Chapter 1

Fundamental Concepts: Vectors

1.1 Scalar and Vector Quantities

5 Km away from School


1

A Scalar Quantity

A scalar is a physical quantity that has **magnitude only (e.g. mass).**

A scalar is completely specified by a **single number** and in **appropriate units**.

A scalar is **independent of any coordinates chosen** to describe the motion of the system.

Examples

temperature.

energy

density

volume

2

A Vector Quantity

A Vector is a physical quantity that has **<u>magnitude and direction</u> (e.g. displacement).**

A Vector quantity depends on the coordinates system.

A Vector quantity represents by the value and direction.

Examples

Velocity

Acceleration

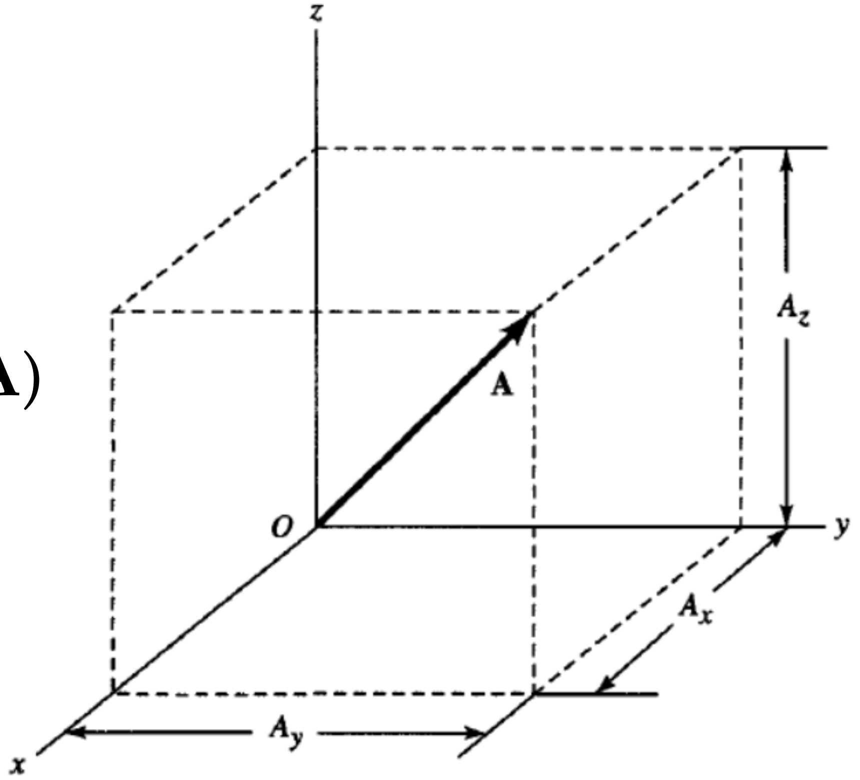
displacement

Force

1.2 Vector Algebra

An arrow, customarily designates a vector.

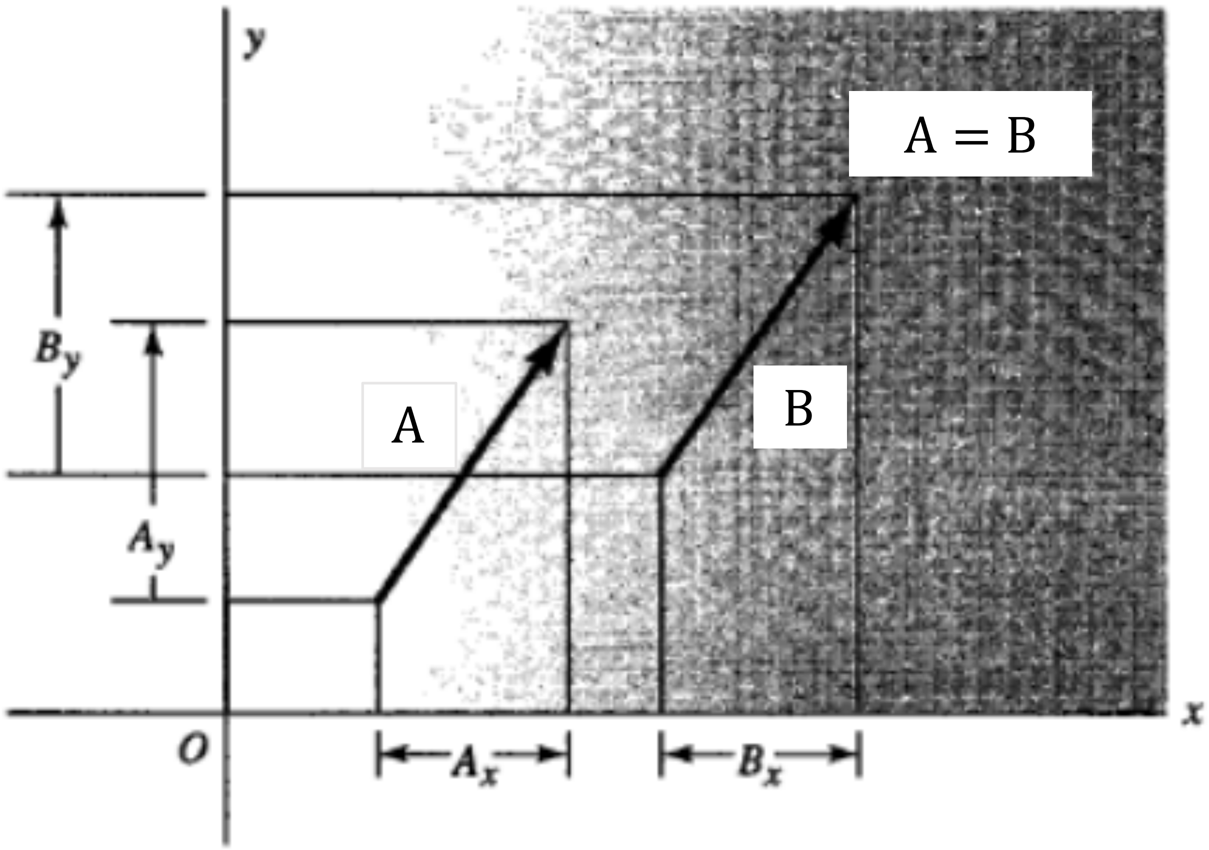
Vector quantities is simply denoted by boldface type.(e.g **A**)



$$\mathrm{A} = \mathrm{A_x} + \mathrm{A_y} + \mathrm{A_z}$$

1 Equality of Vectors

The two vectors are equal only if their respective components are equal.

