1.7: Derivative of a Vector:

Consider a vector A, whose components are functions of single variable u. The paramter u is usually time t. The vector may represent position, velocity and so on.

$$\therefore A(u) = i\,A_x(u) + j\,A_y(u) + k\,A_z(u) \quad \text{---- (1.43)}$$

so the drivative of A can expressed as following:

$$\frac{dA}{du} = \lim_{\Delta u \to 0} \frac{\Delta A}{\Delta u} = \lim_{\Delta u \to 0} \left(i\,\frac{\Delta A_x}{\Delta u} + j\,\frac{\Delta A_y}{\Delta u} + k\,\frac{\Delta A_z}{\Delta u} \right)$$

$$\therefore \frac{dA}{du} = i\,\frac{dA_x}{du} + j\,\frac{dA_y}{du} + k\,\frac{dA_z}{du} \quad \text{---- (1.44)}$$

This mean, the deriviative of avetor is avector whose Cartesian components are ordinary drivatives.

Now, below are the rules of vector differentional.

$$\frac{d}{du}(A+B) = \frac{dA}{du} + \frac{dB}{du}$$

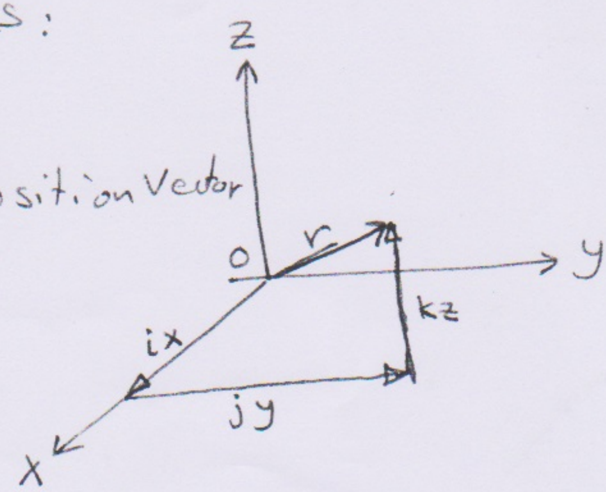
$$\frac{d(nA)}{du} = \frac{dn}{du}A + n\frac{dA}{du}$$

$$\frac{d(A \cdot B)}{du} = \frac{dA}{du} \cdot B + A \cdot \frac{dB}{du}$$

$$\frac{d(A \times B)}{du} = \frac{dA}{du} \times B + A \times \frac{dB}{du}$$

1.8 Position Vector of Particle : Velocity and Acceleration in Rectangular Coordinates :

$r = ix + jy + kz$ ---- The position Vector



as $x = x(t)$, $y = y(t)$, $z = z(t)$ are the components of moving particle. (of the position vector)

So the velocity vector can be written as following:

$$v = \frac{dr}{dt} = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt} \quad \text{---- } (1.45)$$

or

$$v = \dot{r} = i\dot{x} + j\dot{y} + k\dot{z}$$

The velocity value is called the speed ?! and is difiend (1.46)

$$v = |v| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad \text{----}$$

in rectangular components and the second drivite of the volocity is called the acceleration

$$a = \dot{v} = \frac{d^2 r}{dt^2} = i\ddot{x} + j\ddot{y} + k\ddot{z} \quad \text{---- } (1.47)$$

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If we denote the cumulative scalar distance along the path with s, then we can express the speed alternatively as

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}}{\Delta t} \tag{1.10.6}$$

which reduces to the expression on the right of Equation 1.10.5.

