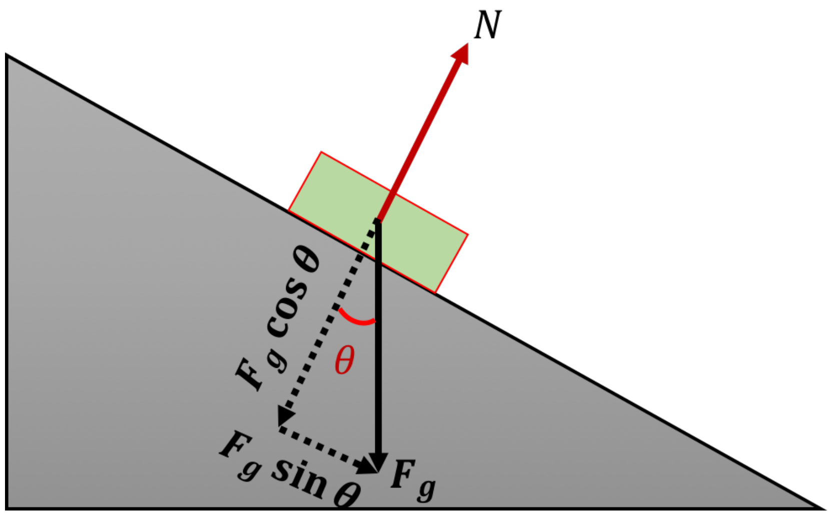
$$F_{net} = m\,\ddot{r}$$

$$m\,\ddot{r} = F_g + N$$

The vector must be applied in two directions *x and y*

x − direction $F_g \sin\theta = m\,\ddot{x}$

y − direction $-F_g \cos\theta + N = 0$



$$\ddot{x} = \frac{F_g}{m} sin\theta \Rightarrow \ddot{x} = \frac{mg}{m} sin\theta$$

$$\ddot{x} = g\ sin\theta \Rightarrow \ddot{x} = (9.8)\sin(30) \Rightarrow \ddot{x} = 4.9\ m/s^2$$

$$[F_g \sin\theta = m\,\ddot{x}] \times 2\dot{x}$$

$$2\dot{x}\ddot{x} = 2\dot{x}\, g \sin\theta$$

$$\frac{d}{dt}(\dot{x}^2) = 2g\sin\theta\,\frac{dx}{dt}$$

$$d(\dot{x}^2) = 2g\sin\theta\, dx$$

$$\int_0^{v_0} d(\dot{x}^2) = 2g\sin\theta \int_0^{x_0} dx$$

$$v_0^2 = 2g\sin\theta\, x_0$$

$$v_0 = \sqrt{2g\sin\theta\, x_0}$$

Example 2

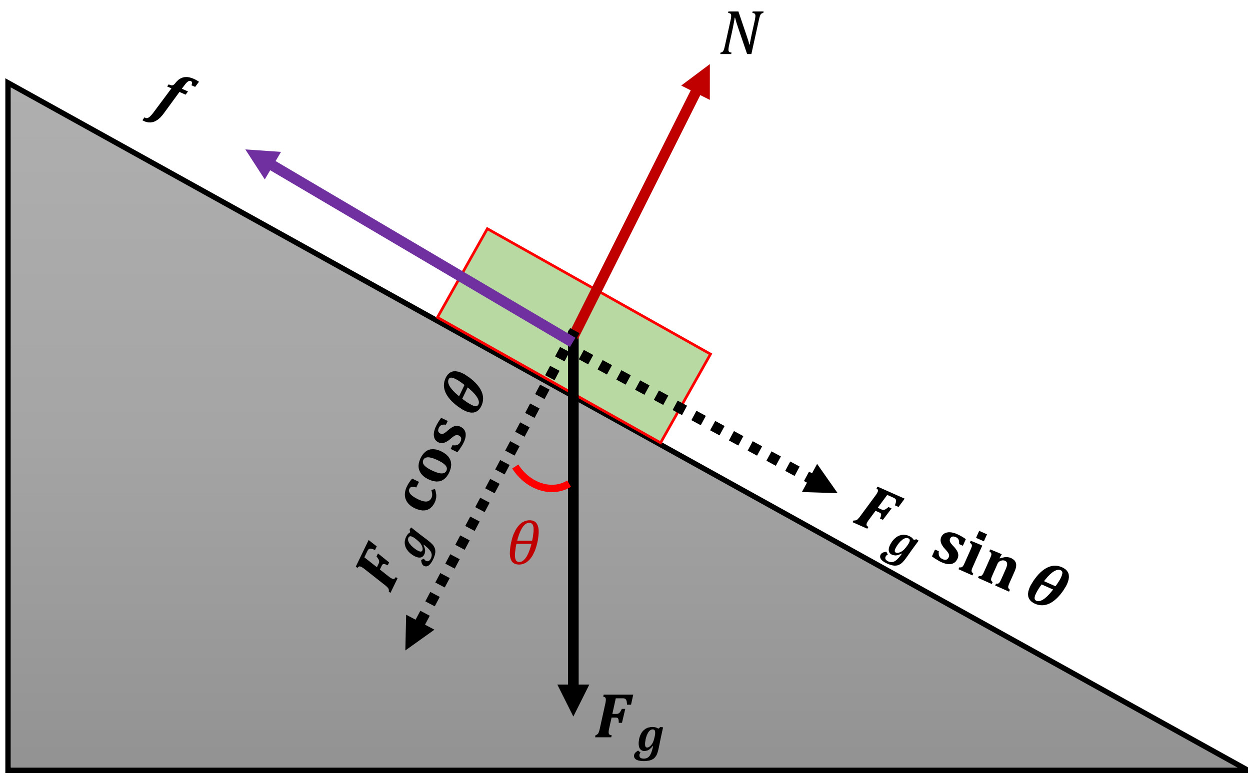
Consider that the coefficient of static fraction between the block and plane in Example 1 is $\boldsymbol{\mu_s = 0.4}$, at what angle $\boldsymbol{\theta}$ will the block start sliding if it is initially at rest.

The static frictional force has the approximate maximum value.

$$f_{max} = \mu_s N$$

$$F_{net} = F_g + N + f$$

$$m\ddot{r} = F_g + N + f$$



$x - direction$

$$m\ddot{x} = F_g sin\theta - f$$

$y - direction$

$$0 = -F_g cos\theta + N$$

As the angle θ increases, the static frictional force will be unable to keep the block at rest.

$x - direction\ equation$

$$f_s = f_{max} = \mu_s N = \mu_s F_g \cos\theta$$

$$m\ddot{x} = F_g sin\theta - f$$

$$m\ddot{x} = F_g \sin\theta - f_{max} \quad \Rightarrow \quad m\ddot{x} = F_g \sin\theta - \mu_s F_g \cos\theta$$

$$\ddot{x} = g(\sin\theta - \mu_s \cos\theta)$$

Just before the block starts to slide, the acceleration $\ddot{x} = 0$

$$\sin\theta = \mu_s \cos\theta \quad \Rightarrow \quad \theta = tan^{-1}\mu_s \quad \Rightarrow \quad \theta = tan^{-1}(0.4) \quad \Rightarrow \quad \theta = 22^o$$

Example 3

After the block in the previous example begins to slide, the coefficient of kinetic (sliding) friction becomes $\boldsymbol{\mu_k = 0.3}$. Find the acceleration for the angle $\boldsymbol{\theta = 30^o}$.