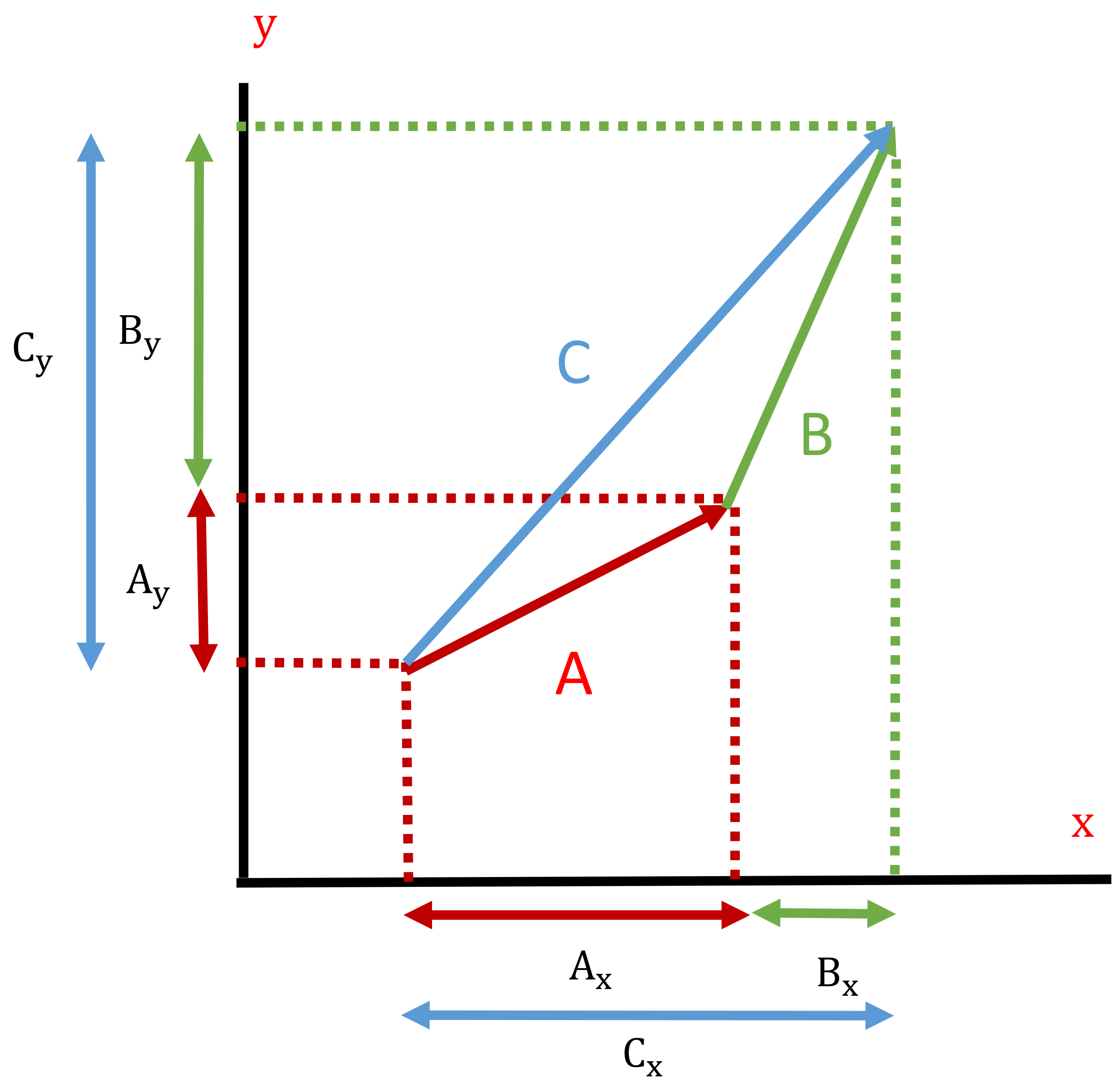


$$A = B$$

$$(A_x, A_y, A_z) = (B_x, B_y, B_z)$$

$$A_x = B_x$$

2 Vector Addition



$$C = A + B$$

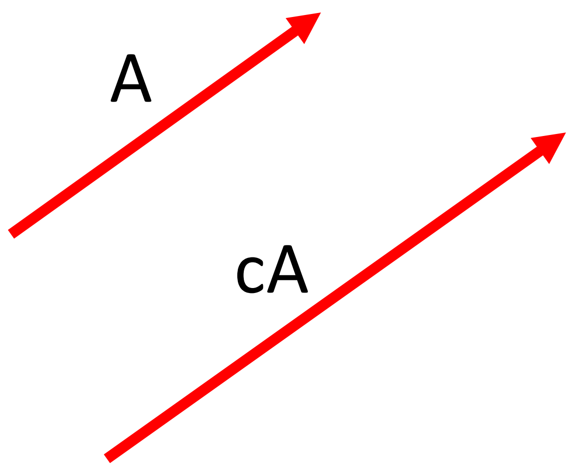
$$\underbrace{A + B}_{C} = \underbrace{(A_x + B_x)}_{C_x} + \underbrace{(A_y + B_y)}_{C_y}$$

3 Multiplication by a Scalar

If c is a scalar and A is a vector, then:

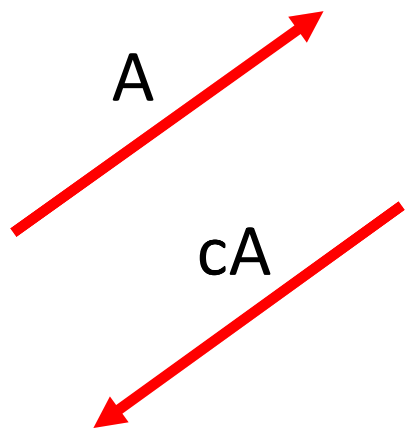
$$\mathrm{cA} = \mathrm{c}\left(\mathrm{A_x} + \mathrm{A_y} + \mathrm{A_z}\right) = \mathrm{cA_x} + \mathrm{cA_y} + \mathrm{cA_z}$$

Example 1 Let $c = 2$



- cA is parallel to A
- c times the length

Example 2 Let $c = -1$



- cA is reverse of A

4 Vector Subtraction

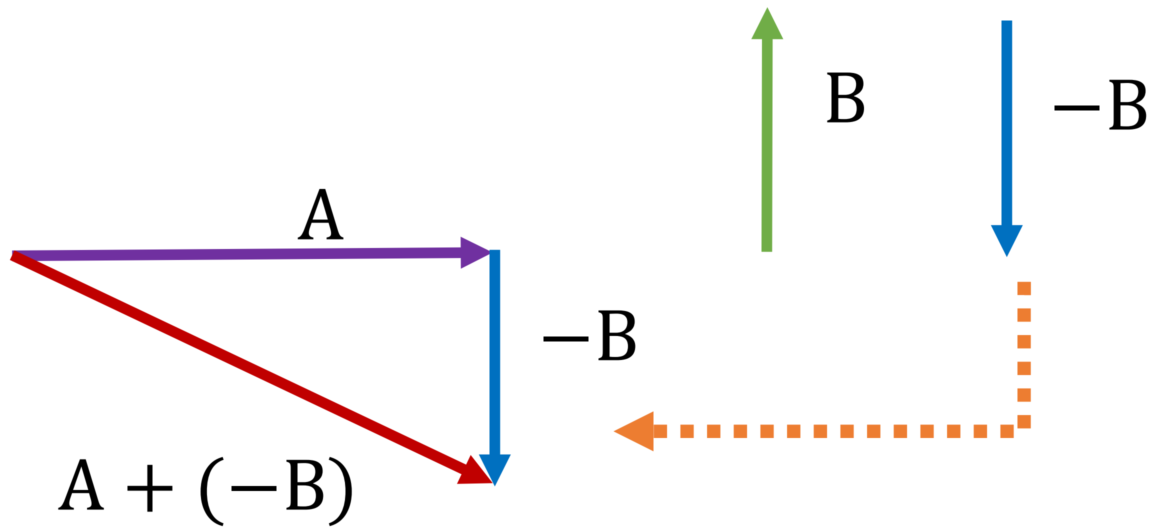
$$6 - 4 = 2$$

$$6 + (-4) = 2$$

Differences between A and B

$$A - B = A + (-1)\,B$$

$$= (A_x - B_x, \quad A_y - B_y, \quad A_z - B_z)$$



5 The Null Vector

The vector **O** = (0,0,0) is called the null vector.

The direction of **O** is undefined.

$A - A = 0$

$\mathbf{O}=0$

6 The Commutative Law of Addition

This law holds for vectors

$$A + B = B + A$$

$$A_x + B_x = B_x + A_x$$

Similarly for the y and z components.

7 The Associative Law

The associative law is also true, because

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A_x} + (\mathbf{B_x} + \mathbf{C_x}); \mathbf{A_y} + (\mathbf{B_y} + \mathbf{C_y}); (\mathbf{A_z} + (\mathbf{B_z} + \mathbf{C_z})$$

$$= (\mathbf{A_x} + \mathbf{B_x}) + \mathbf{C_x}; (\mathbf{A_y} + \mathbf{B_y}) + \mathbf{C_y}; (\mathbf{A_z} + \mathbf{B_z}) + \mathbf{C_z}$$

$$= (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

8 The Distributive Law

$$\mathbf{c}\,(\mathbf{A} + \mathbf{B}) = \mathbf{c}(\mathbf{A_x} + \mathbf{B_x};\ \mathbf{A_y} + \mathbf{B_y};\ \mathbf{A_z} + \mathbf{B_z})$$

$$= \mathbf{c}(\mathbf{A_x} + \mathbf{B_x}), \mathbf{c}(\mathbf{A_y} + \mathbf{B_y}), \mathbf{c}(\mathbf{A_z} + \mathbf{B_z})$$

$$= \mathbf{cA_x} + \mathbf{cB_x}, \mathbf{cA_y} + \mathbf{cB_y}, \mathbf{cA_z} + \mathbf{cB_z}$$

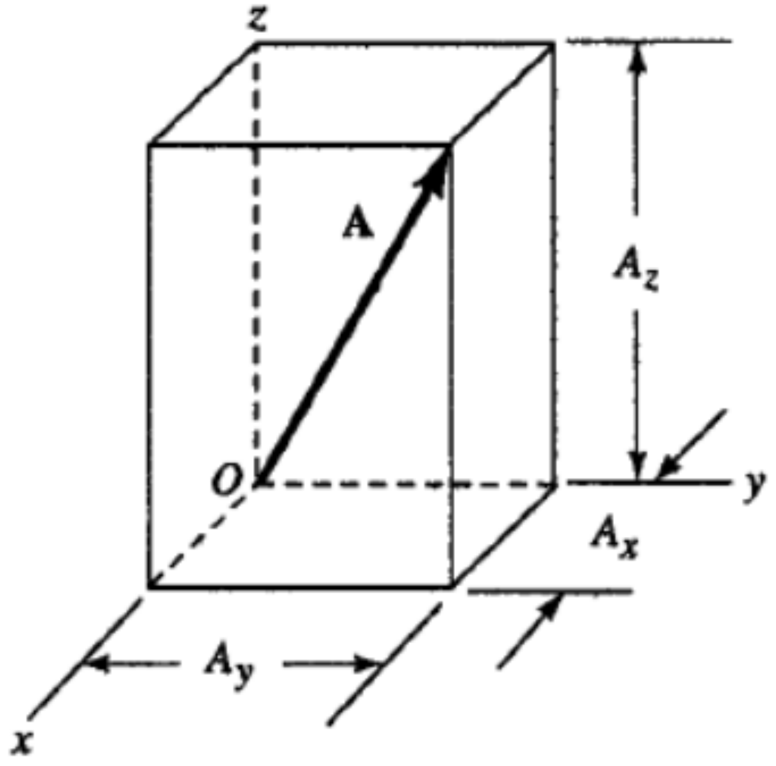
$$= \mathbf{cA} + \mathbf{cB}$$

9 Magnitude of a Vector

The magnitude of a vector A, denoted by |A| or A

The square root of the sum of the squares of the components.

$$A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



10 Unit Coordinate Vectors

A unit vector is a vector whose magnitude is unity.