The time derivative of the velocity is called the *acceleration*. Denoting the acceleration with **a**, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{1.10.7}$$

In rectangular components,

$$\mathbf{a} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z} \tag{1.10.8}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

EXAMPLE 1.10.1

Projectile Motion

Let us examine the motion represented by the equation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity **v** is obtained by differentiating with respect to t, namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The acceleration, likewise, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, **a** is in the negative y direction and has the constant magnitude g. The path of motion is a parabola, as shown in Figure 1.10.3. The speed v varies with t according to the equation

$$v = [b^2 + (c - gt)^2]^{1/2}$$

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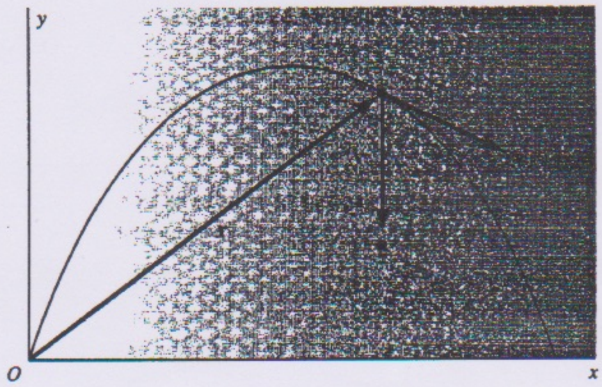

Figure 1.10.3 Position, velocity, and acceleration vectors of a particle (projectile) moving in a parabolic path.

EXAMPLE 1.10.2

Circular Motion

Suppose the position vector of a particle is given by

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating $\mathbf{r}$, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin \omega t - \mathbf{j}b\omega^2 \cos \omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of $\mathbf{v}$ and $\mathbf{a}$ vanishes:

$$\mathbf{v} \cdot \mathbf{a} = (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) + (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) = 0$$

Comparing the two expressions for $\mathbf{a}$ and $\mathbf{r}$, we see that we can write

$$\mathbf{a} = -\omega^2 \mathbf{r}$$

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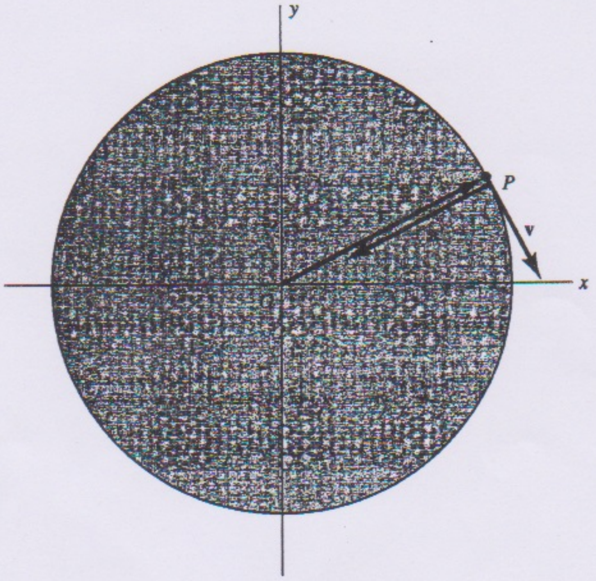

Figure 1.10.4 A particle moving in a circular path with constant speed.

so **a** and **r** are oppositely directed; that is, **a** always points toward the center of the circular path (Fig. 1.10.4).

EXAMPLE 1.10.3

Rolling Wheel

Let us consider the following position vector of a particle P:

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

in which

$$\mathbf{r}_1 = \mathbf{i}b\omega t + \mathbf{j}b$$
$$\mathbf{r}_2 = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

Now $\mathbf{r}_1$ by itself represents a point moving along the line $y = b$ at constant velocity, provided ω is constant; namely,

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} = \mathbf{i}b\omega$$

The second part, $\mathbf{r}_2$, is just the position vector for circular motion, as discussed in Example 1.10.2. Hence, the vector sum $\mathbf{r}_1 + \mathbf{r}_2$ represents a point that describes a circle of radius b about a moving center. This is precisely what occurs for a particle on the rim of a rolling wheel, $\mathbf{r}_1$ being the position vector of the center of the wheel and $\mathbf{r}_2$ being