By following the same steps in example 2, the kinetic friction becomes (approximately)

$$f_k = \mu_k N = \mu_k F_g \cos\theta \quad \Rightarrow \quad m\ddot{x} = F_g \sin\theta - f_k$$

$$m\ddot{x} = mg(\sin\theta - \mu_k \cos\theta) \quad \Rightarrow \quad \ddot{x} = g(\sin 30 - 0.3\ \cos 30) \quad \Rightarrow \quad \boxed{\ddot{x} = 0.24\ g}$$

5. Effects of retarding forces

We should emphasize that the force F in Newton's equation [$\boldsymbol{F} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}(\boldsymbol{mv}) = \boldsymbol{m}\frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \boldsymbol{m\ddot{r}}$] is not necessarily constant, and it may consist of several distinct parts.

For example, if a particle falls in a constant gravitational field, the gravitational force is [$F_g = mg$], where g is the acceleration of gravity.

If, in addition, a **retarding force** F_r exists that in some function of the **instantaneous speed.** Then, the total force is

$$F = F_g + F_r \quad \Rightarrow \quad F = mg + F_r(v)$$

Gravitational force (F_g) — Retarding force (F_r)

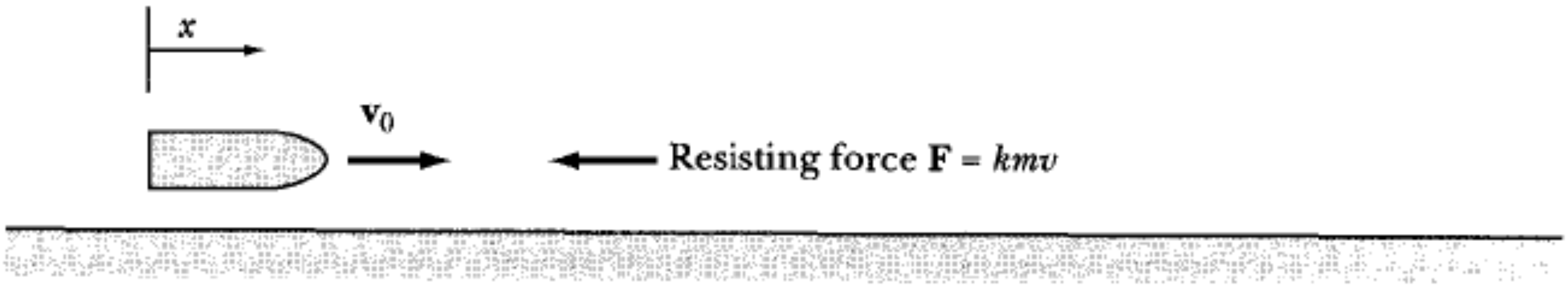
F is proportional to some power of speed. $F_r \propto v^{n}$

$F_r \propto v$ $\Longleftrightarrow$ $[Low\ speed]$

$F_r \propto v^2$ $\Longleftrightarrow$ $[high\ speed]$

Example 4

Find the displacement and velocity of a horizontal motion in a medium which the retarding force is proportional to the velocity. Then, find the velocity as a function of displacement.



$x - direction$

$$ma = m\frac{dv}{dt} = -kmv \quad \Rightarrow \quad \frac{dv}{dt} = -kv \quad \Rightarrow \quad \frac{dv}{v} = -kdt$$

Integrate → Initial condition

$$\int_{v_0}^{v} \frac{dv}{v} = -k\int_0^t dt \;\Rightarrow\; \ln v\Big]_{v_0}^{v} = -\mathrm{k}\, t\Big]_0^t \;\Rightarrow\; \ln v - \ln v_0 = -kt$$

$$\frac{\ln v}{\ln v_0} = -kt \;\Rightarrow\; \frac{v}{v_0} = e^{-kt} \;\Rightarrow\; v = v_0 e^{-kt}$$

Integrate again to find the displacement

$$\frac{dx}{dt} = v_0 e^{-kt} \;\Rightarrow\; dx = v_0 e^{-kt}\, dt \;\Rightarrow\; \int dx = v_0 \int e^{-kt}\, dt$$

Initial condition $x(t=0)=0$

$$x = v_0\left(-\frac{1}{k}e^{-kt}\right)\Bigg]_0^t \quad \Rightarrow \quad x = \frac{v_0}{k}\left[1-e^{-kt}\right]$$

We can obtain the velocity as a function of displacement by writing:

$$\frac{dv}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$$

[chain rule]

$$\frac{dv}{dx} = \frac{dv}{dt}.\frac{1}{v}$$

$$v\frac{dv}{dx} = \frac{dv}{dt} \quad \Rightarrow \quad v\frac{dv}{dx} = -kv_0e^{-kt} \quad \Rightarrow \quad v\frac{dv}{dx} = -kv$$

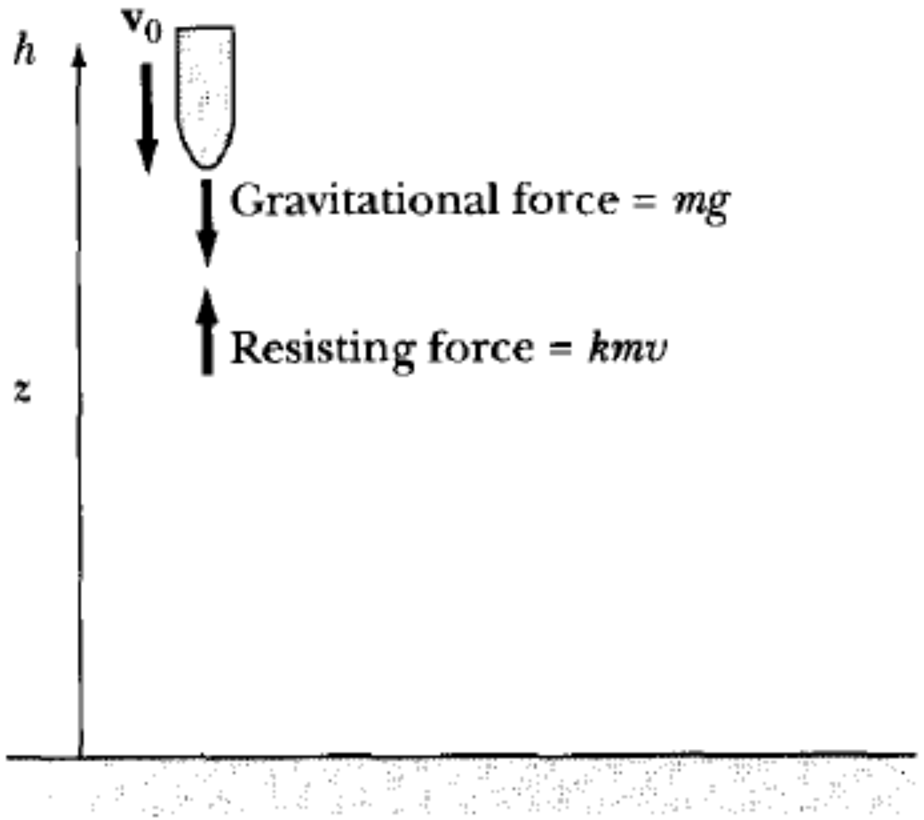
$$\int_{v_0}^{v} dv = -k\int_0^x dx \quad \Rightarrow \quad v = v_0 - kx$$