Unit vectors are often designated by the symbol e.

$$e_x = (1,0,0)$$

$$e_y = (0,1,0)$$

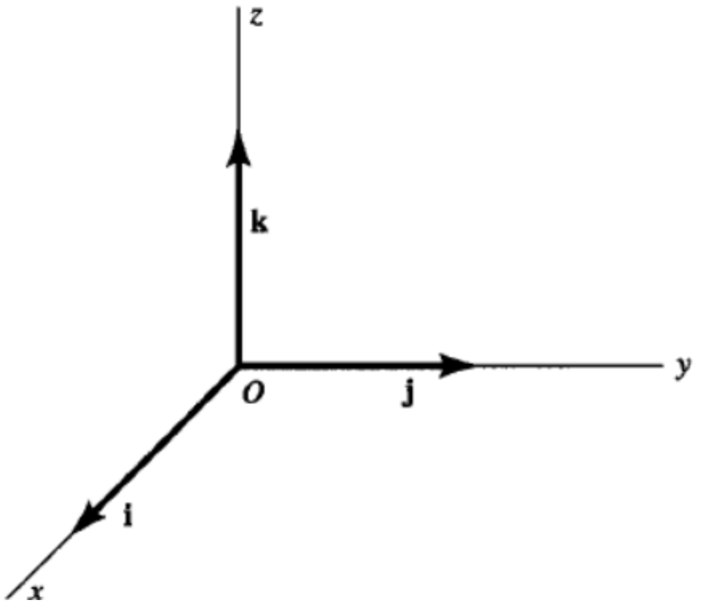
$$e_z = (0,0,1)$$

$$\mathrm{A} = \mathrm{e_x A_x} + \mathrm{e_y A_y} + \mathrm{e_z A_z}$$

$$i = e_x$$

$$\mathrm{j} = e_y$$

$$\mathrm{k} = e_z$$



Example 1.1

Find the sum and the magnitude of the two vectors A=(1,0,2) and B=(0,1,1)

$$\mathrm{A} + \mathrm{B} = (1{,}0{,}2) + (0{,}1{,}1) \quad =(1{,}1{,}3)$$

$$|\mathrm{A} + \mathrm{B}| = \sqrt{1^2 + 1^2 + 3^3} \;= \sqrt{11}$$

H. W

Express the differences in ijk and the magnitude of the new vector $A - B$ of the two vectors A=(1,0,2) and B=(0,1,1)

Example 1.2

A helicopter flies 100 m vertically upward, then 500 m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that started from the same point and flew 200 m upward, 100 m west, and 500 m north?

Choosing up, east, and north as basis directions.

The final position of the first helicopter is expressed vectorially as A = (100,500,1000).

The final position of the second helicopter is expressed vectorially as B =(200, -100, 500).

Hence, the distance between the final positions is given by the expression.

$$
\begin{aligned}
|\mathbf{A}-\mathbf{B}| &= |((100-200), (500+100), (1000-500))| \text{ m} \\
&= (100^2 + 600^2 + 500^2)^{1/2} \text{ m} \\
&= 787.4 \text{ m}
\end{aligned}
$$

1.3 Scalar Product

Given two vectors A and B, the scalar product or "dot" product, A: B, is the scalar defined by the equation

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

Scalar multiplication is commutative $A.B = B.A$

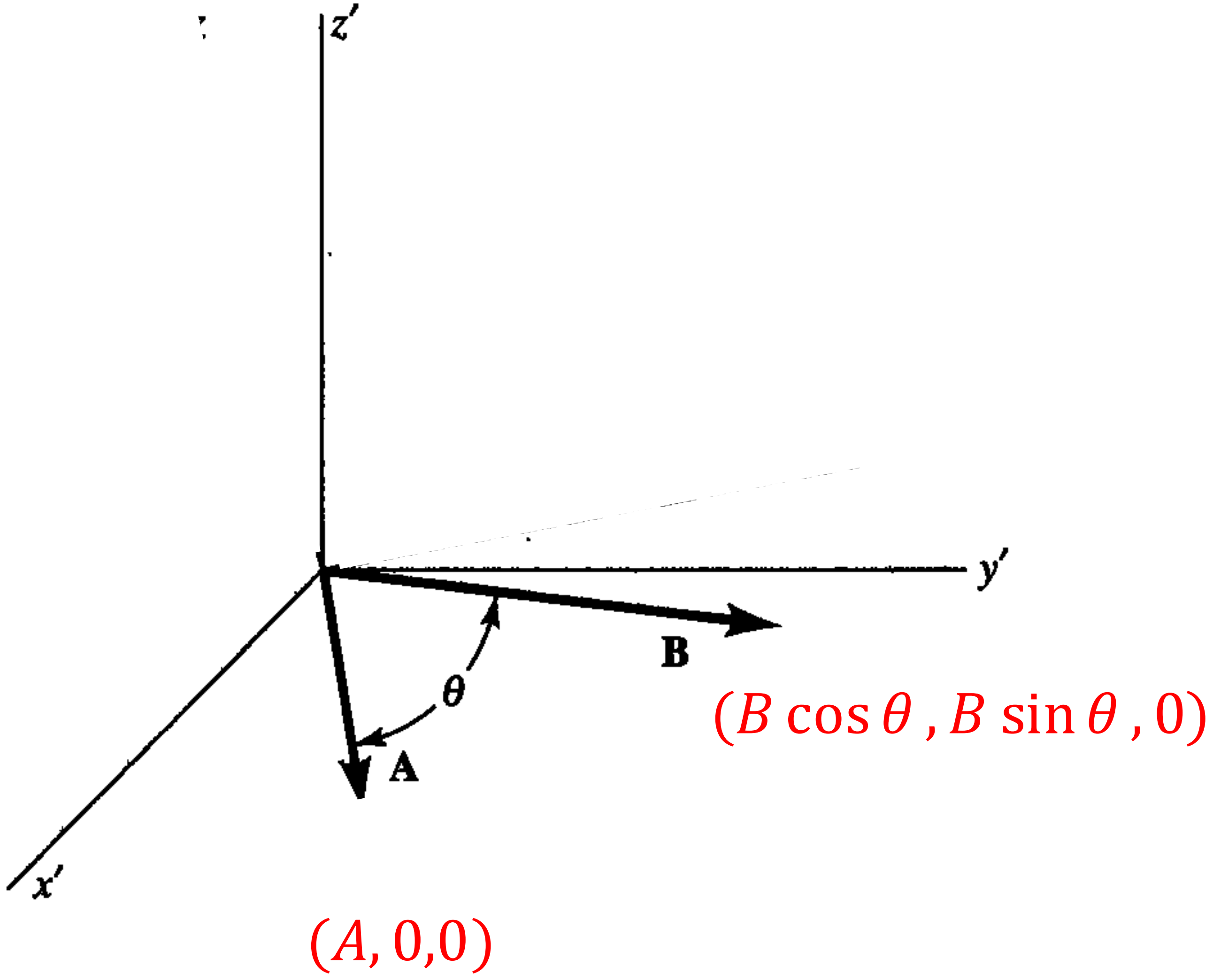
Scalar multiplication is also distributive $A.(B + C) = A.B + A.C$

$$\boxed{A.B = A_x B_x = A\,(B\cos\theta)}$$

$$\boxed{A.B = |A||B|\cos\theta}$$

$$\boxed{\cos\theta = \frac{A.B}{|A||B|}}$$

$$\boxed{\theta = cos^{-1}\frac{A.B}{|A||B|}}$$



Evaluating a dot product between two vectors.