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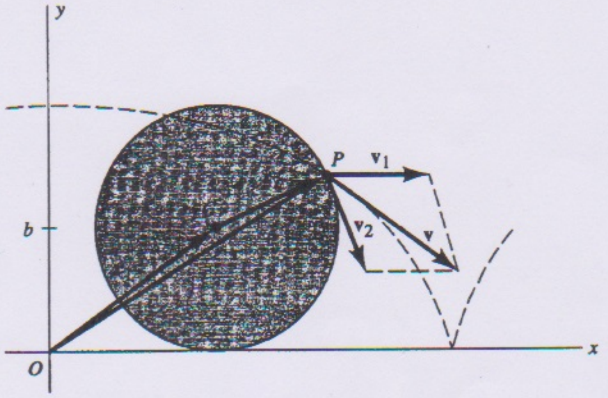


Figure 1.10.5 The cycloidal path of a particle on a rolling wheel.

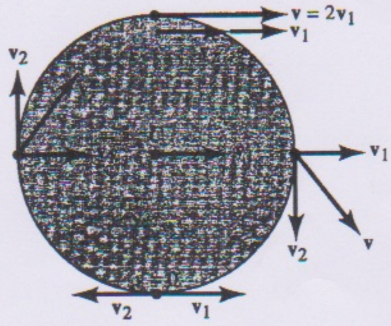


Figure 1.10.6 Velocity vectors for various points on a rolling wheel.

the position vector of the particle P *relative* to the moving center. The actual path is a *cycloid*, as shown in Figure 1.10.5. The velocity of P is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{i}(b\omega + b\omega \cos \omega t) - \mathbf{j} b\omega \sin \omega t$$

In particular, for $\omega t = 0, 2\pi, 4\pi, \ldots$, we find that $\mathbf{v} = \mathbf{i}2b\omega$, which is just twice the velocity of the center C. At these points the particle is at the uppermost part of its path. Furthermore, for $\omega t = \pi, 3\pi, 5\pi, \ldots$, we obtain $\mathbf{v} = 0$. At these points the particle is at its lowest point and is instantaneously in contact with the ground. See Figure 1.10.6.

1.9 ~~1.11~~ Velocity and Acceleration in Plane Polar Coordinates

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector $\mathbf{e}_r$:

$$\mathbf{r} = r\mathbf{e}_r \qquad (1.11.1)$$

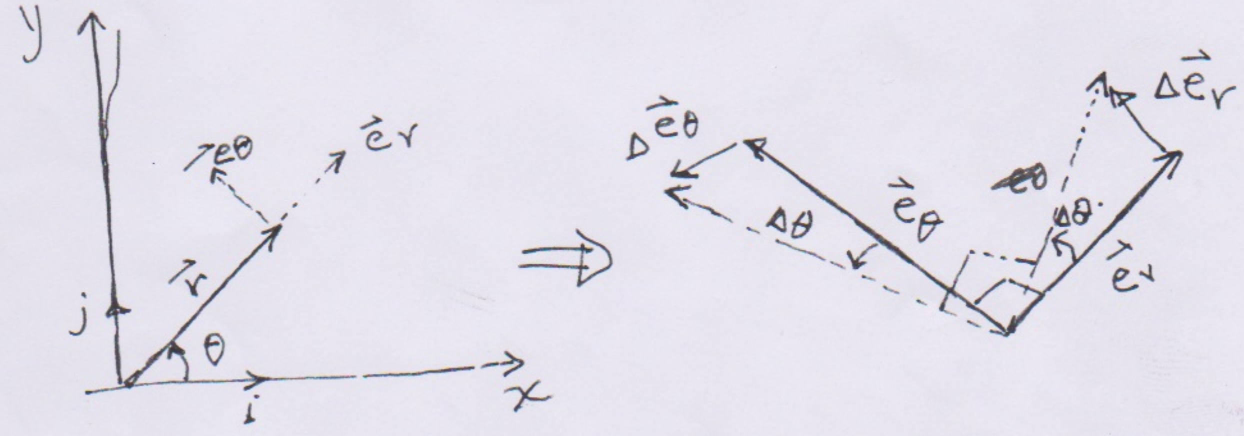
We have

$$r = r\vec{e}_r \quad \text{------} \quad (1.48)$$

$\Rightarrow$ is the position vector in the Polar Coordinates where $\underline{\underline{\vec{e}_r}}$ is the unit radial vector.