The velocity decreases linearly with displacement

Example 5

Find the displacement and velocity of a particle undergoing vertical motion in the medium having a retarding force proportional to velocity.

Let us consider that the particle is falling downward z with initial velocity v_0 from the height h in a constant gravitational field.



The equation of motion is:

$$F = m\frac{dv}{dt} = -mg - kmv$$

$$\frac{m\,dv}{mg + kmv} = -dt \quad \Rightarrow \quad \frac{dv}{g + kv} = -dt$$

Integrating and setting $v(t = 0) = v_0$

$$\frac{1}{k}\ln(g+kv)=-t+c \Rightarrow \ln(g+kv)=k(-t+c) \Rightarrow g+kv=e^{-kt}e^{kc}$$

To find e^{kc}

$$\frac{1}{k}\ln(g+kv)=-t+c \quad \xrightarrow{t\to 0 \qquad v=v_0} \quad c=\frac{1}{k}\ln(g+kv_0)$$

$$ck=\ln(g+kv_0) \Rightarrow e^{ck}=g+kv_0 \Rightarrow \boxed{v=-\frac{g}{k}+\frac{g+kv_0}{k}e^{-kt}}$$

Integrating once more and evaluating the constant by setting $z(t=0)=h$, we find

$$\boxed{z=h-\frac{gt}{k}+\frac{kv_0+g}{k^2}\left(1-e^{-kt}\right)}$$

H. W

HWs

1

A particle of mass m slides down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d after starting from rest is

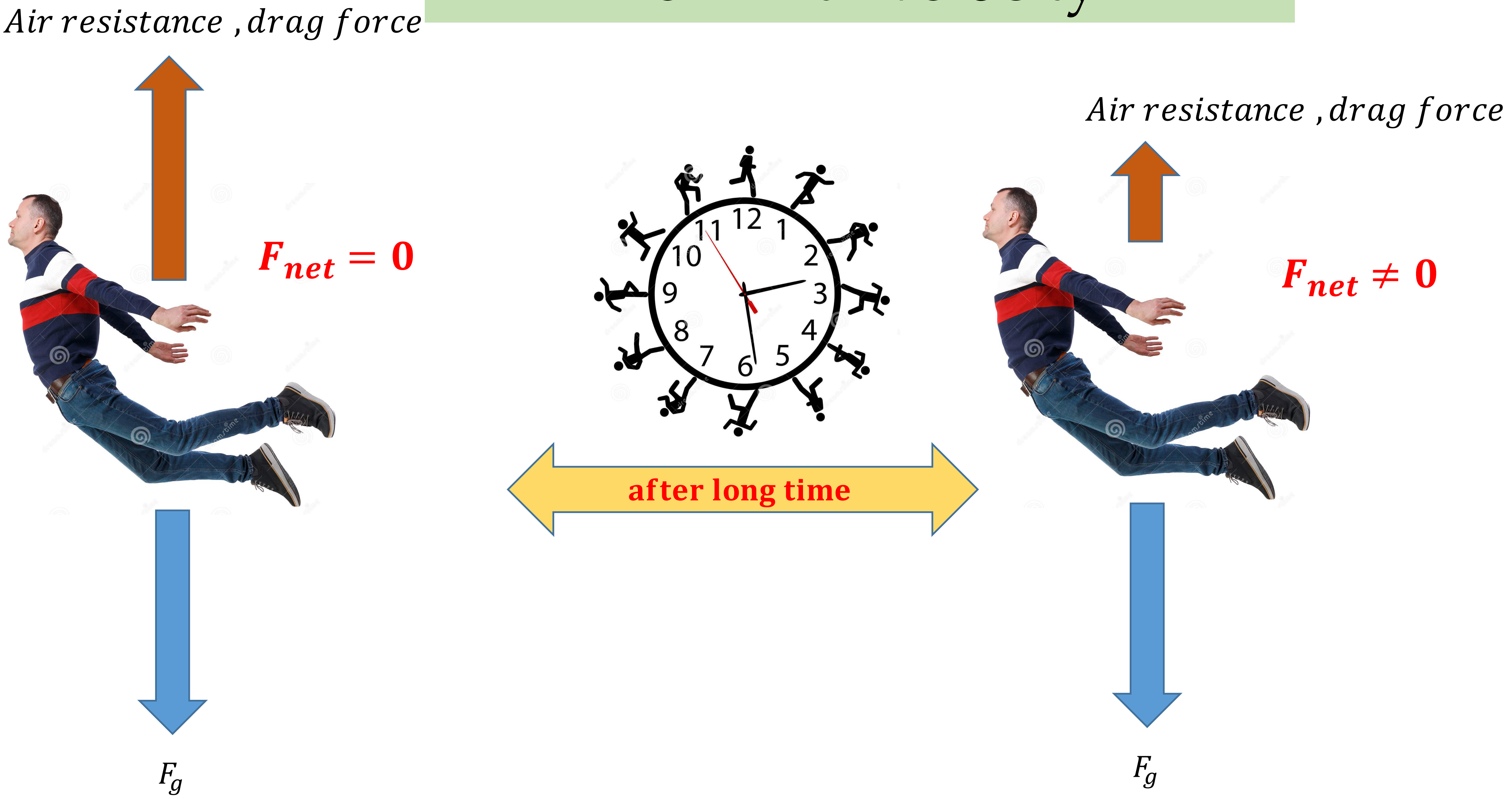
$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg \sin \theta}}$$

where θ is the angle of inclination of the plane.

2

A skier weighing 90 kg starts from rest down a hill inclined at 17°. He skis 100 m down the hill and then coasts for 70 m along level snow until he stops. Find the coefficient of kinetic friction between the skis and the snow. What velocity does the skier have at the bottom of the hill?