From the definitions of the unit coordinate vectors $i, j, and\ k,$ it is clear that the following relations hold

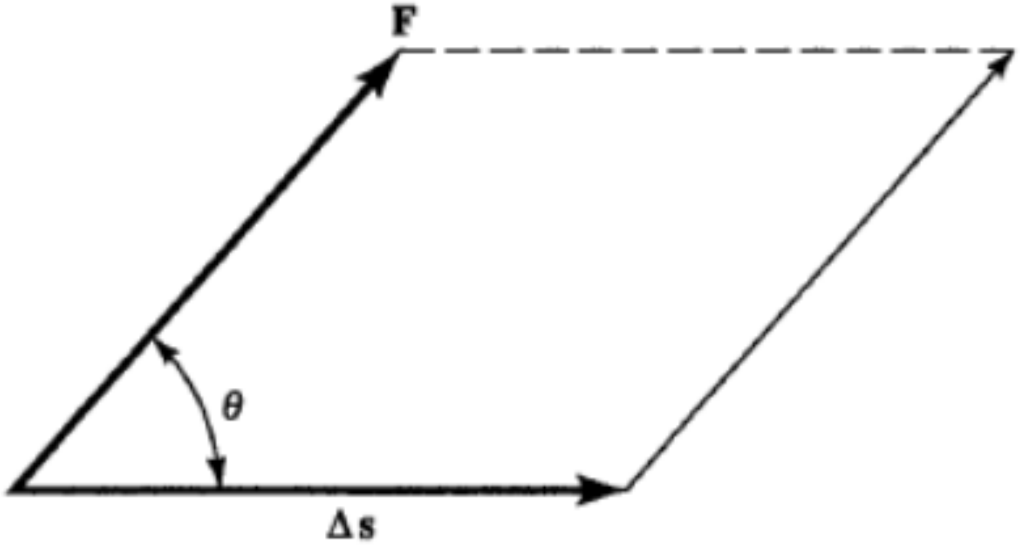
$$\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = \mathbf{1}$$
$$\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{i}.\boldsymbol{k} = \boldsymbol{j}.\boldsymbol{k} = \mathbf{0}$$

In addition , we can write any vector associated with its unit vectors by this form:

$$A = iA_x + jA_y + \mathrm{k}A_z$$

Example 1.3.1

Suppose that an object under the action of a constant force undergoes a linear displacement. By definition, the work ΔW done by the force is given by the product of the component of the force F in the direction of multiplied by the magnitude of the displacement; that is,

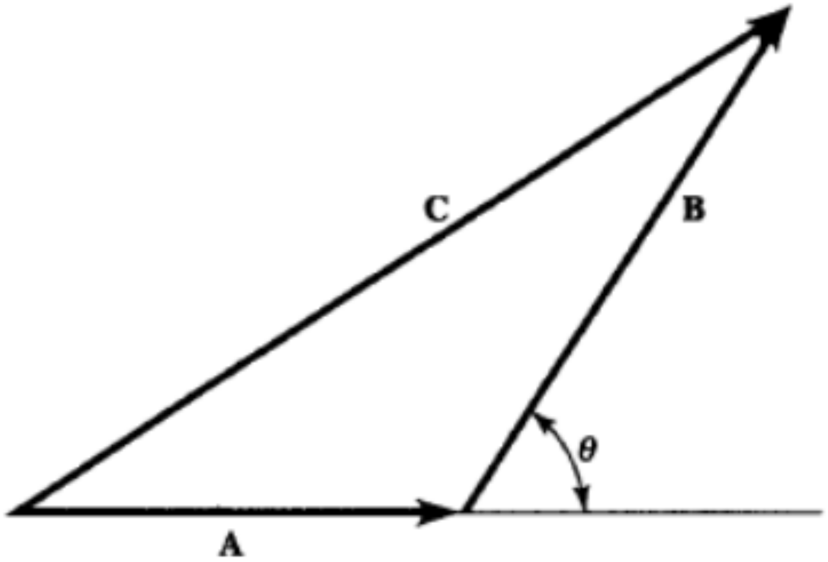


$$\Delta W = (F\cos\theta)\,\Delta s$$

$$\Delta W = F.\Delta s$$

Example 1.3.2

Consider the triangle whose sides are A, B, and C, as shown in the Figure below. Then C = A + B. Take the dot product of C with itself,



$$C.C = (A + B).(A + B)$$

$$C.C = A.A + 2A.B + B.B$$

By Replacing $A.B$ with $AB\cos\theta$ which is the familiar law of cosines.

$$C^2 = A^2 + 2AB\cos\theta + B^2$$

Example 1.3.3

Find the cosine of the angle between a long diagonal and an adjacent face diagonal of a cube.

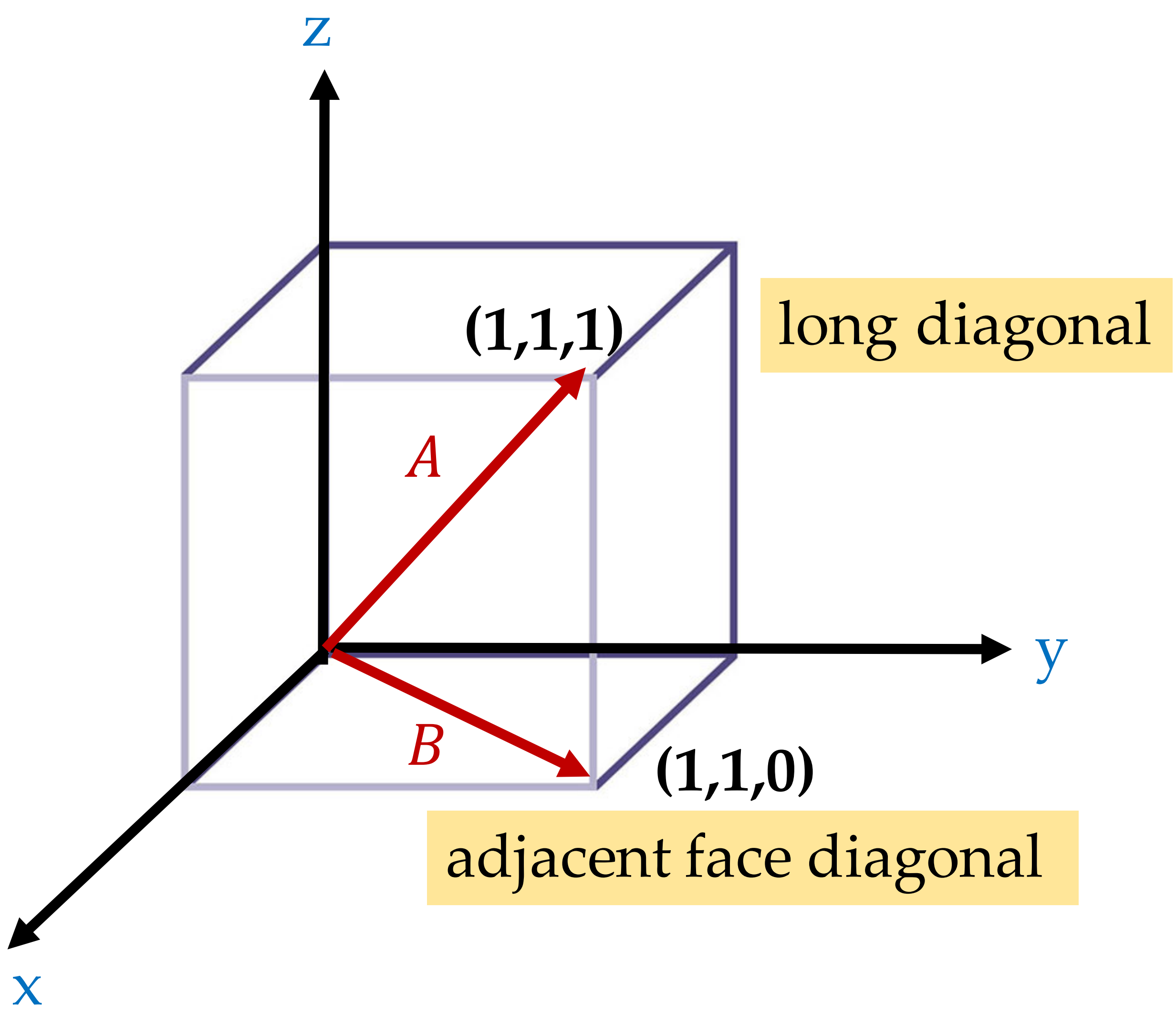
$$\cos\theta = \frac{A.B}{|A||B|}$$

$$|A| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|B| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$A.B = (i + j + k).(i + j) = 1 + 1 + 0 = 2$$

$$\cos\theta = \frac{2}{\sqrt{2}\sqrt{3}} = 0.865$$



Example 1.3.4

The vector $\boldsymbol{ai + j - k}$ is perpendicular to the vector $\boldsymbol{i + 2j - 3k}$. What is the value of a?