Now, if we differentiate with respect to t, we have

$$\vec{v} = \frac{dr}{dt} = \dot{r}\,\vec{e}_r + r\frac{d\vec{e}_r}{dt} \quad \text{----} \quad (1.49)$$



To calculate $\frac{de_r}{dt}$, let us consider the vector digram of r changes shown above. Now, when the direction changes by $\Delta\theta$, then the change in $\vec{e}_r$ can be explained as following:

and introduce $e_\theta \perp e_r$

$$|\Delta e_r| \cong \Delta\theta$$

بتساوي

$$\therefore \boxed{\Delta e_r = e_\theta \Delta\theta}$$

ضرب $\Delta\theta$

If we divide by Δt and take the limit, we get

$$\frac{de_r}{dt} = e_\theta \frac{d\theta}{dt} \quad \text{----} \quad (1.50)$$

By using a similar way, we get (27)

$\Delta e_\theta = - e_r \Delta \theta$; a minus sign indicates to the opposite direction of e_θ in respet to e_r.

$$\therefore \frac{d e_\theta}{dt} = - e_r \frac{d\theta}{dt} \quad \text{-----} (1.51)$$

Thus, the equation (1.49) can be written

$$\boxed{\vec{V} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta} \quad \text{----} (1.52)$$

as $\vec{V}_r = \dot{r}\,\vec{e}_r$; $\vec{V}_\theta = r\dot{\theta}\,\vec{e}_\theta$ are the velocity componentes in Polar direction (r, θ).

Additionally, to find the acceleration vector, we will take the volocity ~~drivte~~ derivative with respect to time.

$$\vec{a} = \frac{d\vec{V}}{dt} = \ddot{r}\,\vec{e}_r + \dot{r}\,\frac{d\vec{e}_r}{dt} + (r\ddot{\theta} + \dot{r}\dot{\theta})\vec{e}_\theta + (r\dot{\theta})\frac{d\vec{e}_\theta}{dt}$$

$$= \ddot{r}\,\vec{e}_r + \dot{r}\dot{\theta}\,\vec{e}_\theta + r\ddot{\theta}\,\vec{e}_\theta + \dot{r}\dot{\theta}\,\vec{e}_\theta - r\dot{\theta}^2\,\vec{e}_r$$

$$\therefore \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{e}_\theta}$$

and $\vec{a}_r = (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r$

$\vec{a}_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\vec{e}_\theta$

EXAMPLE 1.11.1 مهم

A honeybee hones in on its hive in a spiral path in such a way that the radial distance decreases at a constant rate, $r = b - ct$, while the angular speed increases at a constant rate, $\dot{\theta} = kt$. Find the speed as a function of time.

Solution:

We have $\dot{r} = -c$ and $\ddot{r} = 0$. Thus, from Equation 1.11.7,

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

so

$$v = [c^2 + (b - ct)^2k^2t^2]^{1/2}$$

which is valid for $t \le b/c$. Note that $v = c$ both for $t = 0$, $r = b$ and for $t = b/c$, $r = 0$.

EXAMPLE 1.11.2

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratically with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

We have $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$. Substituting into Equation 1.11.9, we find

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2\omega^2) + \mathbf{e}_\theta[0 + 2(2bt)\omega] \\ &= b(2 - t^2\omega^2)\mathbf{e}_r + 4b\omega t\mathbf{e}_\theta\end{aligned}$$

Note that the radial component of the acceleration becomes negative for large t in this example, although the radius is always increasing monotonically with time.