Terminal velocity



Terminal velocity

$$v = -\frac{g}{k} + \frac{g + kv_0}{k} e^{-kt}$$

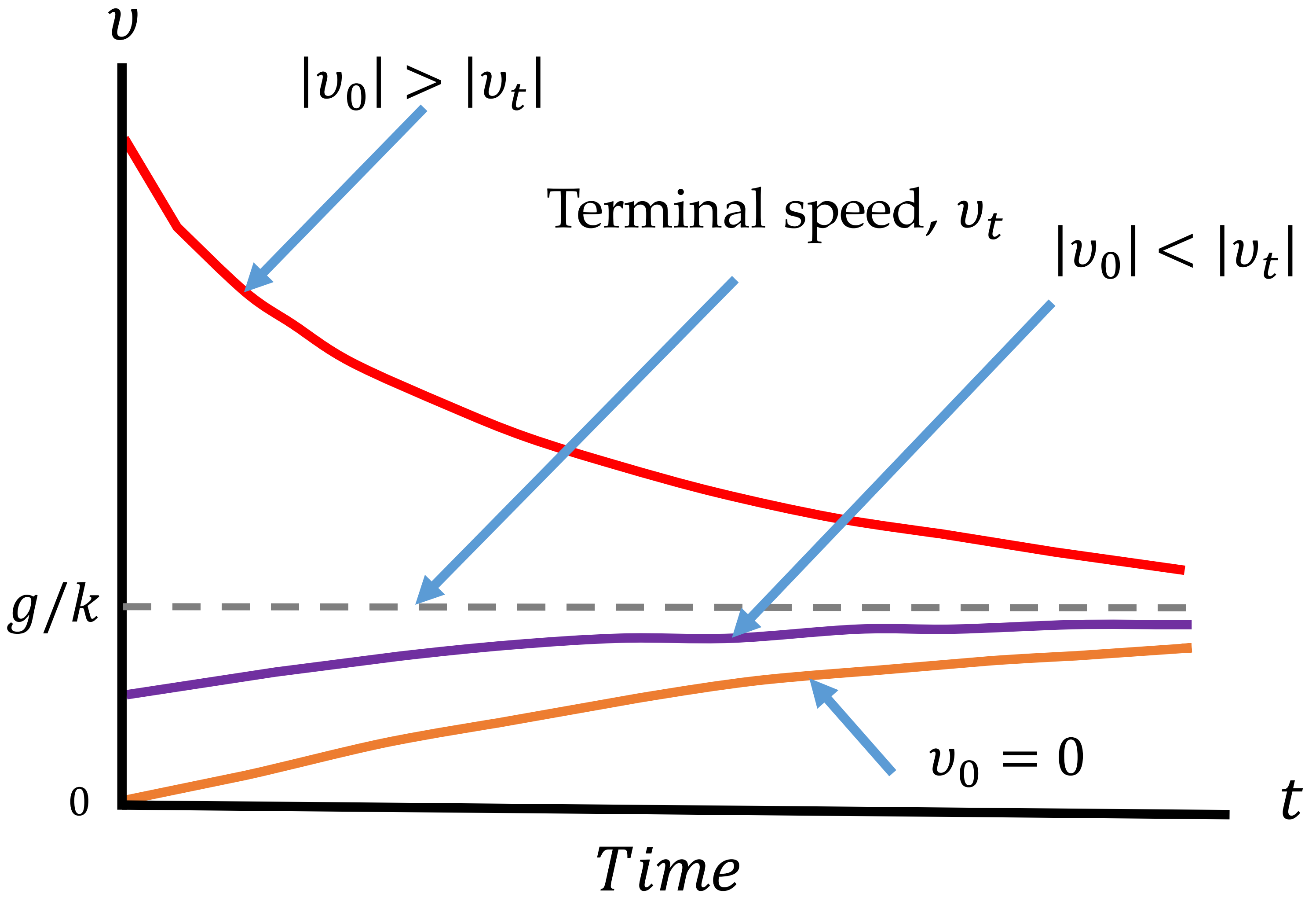
velocity of a particle undergoing vertical motion

$t \rightarrow \infty$

$$v_t = -\frac{g}{k}$$

Force will vanish

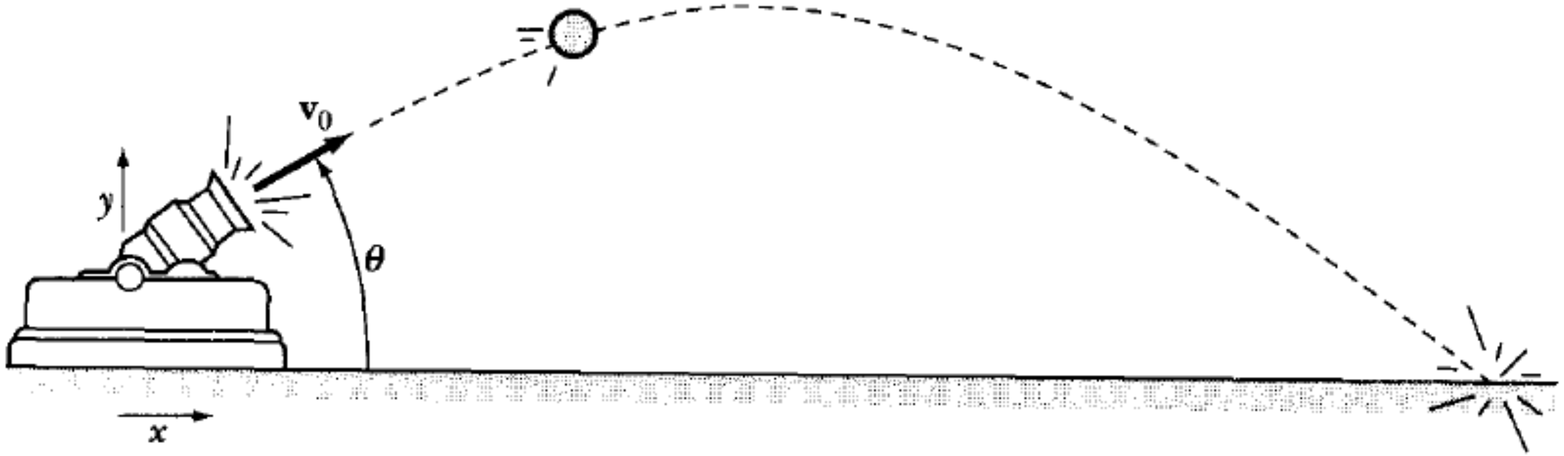
$$F = -mg - kmv$$



Example 6

Consider a projectile motion in two dimension, without considering air resistance. Let the muzzle velocity of the projectile be v_0 and the angle of the elevation be θ. Calculate the projectile's displacement, velocity, and range. Neglect the height of the gun.