If the vectors are perpendicular to each other, their dot product must vanish ($\cos 90 = 0$).

$$(ai + j - k).(i + 2j - 3k)\text{=0}$$

$$a + 2 + 3 = 0$$

$$a = -5$$

1.4 The Vector Product

$$A \times B = \begin{pmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}$$

$$A \times B = i(A_y B_z - A_z B_y) + j(A_z B_x \text{-} A_x B_z) + k(A_x B_y \text{-} A_y B_x)$$

It can be shown that the following rules hold for cross multiplication:

$$(A \times B) = -(B \times A)$$

$$A \times (B + C) = A \times B + A \times C$$

$$\mathrm{n}(A \times B) = (nA) \times B = A \times (nB)$$

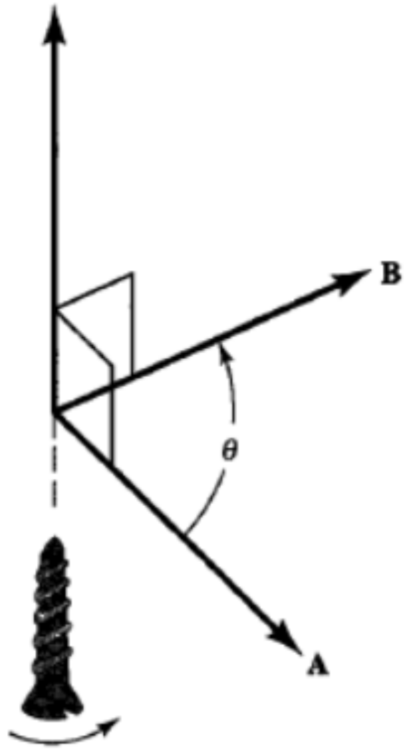
$$\mathrm{i}\times\mathrm{i} = \mathrm{j}\times\mathrm{j} = \mathrm{k}\times\mathrm{k} = 0$$

$$\mathrm{j}\times\mathrm{k} = \mathrm{i} = -\mathrm{k}\times\mathrm{j}$$

$$\mathrm{i}\times\mathrm{j} = \mathrm{k} = -\mathrm{j}\times\mathrm{i}$$

$$\mathrm{k}\times\mathrm{i} = \mathrm{j} = -\mathrm{i}\times\mathrm{k}$$

$$A\times B = AB\sin\theta$$



The cross product of two vectors.

Example 1.4.1

Given the two vectors $A = 2i + j - k$, $B = i - j + 2k$, find $A \times B$.

In this case it is convenient to use the determinant form

$$A \times B = \begin{pmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} = i\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} - j\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + k\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A \times B = i(2 - 1) - j(4 + 1) + k(-2 - 1)$$

$$A \times B = i - 5j - 3k$$

Example 1.4.2

Find a unit vector normal to the plane containing the two vectors A and B
$A = 2i + j - k$, $B = i - j + 2k$

$$n = \frac{A \times B}{|A \times B|} = \frac{i - 5j - 3k}{\sqrt{1^2 + 5^2 + 3^2}} = \frac{i - 5j - 3k}{\sqrt{35}}$$

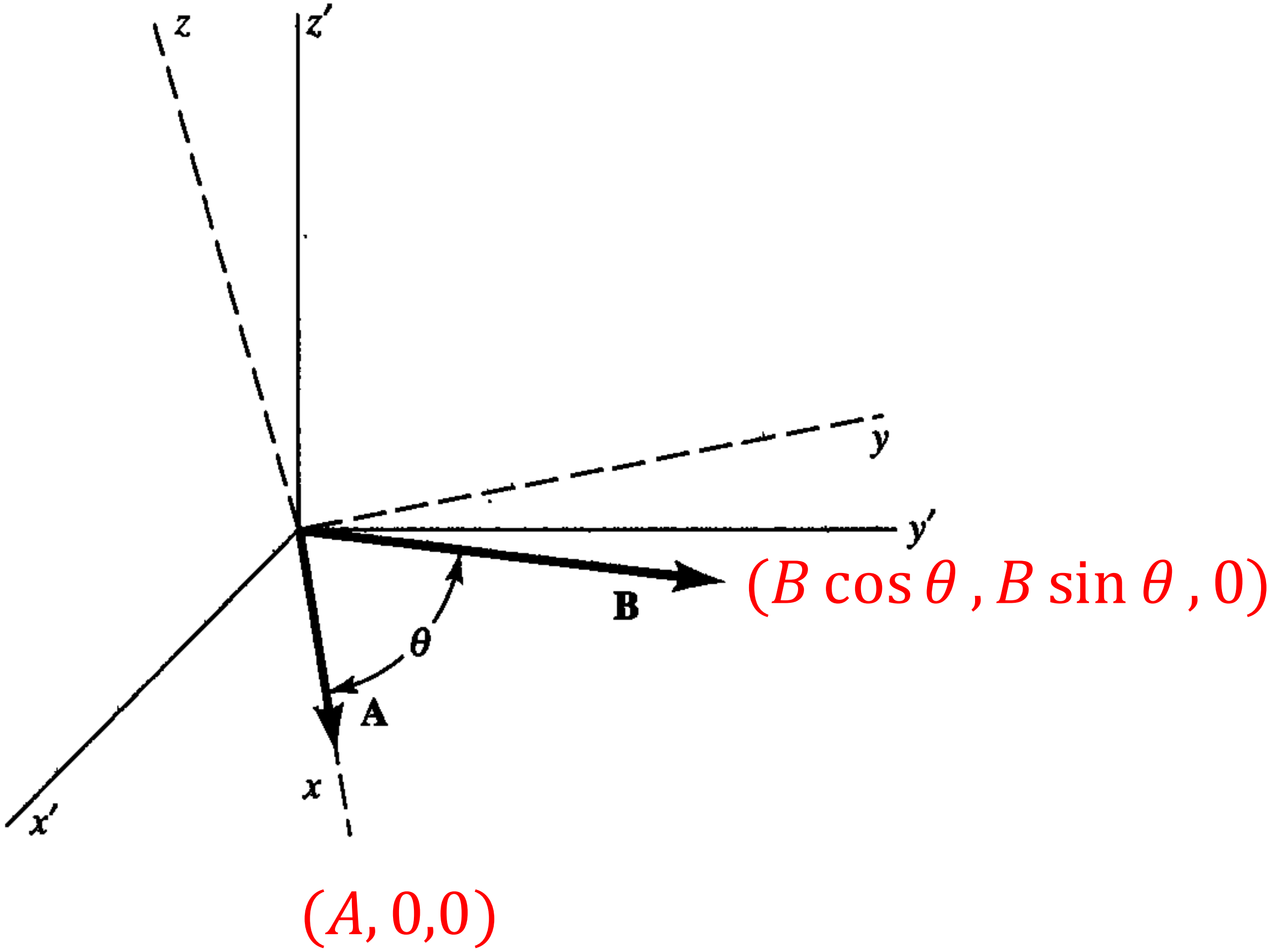
$$= \frac{i}{\sqrt{35}} - \frac{5j}{\sqrt{35}} - \frac{3k}{\sqrt{35}}$$

Example 1.4.3

Show by direct evaluation that A x B is a vector with direction perpendicular to A and B and magnitude AB sin θ.

$$\mathrm{A}\times\mathrm{B} = \begin{pmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \mathrm{A} & 0 & 0 \\ \mathrm{B}\cos\theta & \mathrm{B}\sin\theta & 0 \end{pmatrix}$$

$$= \mathrm{kAB}\sin\theta$$



HWs

1

For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

(a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$ (b) component of $\mathbf{B}$ along $\mathbf{A}$ (c) angle between $\mathbf{A}$ and $\mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

2

Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:

(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$

(b) $3\mathbf{A} - 2\mathbf{B}$

(c) $\mathbf{A} \cdot \mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

3

For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1.5 Triple Products

The expression $A \,.\, (B \times C)$ is called the **scalar triple product** of A, B, and C.