1.10 $\vec{v}$ and $\vec{a}$ in Cylindrical Coordinates:

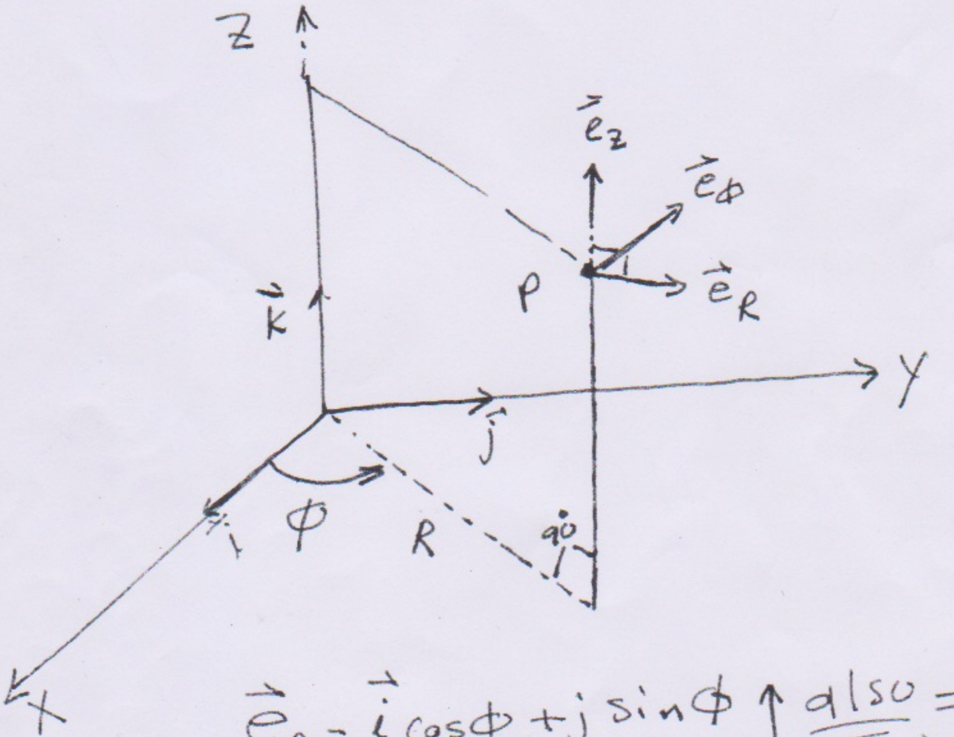
The Position vector is given by

$$r = R\,e_R + z\,e_z \quad \text{......} \quad (1.52)$$

where e_R: is a unit radial vector in xy plane

e_z: is " in the z-direction.

Now, we need a third unit vector e_ϕ is needed so that, the three unit vectors $e_R\,e_\phi\,e_z$ constitute a right handed tried, please see the Figure below:



$$\vec{e}_R = \vec{i}\cos\phi + j\sin\phi$$

$$\vec{e}_\phi = -i\sin\phi + j\cos\phi$$

$$\vec{e}_z = \vec{k}$$

also ⇒

$$\frac{d\vec{e}_R}{dt} = \vec{e}_\theta\,\dot{\phi}$$

$$\frac{d\vec{e}_\phi}{dt} = -\vec{e}\,\dot{\phi}$$

$$\therefore \vec{V} = \frac{d\vec{r}}{dt} = \dot{R}\,e_R + R\frac{d\vec{e}_R}{dt} + \dot{z}\,\vec{e}_z$$

$$\boxed{\vec{V} = \dot{R}\,e_R + R\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\vec{e}_z} \quad \text{.....} \quad (1.53)$$