$$F = mg$$



$x - direction$

$$0 = m\ddot{x}$$

$y - direction$

$$-mg = m\ddot{y}$$

$Neglect\ the\ height\ of\ the\ gun$ $\qquad x = y = 0$ at $t = 0$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

$$\dot{x} = v_0 \cos\theta \qquad \dot{y} = -gt + v_0 \sin\theta$$

$$x = v_0 t \cos\theta \qquad y = -\frac{gt^2}{2} + v_0 t \sin\theta$$

The speed and total displacement as functions of time are found to be

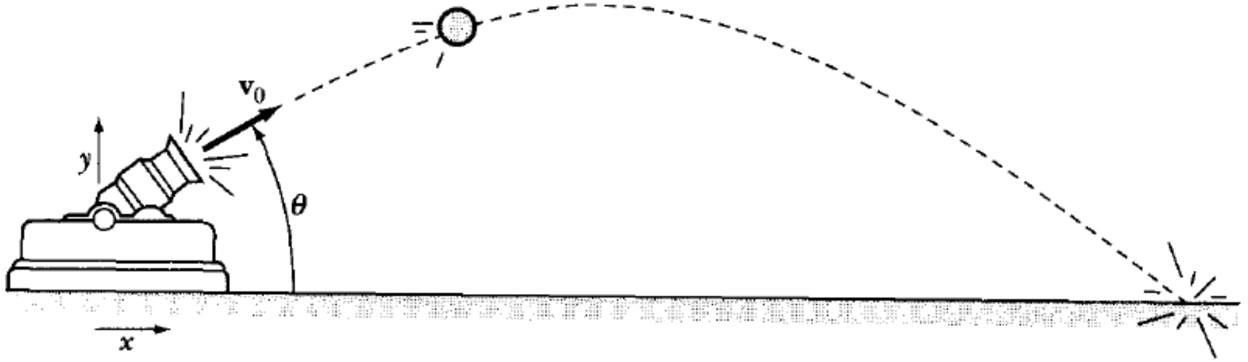
$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(v_0 \cos\theta)^2 + (-gt + v_0 \sin\theta)^2} = \sqrt{v_0^2 + g^2 t^2 - 2v_0 t \sin\theta}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(v_0 t \cos\theta)^2 + (-\frac{gt^2}{2} + v_0 t \sin\theta)^2} = \sqrt{v_0^2 t^2 + \frac{g^2 t^2}{4} - 2v_0 g t^3 \sin\theta}$$

We can find the range by determining the value of x when the projectile falls back to ground (when $y = 0$).

$$y = -\frac{gt^2}{2} + v_0 t \sin\theta \quad \Rightarrow \quad \frac{gT^2}{2} = v_0 T \sin\theta \quad \Rightarrow \quad T = \frac{2v_0 \sin\theta}{g}$$

The range R is found from



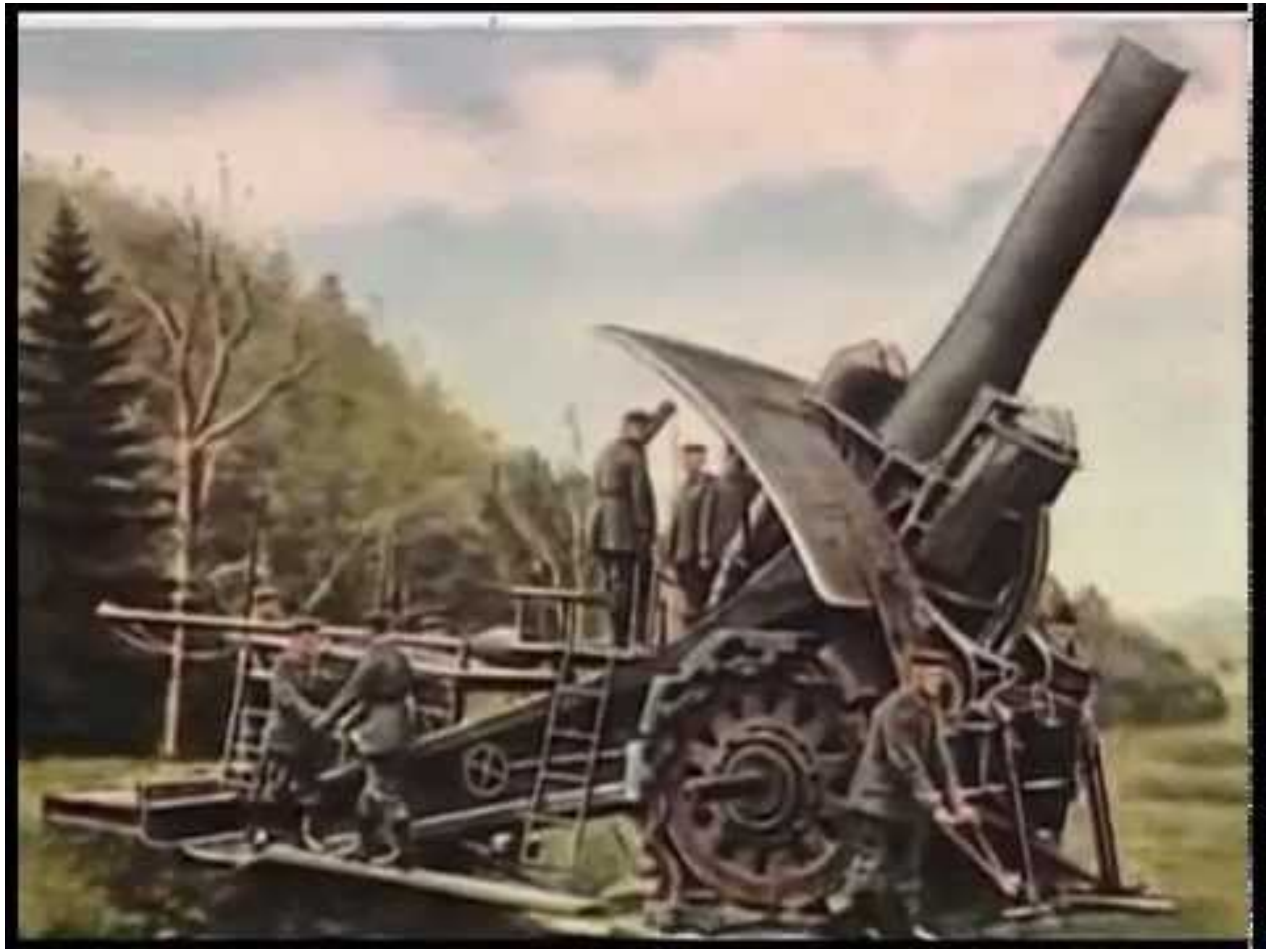
$y = 0$ $t = 0$ $y = 0$ $t = T$

$$x(t = T) = Range = v_0 \boldsymbol{T} \cos\theta$$

$$x(t = T) = v_0 \left(\frac{2v_0 \sin\theta}{g}\right) \cos\theta \quad \Rightarrow \quad R = Range = \frac{v_0^2}{g} \sin 2\theta$$

Notice that the maximum range occurs for $\theta = 45^0$.

Let us consider some actual numbers in these calculations. The Germans long-range gun named *Big Bertha* in World War I to bombard Paris.