We can see that the scalar triple product may be written as matrix

$$\boxed{A.\,(B \times C) = \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}}$$

$$A.\,(B \times C) = (A \times B).\,C$$

The expression $A \times (B \times C)$ is called the **Vector triple product** of A, B, and C.

$$\boxed{A \times (B \times C) = B(A.\,C) - C(A.\,B)}$$

Example 1.5.1

Given the three vectors $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1 + 0) = -1$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{i} - \mathbf{j})0 - \mathbf{k}(1 - 0) = -\mathbf{k}$$

1.6 Change of Coordinate System: The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. Consider the vector A expressed relative to the triad ijk:

$$\mathrm{A} = \mathrm{i}\mathrm{A}_{\mathrm{x}} + \mathrm{j}\mathrm{A}_{\mathrm{y}} + \mathrm{k}\mathrm{A}_{\mathrm{z}}$$

Relative to a new triad $i'j'k'$ having a different orientation from that of ijk, the same vector A is expressed as

$$\mathrm{A} = \mathrm{i}'\mathrm{A}_{\mathrm{x}'} + \mathrm{j}'\mathrm{A}_{\mathrm{y}'} + \mathrm{k}'\mathrm{A}_{\mathrm{z}'}$$

Now the dot product $A.i'$ is just $A_{x'}$: the projection of A on the unit vector i'.

$$\mathrm{A}_{\mathrm{x}'} = \mathrm{A}.\mathrm{i}' = (\mathrm{i}.\mathrm{i}')\mathrm{A}_{\mathrm{x}} + (\mathrm{j}.\mathrm{i}')\mathrm{A}_{\mathrm{y}} + (\mathrm{k}.\mathrm{i}')\mathrm{A}_{\mathrm{z}}$$

$$\mathrm{A}_{\mathrm{y}'} = \mathrm{A}.\mathrm{j}' = (\mathrm{i}.\mathrm{j}')\mathrm{A}_{\mathrm{x}} + (\mathrm{j}.\mathrm{j}')\mathrm{A}_{\mathrm{y}} + (\mathrm{k}.\mathrm{j}')\mathrm{A}_{\mathrm{z}}$$

$$\mathrm{A}_{\mathrm{z}'} = \mathrm{A}.\mathrm{k}' = (\mathrm{i}.\mathrm{k}')\mathrm{A}_{\mathrm{x}} + (\mathrm{j}.\mathrm{k}')\mathrm{A}_{\mathrm{y}} + (\mathrm{k}.\mathrm{k}')\mathrm{A}_{\mathrm{z}}$$

The scalar product $(\mathrm{i.\,i'}), (\mathrm{i.\,j'})$ and so on are called the **coefficients of transformation.**

The unprimed components are similarly expressed as

$$A_x = A.\,i = (i'.\,i)A_{x'} + (j'.\,i)A_{y'} + (k'.\,i)A_{z'}$$

$$A_y = A.\,j = (i'.\,j)A_{x'} + (j'.\,j)A_{y'} + (k'.\,j)A_{z'}$$

$$A_z = A.\,k = (i'.\,k)A_{x'} + (j'.\,k)A_{y'} + (k'.\,k)A_{z'}$$

The equation of transformation are conveniently expressed in matrix notation.

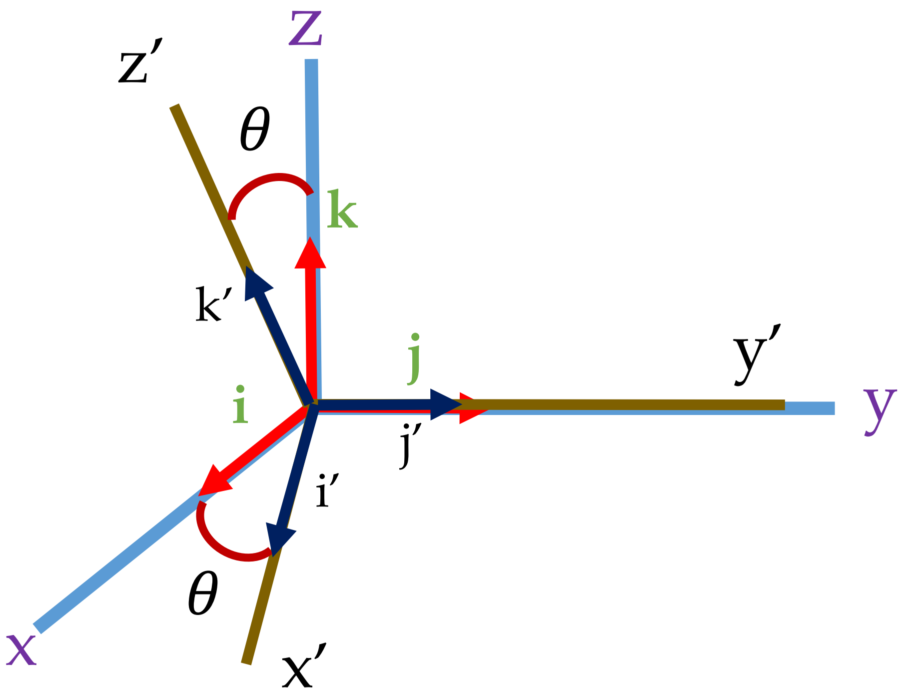
$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} i.\,i' & j.\,i' & k.\,i' \\ i.\,j' & j.\,j' & k.\,j' \\ i.\,k' & j.\,k' & k.\,k' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \underbrace{\begin{pmatrix} i.\,i' & j.\,i' & k.\,i' \\ i.\,j' & j.\,j' & k.\,j' \\ i.\,k' & j.\,k' & k.\,k' \end{pmatrix}}_{\text{Transformation matrix}} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$i.\,i' = (1)(1)\cos\theta = \cos\theta$$

$$j.\,i' = (1)(1)\cos 90 = 0$$

$$\begin{aligned} k.\,i' &= (1)(1)\cos(90+\theta) \\ &= \cos 90 \cos\theta - \sin 90 \sin\theta \\ &= -\sin\theta \end{aligned}$$



HWs

1 Show that for the **rotation around z**, the transformation matrix is given by

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Show that for the **rotation around x**, the transformation matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

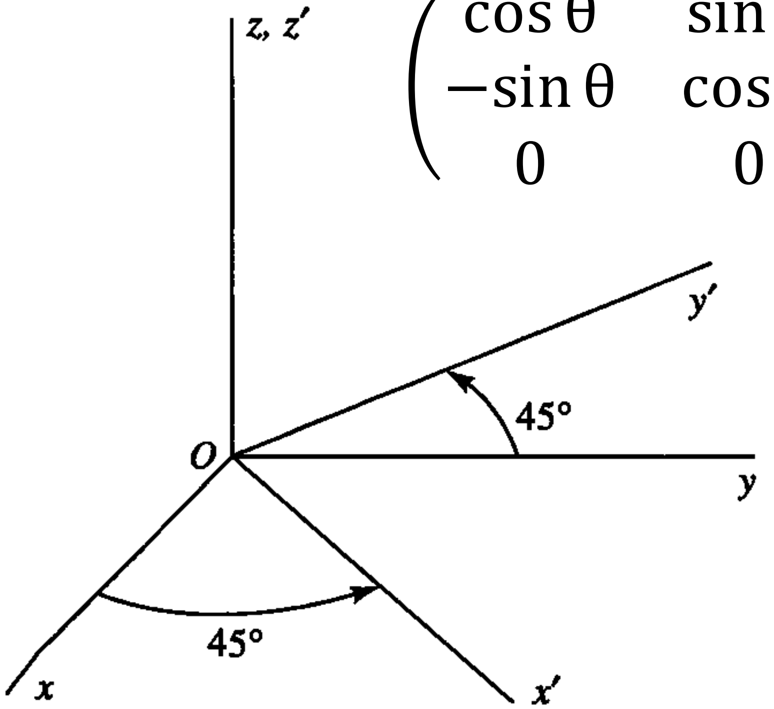
Example 1.6.1

Express the vector $A = 3i + 2j + k$ in terms of the triad $i'j'k'$, where the $x'y' - axes$ are rotated 45^o around z-axis, with the z- and z'-axes coinciding as shown in the figure below, we have for the coefficients of transformation $i.i' = \cos 45^o$ and so on;

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} i.i' & j.i' & k.i' \\ i.j' & j.j' & k.j' \\ i.k' & j.k' & k.k' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Around z

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$i.i' = \frac{1}{\sqrt{2}} \qquad j.i' = \frac{1}{\sqrt{2}} \qquad k.i' = 0$$

$$i.j' = -\frac{1}{\sqrt{2}} \qquad j.j' = \frac{1}{\sqrt{2}} \qquad k.j' = 0$$

$$i.k' = 0 \qquad j.k' = 0 \qquad k.k' = 1$$

Before rotation $\boxed{A = 3i + 2j + k}$

After rotation $\boxed{A = \frac{5}{\sqrt{2}} i' - \frac{1}{\sqrt{2}} j' + k'}$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A_{x'} = \frac{1}{\sqrt{2}}(3) + \frac{1}{\sqrt{2}}(2) + (0)(1) \qquad A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} \qquad A_{x'} = \boxed{\frac{5}{\sqrt{2}}}$$

$$A_{y'} = -\frac{1}{\sqrt{2}}(3) + \frac{1}{\sqrt{2}}(2) + (0)(1) \qquad A_{y'} = -\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} \qquad A_{y'} = \boxed{-\frac{1}{\sqrt{2}}}$$

$$A_{z'} = (0)(3) + (0)(2) + (1)(1) \qquad A_{z'} = \boxed{1}$$

HWs

Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of $30°$.