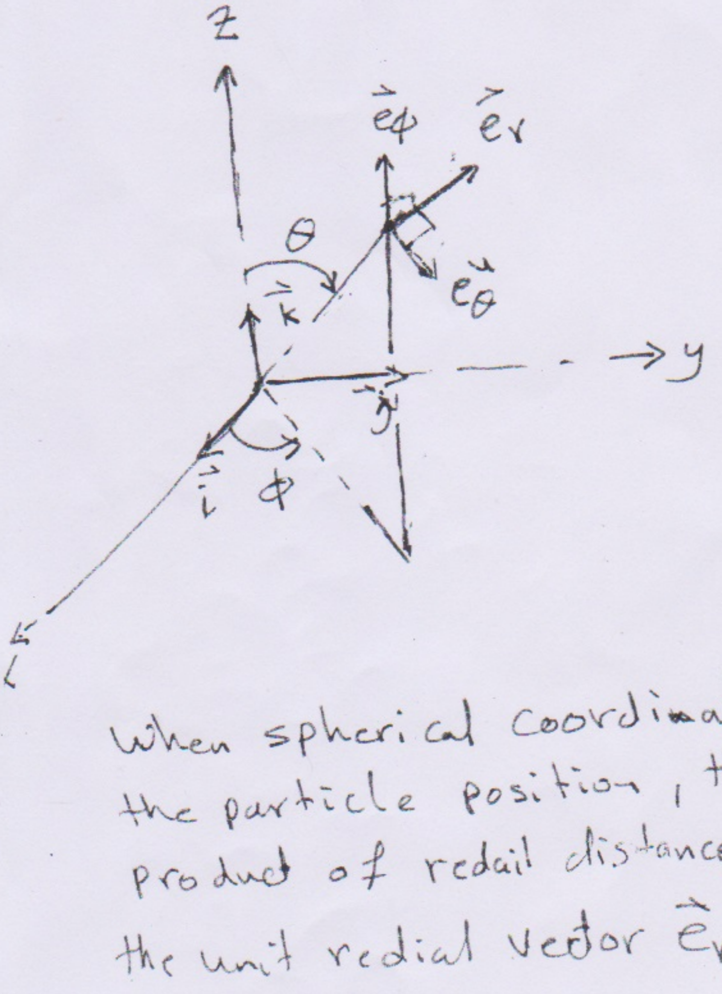
and $\vec{a} = \frac{d\vec{v}}{dt} = \ddot{R}\,\vec{e}_R + \dot{R}\,\frac{d\vec{e}_R}{dt} + (\dot{R}\dot{\Phi} + R\ddot{\Phi})\,\vec{e}_\Phi$

$+ R\dot{\Phi}\,\frac{d\vec{e}_\Phi}{dt} + \ddot{z}\,\vec{e}_z$

$\therefore \quad = \ddot{R}\,\vec{e}_R + \dot{R}\dot{\Phi}\,\vec{e}_\Phi + R\ddot{\Phi}\,\vec{e}_\Phi - R\dot{\Phi}^2\,\vec{e}_R + \ddot{z}\,\vec{e}$

$$\boxed{\vec{a} = (\ddot{R} - R\dot{\Phi}^2)\,\vec{e}_R + (2\dot{R}\dot{\Phi} + R\ddot{\Phi})\,\vec{e}_\Phi + \ddot{z}\,\vec{e}_z} \quad \text{---}(1.54)$$

1.10 $\vec{v}$ and $\vec{a}$ in spherical coordinate:



$\Rightarrow$

$\vec{e}_r = \vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta$

$\vec{e}_\theta = \vec{i}\cos\theta\cos\phi + \vec{j}\cos\theta\sin\phi - \vec{k}\sin\theta$

$e_\phi = -\vec{i}\sin\phi + \vec{j}\cos\phi$

(*)