Muzzle velocity = $1450\ m/s$ $\theta = 55^0$

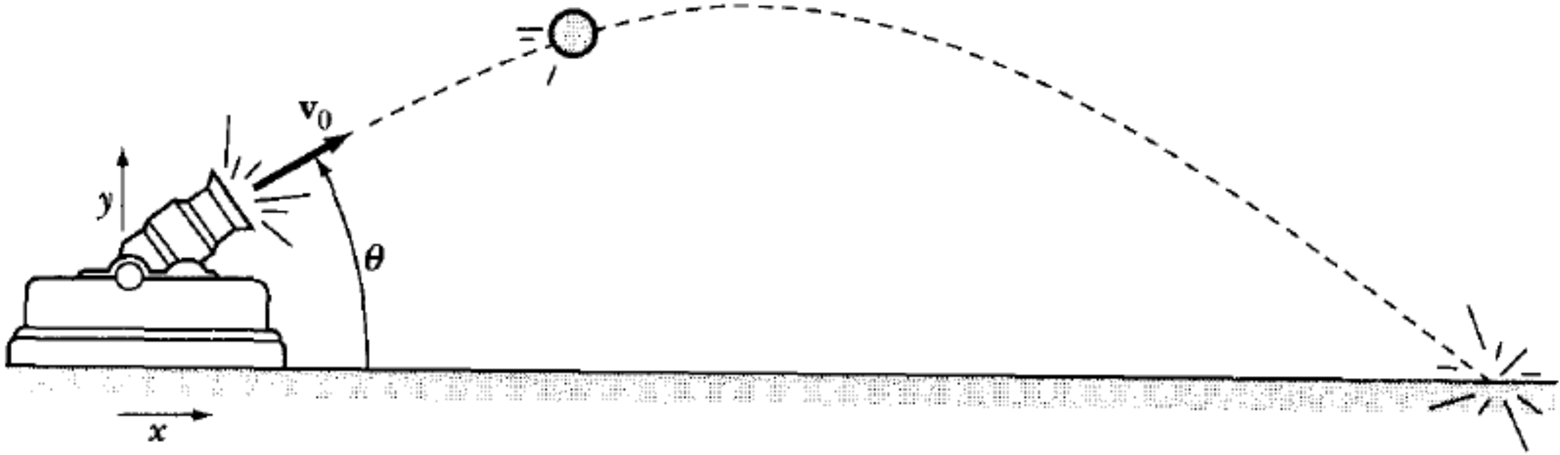
$$R = Range = \frac{v_0^2}{g}\sin 2\theta \Rightarrow R = \frac{(1450\ m/s)^2}{9.8\ m/s^2}\sin(110) \Rightarrow R = 202\ Km$$

Big Bertha ***actual range was 120 km***. The differences is due to the real effect of air resistance.

$$T = \frac{2v_0 \sin\theta}{g} \Rightarrow T = \frac{2v_0 \sin\theta}{g} \Rightarrow T = \frac{2(1450\ m/s)\sin 55}{9.8\ m/s^2}$$

$$\Downarrow$$

$$T = 242\ s$$



$$y_{max}(t=\frac{T}{2}) = -\frac{gT^2}{8} + \frac{v_0 T}{2}\sin\theta$$

$$y_{max}(t=\frac{T}{2}) = -\frac{(9.8\frac{m}{s})(242\ s)^2}{8} + \frac{(1450\frac{m}{s})(242\ s)}{2}\sin(55)$$

$$y_{max} = 72\ Km$$

Example 7

Add the effect of air resistance to the motion of projectile in Example 6. Calculate the decrease in range under the assumption that the force caused by air resistance is directly proportional to the projectile's velocity.

Without air resistance

$$x = v_0 t \cos\theta$$

$$y = -\frac{gt^2}{2} + v_0 t \sin\theta$$

The initial conditions

$$x(t=0) = 0 = y(t=0)$$

$$\dot{x}(t=0) = v_0 \cos\theta = U$$

$$\dot{y}(t=0) = v_0 \sin\theta = V$$

The equation of motion with air resistance is written as:

$$m\ddot{x} = -km\dot{x}$$

$$m\ddot{y} = -km\dot{y} - mg$$

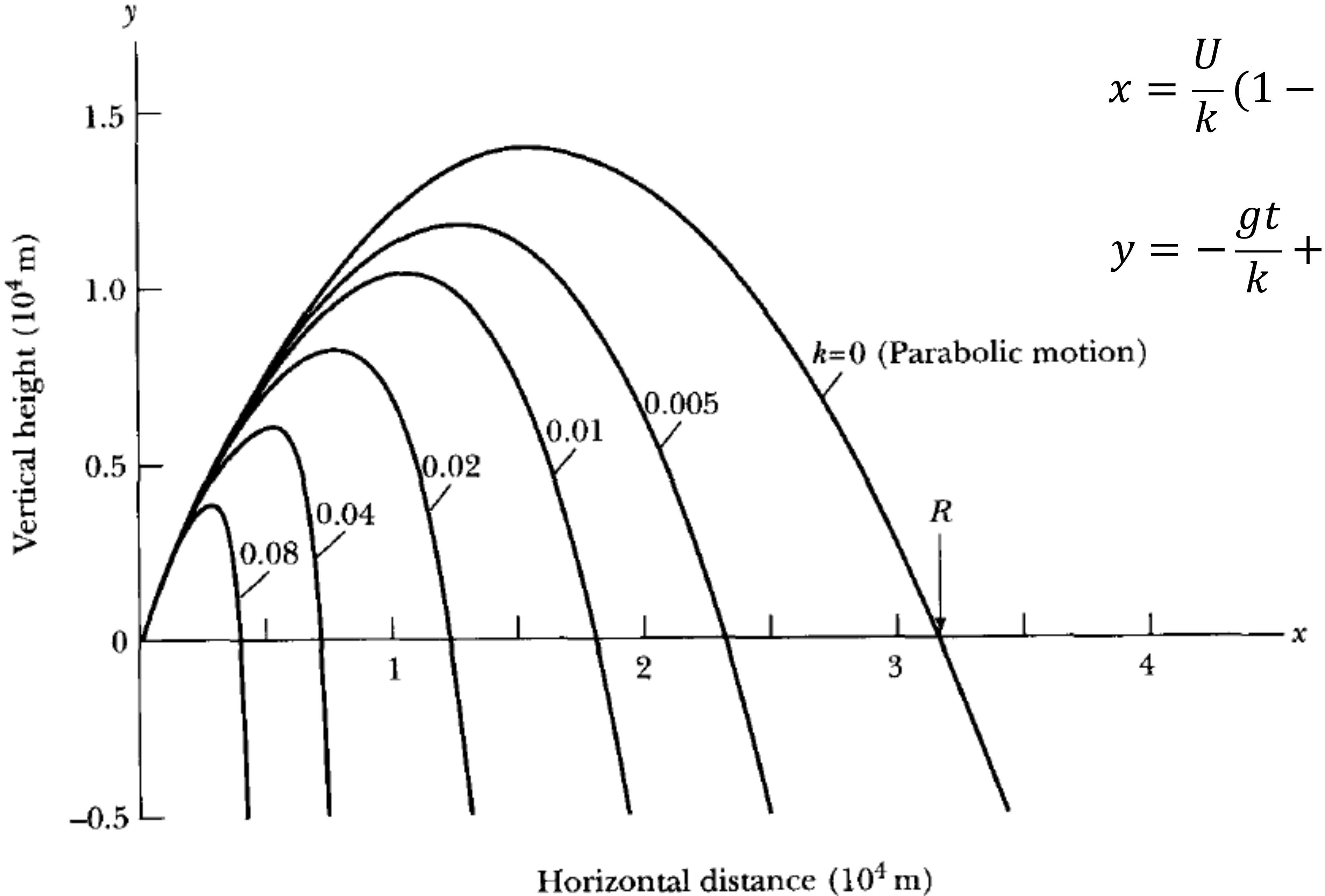
H.W

$$x = \frac{U}{k}(1 - e^{-kt})$$

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt})$$

With air resistance

Trajectories of a particle in air resistance for various values of k. [$\theta = 60^0, v_0 = 600\ m/s$]



$$x = \frac{U}{k}(1 - e^{-kt})$$

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt})$$

The range R', which is the range including air resistance, can be found as previously by calculation the time T required for the entire trajectory and then substituting this value into equation x.

The time T is found as previously by finding $t = T$ when $y = 0$.

$$y = -\frac{gt}{k} + \frac{kV+g}{k^2}(1 - e^{-kt}) \quad \Rightarrow \quad \boxed{T = \frac{kV+g}{gk}(1 - e^{-kT})}$$

This is a transcendental equation, and therefore we cannot obtain an analytic expression for T.

To solve this equation, we need to follow the perturbation method.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore, $T = \frac{kV+g}{gk}(1-e^{-kT})$ $\qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$T = \frac{kV+g}{gk}\left(kT - \frac{1}{2}k^2T^2 + \frac{1}{6}k^3T^3 - \cdots\right) \Rightarrow T = \frac{kV+g}{gk}kT\left(1 - \frac{1}{2}kT + \frac{1}{6}k^2T^2 - \cdots\right)$$

$$1 = \frac{kV+g}{g}\left(1 - \frac{1}{2}kT + \frac{1}{6}k^2T^2 - \cdots\right) \Rightarrow \frac{g}{g+kV} = 1 - \frac{1}{2}kT + \frac{1}{6}k^2T^2$$

$$\frac{2g}{g+kV} = 2 - kT + \frac{1}{3}k^2T^2 \Rightarrow kT = \frac{2kV}{g+kV} + \frac{1}{3}k^2T^2$$

$$T = \frac{2V}{g+kV} + \frac{1}{3}kT^2 \Rightarrow T = \frac{2V/g}{1+\frac{kV}{g}} + \frac{1}{3}kT^2$$

$$\frac{1}{1+\frac{kV}{g}} = 1 - \left(\frac{kV}{g}\right) + \left(\frac{kV}{g}\right)^2 - \cdots$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots$$

…..Taylor series

Where, we keep only terms through k^2

H.W

$$T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2}\right) k + O(k^2)$$

If we choose to neglect $O(k^2)$ and the terms of order higher. In this limit $k \to 0$ (no air resistance)

$$T(k=0) = T_0 = \frac{2V}{g} = \frac{2v_0 \sin\theta}{g}$$

If k is small (but nonvanishing), the flight time will approximately equal to T_0. If we then use this approximate value for $T = T_0$ in the right-hand side of this equation

$$\frac{2V}{g}$$

$$T = \frac{2V}{g} + \left(\frac{T^2}{3} - \frac{2V^2}{g^2}\right)k + O(k^2)$$

$$T \cong \frac{2V}{g}\left(1 - \frac{kV}{3g}\right)$$

This is the desired approximate expression for the flight time.

Next, we write the equation for x in the expanded form

$$x = \frac{U}{k}(1 - e^{-kt})$$

$$x = \frac{U}{k}\left(kt - \frac{1}{2}k^2t^2 + \frac{1}{6}k^3t^3 - \right)$$

Because $x(t = T) = R'$, we have approximately for range

$$R' \cong U(T - \frac{1}{2}kT^2)$$

(keep terms only thought the first order of k)

Evaluate this expression by using the value of T from equation

$$T \cong \frac{2V}{g}\left(1 - \frac{kV}{3g}\right)$$

$$R' \cong \frac{2UV}{g}(1 - \frac{4kV}{3g})$$

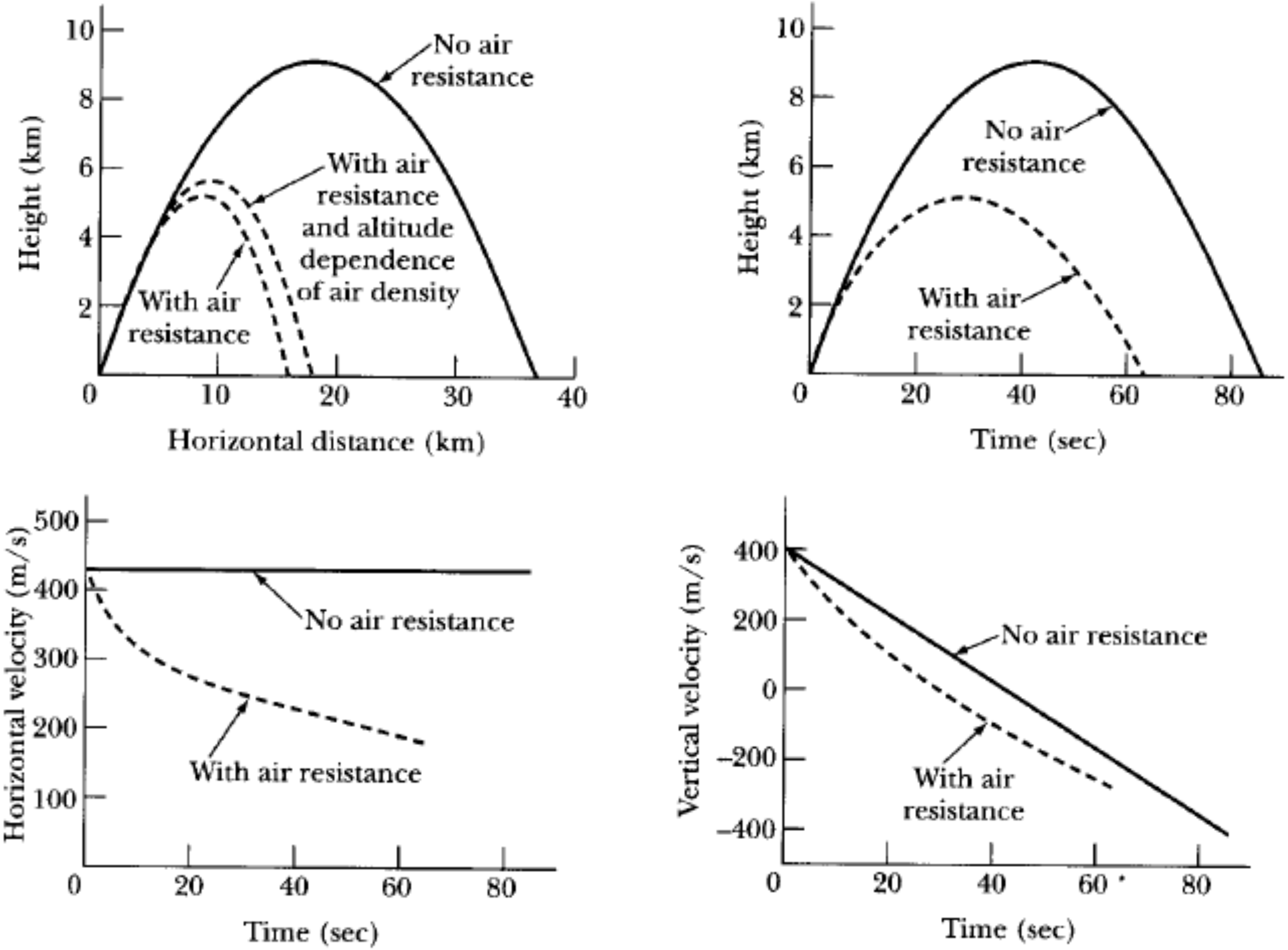
$$R' \cong \frac{2UV}{g}(1 - \frac{4kV}{3g})$$