1.7 Derivative of a vector

Consider a vector A, whose components are function of single variable u, is usually time t, the vector may represent position, velocity, and so on.

$$A(u) = i\,A_x(u) + jA_y(u) + kA_z(u)$$

So the derivative of A can be expressed as following

$$\frac{dA}{du} = \lim_{\Delta u \to 0}(i\,\frac{\Delta A_x}{\Delta u} + j\,\frac{\Delta A_y}{\Delta u} + k\,\frac{\Delta A_z}{\Delta u})$$

$$\frac{dA}{du} = i\,\frac{dA_x}{du} + j\,\frac{dA_y}{du} + k\,\frac{dA_z}{du}$$

This means, the derivative of a vector is a vector whose Cartesian components are ordinary derivatives.

Now, below are the vector rules of vector differential.

$$\boxed{\frac{d}{du}(A+B)=\frac{dA}{du}+\frac{dB}{du}}$$

$$\boxed{\frac{d\,(nA)}{du}=\frac{dn}{du}A+n\frac{dA}{du}}$$

$$\boxed{\frac{d\,(A.B)}{du}=\frac{dA}{du}.B+A.\frac{dB}{du}}$$

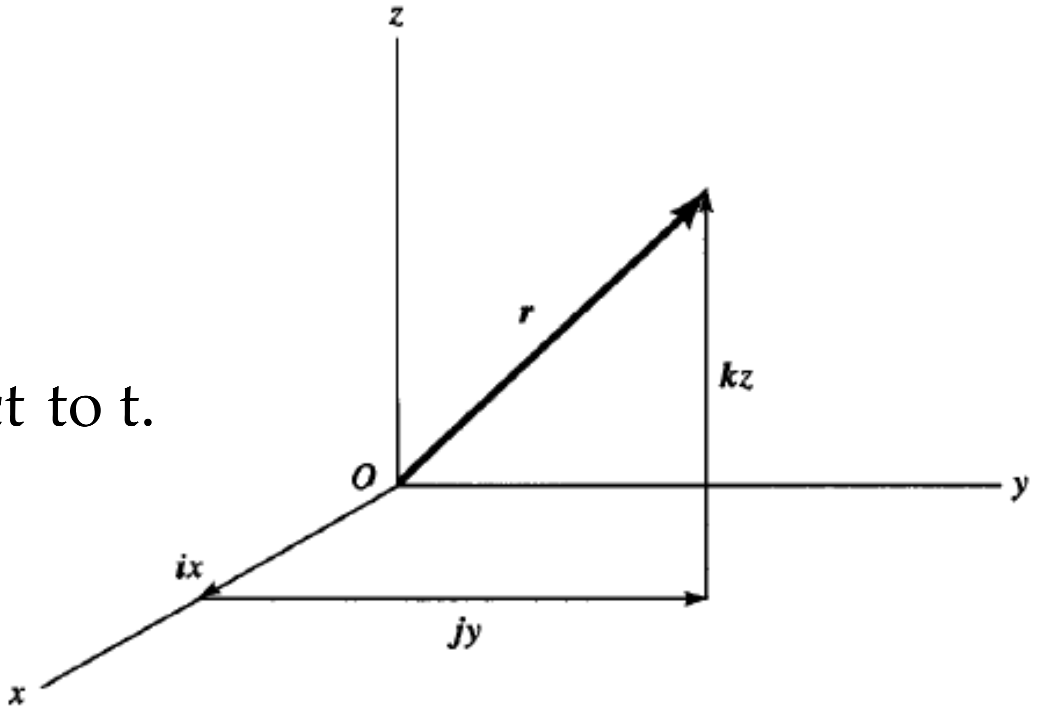
$$\boxed{\frac{d\,(A\times B)}{du}=\frac{dA}{du}\times B+A\times\frac{dB}{du}}$$

1.8 Position of a vector: velocity and Acceleration in Rectangular coordinates

$$\boxed{r = i\,x + jy + kz}$$ ***The position vector***

As $x = x(t), y = y(t), z = z(t)$

are the components (of the position vector) of moving particle



So the **velocity** can be written as following :

$$V = \frac{dr}{dt} = i\frac{dx}{dt} + j\frac{dy}{dt} + k\frac{dz}{dt}$$

where the dots indicate differentiation with respect to t.

$$V = \dot{r} = i\dot{x} + j\dot{y} + k\dot{z}$$

The magnitude of the velocity is called the **speed**

components the speed is just

$$v = |V| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

The time derivative of the velocity is called the **acceleration**. Denoting the acceleration with a, we have

$$a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$$

In rectangular components,

$$a = i\ddot{x} + j\ddot{y} + k\ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

Example 1.8.1

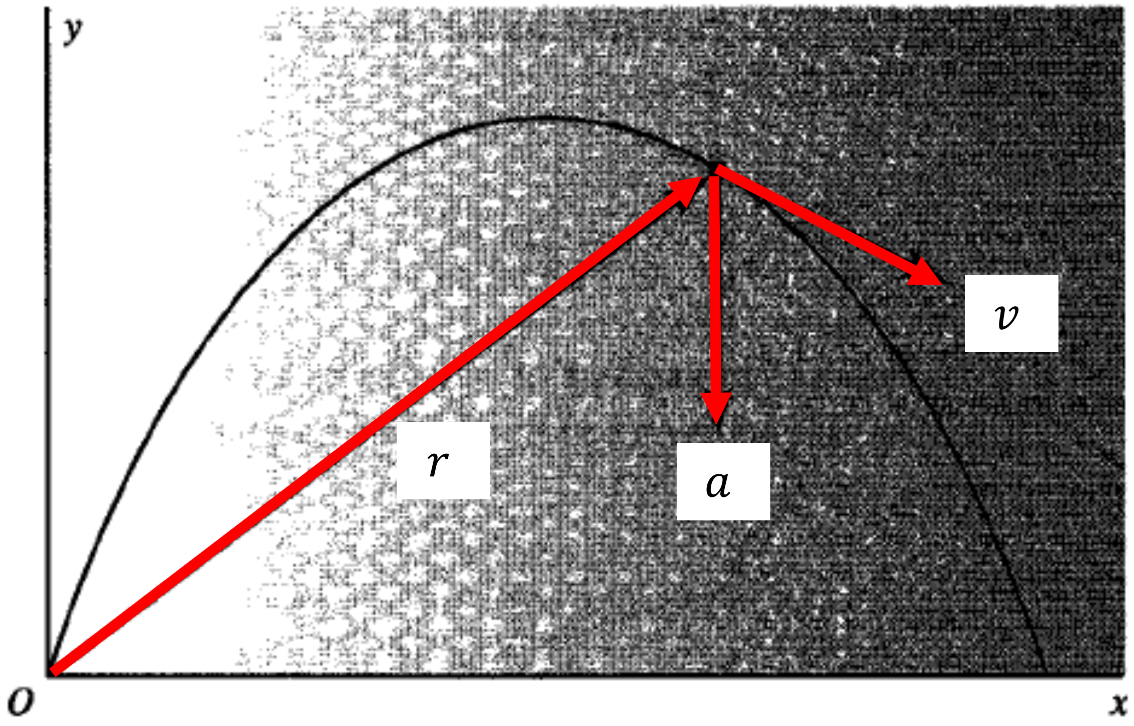
Consider the motion of a particle (projectile) in a parabolic path which is represented by the equation

$$r(t) = ibt + j\left(ct - \frac{gt^2}{2}\right) + k0$$

Find the velocity and acceleration vectors as well as the speed of this particle.

This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity v is obtained by differentiating with respect to t, namely,

$$v = \frac{dr}{dt} = ib + j(c - gt)$$



The acceleration, likewise, is given by

$$\boxed{a = \frac{dv}{dt} = -jg}$$

Thus, a is in the negative y direction and has the constant magnitude g.