When spherical coordinates r, θ, ϕ are used to discribe the particle position, the position vector is written as product of redail distance r and the unit vector e_r, and the unit redial vector $\vec{e}_r$ as with polar coordinates:

$$r = r\,\vec{e}_r$$

From eq. (*) we can

$\frac{d\vec{e}_r}{dt} = \vec{i}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)$
$+ \vec{j}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \vec{k}\,\dot{\theta}\sin\theta$

$$= \dot{\theta}\,(\underbrace{\vec{i}\cos\theta\cos\phi + \vec{j}\cos\theta\sin\phi - \vec{k}\sin\theta}_{\vec{e}_\theta})$$
$$+ \dot{\phi}\sin\theta\,(\underbrace{-\vec{i}\sin\phi + \vec{j}\cos\phi}_{\vec{e}_\phi})$$

$$\boxed{\frac{d\vec{e}_r}{dt} = \dot{\theta}\,\vec{e}_\theta + \dot{\phi}\sin\theta\,\vec{e}_\phi}$$

By using a similar procedure, we can write

$$\frac{d\vec{e}_\theta}{dt} = -\vec{e}_r\,\dot{\theta} + \vec{e}_\phi\,\dot{\phi}\cos\theta$$

$$\frac{de_\phi}{dt} = -\vec{e}_r\,\dot{\phi}\sin\theta - \vec{e}_\phi\,\dot{\phi}\cos\theta$$

$\longrightarrow$ H.W

To get the velocity vector, we need to drive the position vector in this system:

$$\therefore\ \vec{V} = \frac{d\vec{r}}{dt} = \dot{r}\,\vec{e}_r + r\,\frac{d\vec{e}_r}{dt}$$

$$\boxed{\vec{V} = \dot{r}\,\vec{e}_r + r\,\dot{\phi}\sin\theta\,\vec{e}_\phi + r\,\dot{\theta}\,\vec{e}_\theta} \quad \cdots (1.55)$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \vec{e}_r\,\ddot{r} + \dot{r}\,\frac{d\vec{e}_r}{dt} + \vec{e}_\phi\,\frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\,\frac{d\vec{e}_\phi}{dt}$$
$$+ \vec{e}_\theta\,\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\,\frac{d\vec{e}_\theta}{dt}$$

$$= \ddot{r}\,\vec{e}_r + \dot{r}\dot{\phi}\sin\theta\,\vec{e}_\phi + \dot{r}\dot{\theta}\,\vec{e}_\theta + (\dot{r}\dot{\phi}\sin\theta$$
$$+ r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta)\,\vec{e}_\phi - r\dot{\phi}^2\sin^2\theta\,\vec{e}_r$$
$$- r\dot{\phi}^2\sin\theta\cos\theta\,\vec{e}_\theta + (\dot{r}\dot{\theta} + r\ddot{\theta})\,\vec{e}_\theta$$
$$+ r\dot{\theta}\dot{\phi}\cos\theta\,\vec{e}_\phi - r\dot{\theta}^2\,\vec{e}_r$$

(32)

$$\boxed{\begin{aligned}\vec{a} = &(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\,\vec{e_r} \\ &+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\,\vec{e_\phi} \\ &+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\vec{e_\theta}\end{aligned}}$$

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Thus, in this case both velocity and acceleration are constant in magnitude, but they vary in direction because both $\mathbf{e}_\phi$ and $\mathbf{e}_r$ change with time as the bead moves.

EXAMPLE 1.12.2

A wheel of radius b is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed ω_1 about its own axis, which in turn rotates with constant angular speed ω_2 about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the rim of the wheel. In particular, find the acceleration of the highest point on the wheel.

Solution:

We can use the fact that spherical coordinates can be chosen such that $r = b$, $\theta = \omega_1 t$, and $\phi = \omega_2 t$ (Fig. 1.12.3). Then we have $\dot{r} = \quad = 0$, $\dot{\theta} = \omega_1$, $\ddot{\theta} = 0$, $\dot{\phi} = \omega_2$, $\ddot{\phi} = 0$. Equation 1.12.14 gives directly

$$\mathbf{a} = (-b\omega_2^2 \sin^2\theta - b\omega_1^2)\mathbf{e}_r - b\omega_2^2 \sin\theta\cos\theta\,\mathbf{e}_\theta + 2b\omega_1\omega_2 \cos\theta\,\mathbf{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$\mathbf{a} = -b\omega_1^2\mathbf{e}_r + 2b\omega_1\omega_2\mathbf{e}_\phi$$

The first term on the right is the centripetal acceleration, and the last term is a transverse acceleration normal to the plane of the wheel.