The quantity $\frac{2UV}{g}$ can be written as

$$\frac{2UV}{g} = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin 2\theta = R$$

Which will be recognized as the range R of the projectile when air resistance is neglected.

$$R' \cong R(1 - \frac{4kV}{3g})$$


No air resistance
With air resistance and altitude dependence of air density
With air resistance
Height (km)
Horizontal distance (km)
Time (sec)
Horizontal velocity (m/s)
Vertical velocity (m/s)

HWs

1 A student drops a water-filled balloon from the roof of the tallest building in town trying to hit her roommate on the ground (who is too quick). The first student ducks back but hears the water splash 4.021 s after dropping the balloon. If the speed of sound is 331 m/s, find the height of the building, neglecting air resistance.

2 A particle is released from rest ($y = 0$) and falls under the influence of gravity and air resistance. Find the relationship between v and the distance of falling y when the air resistance is equal to **(a)** αv and **(b)** βv^2.

3 A gun fires a projectile of mass 10 kg. The muzzle velocity is 140 m/s. Through what angle must the barrel be elevated to hit a target on the same horizontal plane as the gun and 1000 m away? Compare the results with those for the case of no retardation.

4 Consider a projectile fired vertically in a constant gravitational field. For the same initial velocities, compare the times required for the projectile to reach its maximum height **(a)** for zero resisting force, **(b)** for a resisting force proportional to the instantaneous velocity of the projectile.

HWs

5

Consider a particle of mass m whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity (i.e., kmv^2) is encountered, show that the distance s the particle falls in accelerating from v_0 to v_1 is given by

$$s(v_0 \rightarrow v_1) = \frac{1}{2k} \ln \left[\frac{g - kv_0^2}{g - kv_1^2} \right]$$

6

A particle is projected vertically upward in a constant gravitational field with an initial speed v_0. Show that if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it returns to the initial position is

$$\frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}$$

where v_t is the terminal speed.

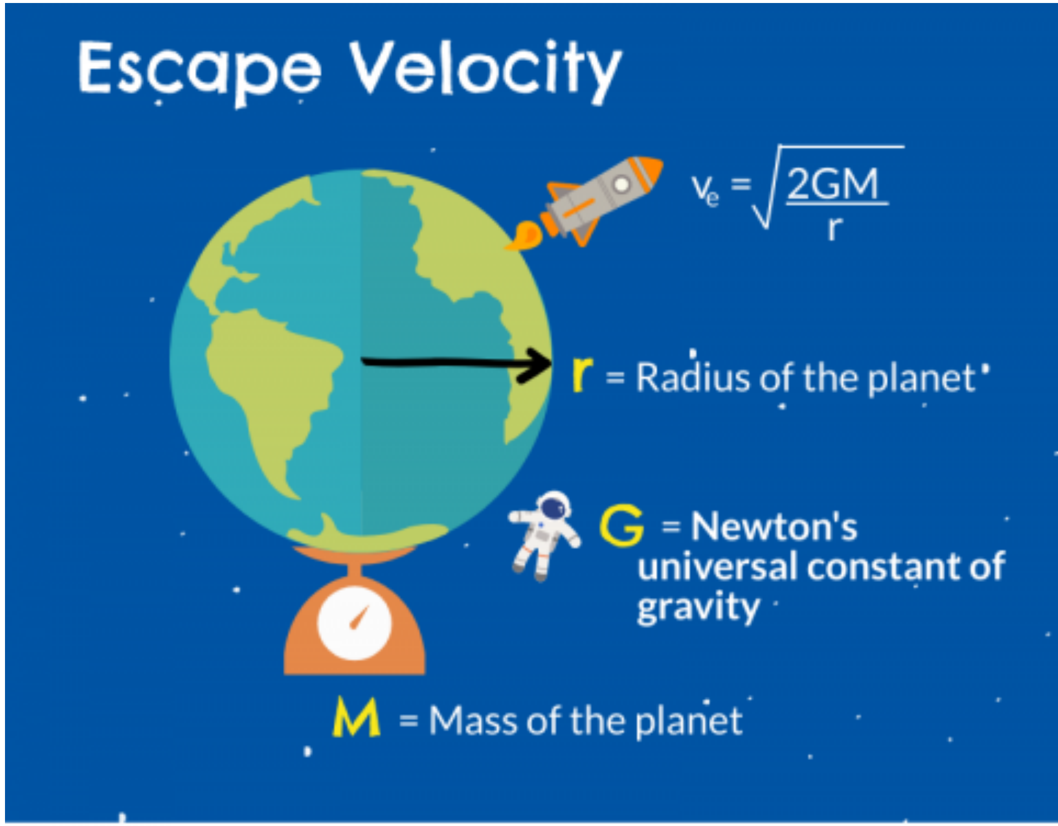
Escape velocity

The escape velocity of a planet is the minimum speed at which an object can be lunched such that it would take an infinite amount of time to slow the object to stop.

Consider **No force applied** and **no air resistance**

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

H.W



Example 8

Determine the escape velocity of the Earth. Assume no air resistance and no planet rotation. Known $M_{Earth} = 5.97\times10^{24}\ kg$; $R_{Earth} = 6.37\times10^{6}\ m$.

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

$$v_{esc} = \sqrt{\frac{2(6.67\times10^{-11})(5.97\times10^{24})}{6.37\times10^{6}}}$$

$$v_{esc} = 11181.38\ \frac{m}{s}$$

$$v_{esc} \approx 11.2\ \frac{km}{s}$$