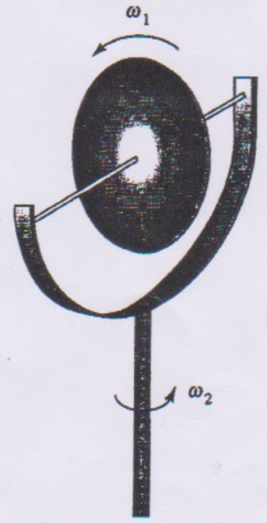

Figure 1.12.3 A rotating wheel on a rotating mount.

(34)

Problems

1.1 Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:
(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$
(b) $3\mathbf{A} - 2\mathbf{B}$
(c) $\mathbf{A} \cdot \mathbf{B}$
(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

1.2 Given the three vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = 4\mathbf{j}$, find the following:
(a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$
(b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

1.3 Find the angle between the vectors $\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$. (*Note:* These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, $2a$, and $3a$.)

1.4 Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends
(a) along a major diagonal of the cube;
(b) along the diagonal of the lower face of the cube.
(c) Calling these vectors $\mathbf{A}$ and $\mathbf{B}$, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.
(d) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

1.5 Assume that two vectors $\mathbf{A}$ and $\mathbf{B}$ are known. Let $\mathbf{C}$ be an unknown vector such that $\mathbf{A} \cdot \mathbf{C} = u$ is a known quantity and $\mathbf{A} \times \mathbf{C} = \mathbf{B}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$, $\mathbf{B}$, u, and the magnitude of $\mathbf{A}$.

1.6 Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

1.7 For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1.8 Give an algebraic proof and a geometric proof of the following relations:

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$$
$$|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}|\,|\mathbf{B}|$$

1.9 Prove the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

1.10 Two vectors $\mathbf{A}$ and $\mathbf{B}$ represent concurrent sides of a parallelogram. Show that the area of the parallelogram is equal to $|\mathbf{A} \times \mathbf{B}|$.

1.11 Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is not equal to $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.

1.12 Three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ represent three concurrent edges of a parallelepiped. Show that the volume of the parallelepiped is equal to $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

1.13 Verify the transformation matrix for a rotation about the z-axis through an angle ϕ followed by a rotation about the y'-axis through an angle θ, as given in Example 1.8.2.

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1.14 Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

1.15 Consider two Cartesian coordinate systems xyz and $x'\,y'\,z'$ that initially coincide. The $x'\,y'\,z'$ undergoes three successive counterclockwise 45° rotations about the following axes: first, about the fixed z-axis; second, about its own x'-axis (which has now been rotated); finally, about its own z'-axis (which has also been rotated). Find the components of a unit vector $\mathbf{X}$ in the xyz coordinate system that points along the direction of the x'-axis in the rotated $x'\,y'\,z'$ system. (*Hint: It would be useful to find three transformation matrices that depict each of the above rotations. The resulting transformation matrix is simply their product.*)

1.16 A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$, where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time $t = \sqrt{b/c}$. (*Hint: At this time the tangential and normal components of the acceleration are equal in magnitude.*)

1.17 A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\cos\omega t + \mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

1.18 A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\sin\omega t + \mathbf{j}b\cos\omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

1.19 A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v}\cdot\mathbf{a}/va$.)

1.20 Work Problem 1.18 using cylindrical coordinates where $R = b$, $\phi = \omega t$, and $z = ct^2$.

1.21 The position of a particle as a function of time is given by

$$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$$

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

1.22 An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?