The speed v varies with t according to the equation

$$\boxed{v = [b^2 + (c - gt)^2]^{1/2}}$$

Example 1.8.2

Circular Motion

Consider a particle moving in a circular path with constant speed, the position vector is given by

$$r = ib\sin\omega t + jb\cos\omega t$$

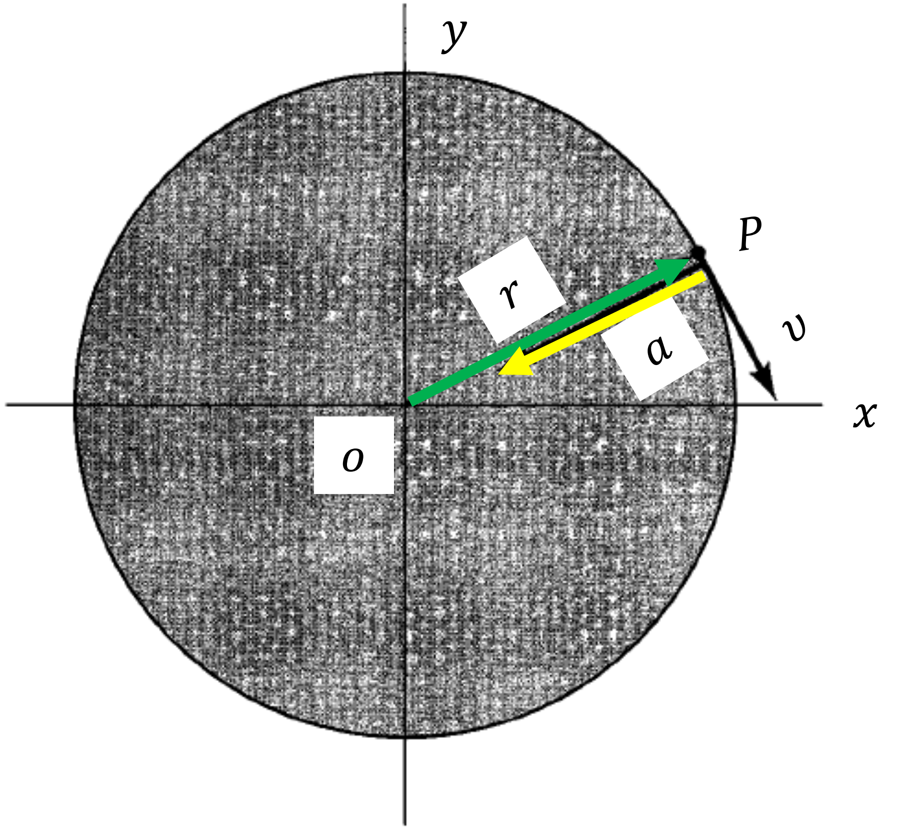
Where ω is constant. Find the **distance**, **velocity vector**, **speed**, and **acceleration** for this particle. Show that, in the case of circular motion, **the acceleration is perpendicular to the velocity**. Write ***a* in terms of *r***.

The distance from the origin remains constant:

$$|r| = r = (b^2 sin^2\omega t + b^2 cos^2\omega t)^{1/2}$$

$$|r| = r = (b^2)^{1/2}\,(sin^2\omega t + cos^2\omega t)^{1/2}$$

$$|r| = b$$ **distance**



So the path is a circle of radius b centered at the origin

Differentiating r, we find the velocity vector

$$\boxed{v = \frac{dr}{dt} = ib\omega\cos\omega t - jb\omega\sin\omega t}$$

Velocity vector

The particle traverses its path with constant speed:

$$v = |v| = (b^2\omega^2 cos^2\omega t + b^2\omega^2 sin^2\omega t)^{1/2} \Rightarrow \boxed{|v| = b\omega}$$

Speed

The acceleration is

$$\boxed{a = \frac{dv}{dt} = -ib\omega^2\sin\omega t - jb\omega^2\cos\omega t}$$

Acceleration

In this case the acceleration is perpendicular to the velocity, because the dot product of v and a vanishes:

$$v.a = (ib\omega\cos\omega t - jb\omega\sin\omega t).(-ib\omega^2\sin\omega t - jb\omega^2\cos\omega t)$$

$$v.a = (ib\omega\cos\omega t)(-ib\omega^2\sin\omega t) + (-jb\omega\sin\omega t)(-jb\omega^2\cos\omega t)$$

$$v.a = -b^2\omega^3\sin\omega t\cos\omega t + b^2\omega^3\sin\omega t\cos\omega t$$

$$\boxed{v.a = 0}$$

The acceleration is perpendicular to the velocity

$$a = \frac{dv}{dt} = -ib\omega^2\sin\omega t - jb\omega^2\cos\omega t$$

Known

$$\boxed{r = ib\sin\omega t + jb\cos\omega t}$$

$$a = -\omega^2(ib\sin\omega t + jb\cos\omega t)$$

$$\boxed{a = -\omega^2 r}$$

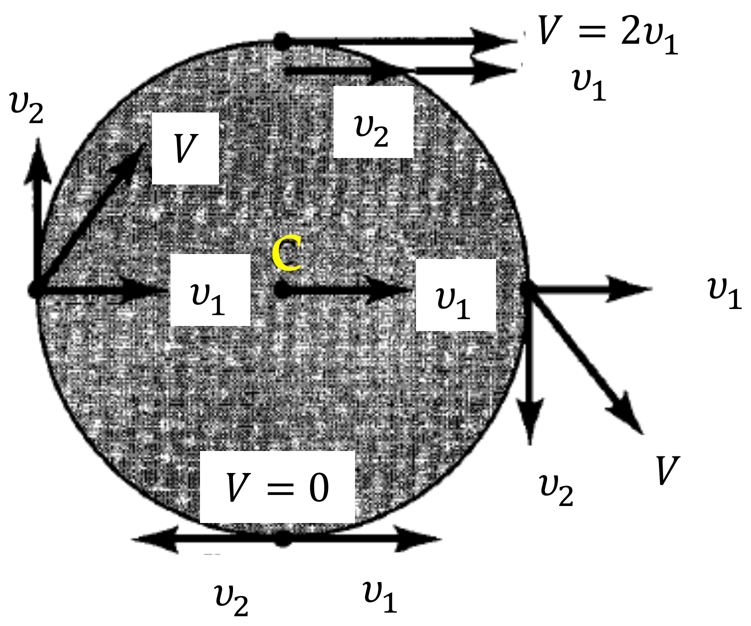
a* in terms of *r

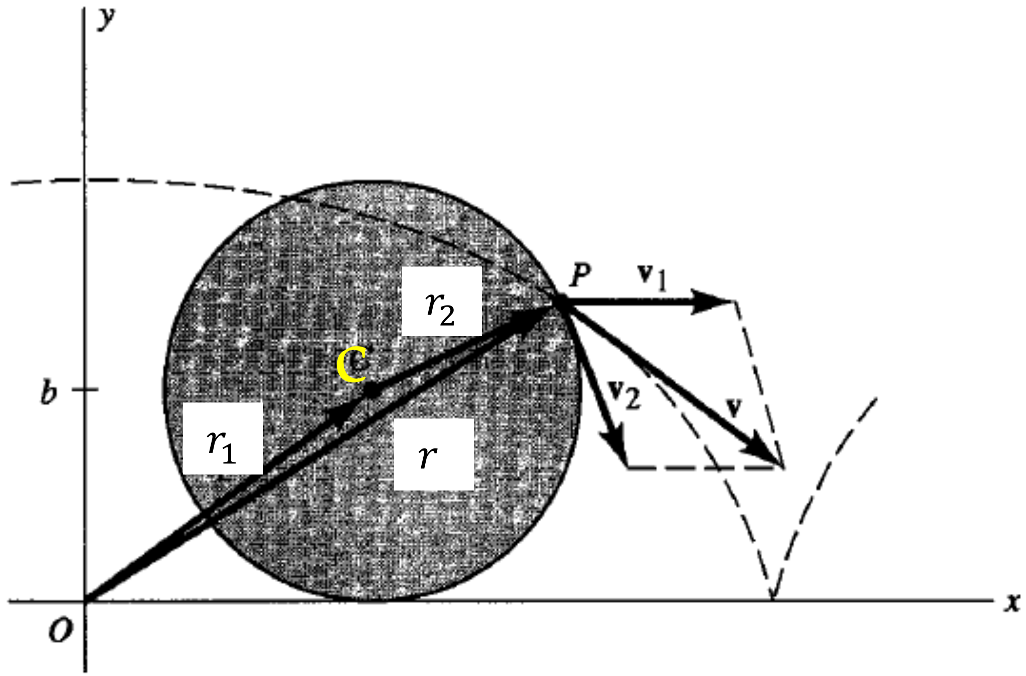
So a and r are oppositely directed, that is , a always points toward the center of the circular path.

Example 1.8.3

Rolling Wheel

Consider a rolling wheel following position vector $r = r_1 + r_2$ in which $r_1 = ib\omega t + jb$ and $r_2 = ib \sin \omega t + jb \cos \omega t$, where r_1 represents a point moving along the line $y = b$ at constant velocity and r_2 is just the position vector for the circular motion (see example 1.8.2). Find the velocity of the point P .





$$r_1 = ib\omega t + jb$$

$$r = r_1 + r_2$$

$$r_2 = ib\sin\omega t + jb\cos\omega t$$

$$v_1 = \frac{dr_1}{dt} = \boxed{ib\omega}$$

$$v_2 = \frac{dr_2}{dt} = \boxed{ib\omega\cos\omega t - jb\omega\sin\omega t}$$

$$v = v_1 + v_2 \Rightarrow v = ib\omega + ib\omega\cos\omega t - jb\omega\sin\omega t$$

$$\boxed{v = i(b\omega + b\omega\cos\omega t) - jb\omega\sin\omega t}$$

H.W. **Accelecration** ?

HWs

1

A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\cos\omega t + \mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

2

A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\sin\omega t + \mathbf{j}b\cos\omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

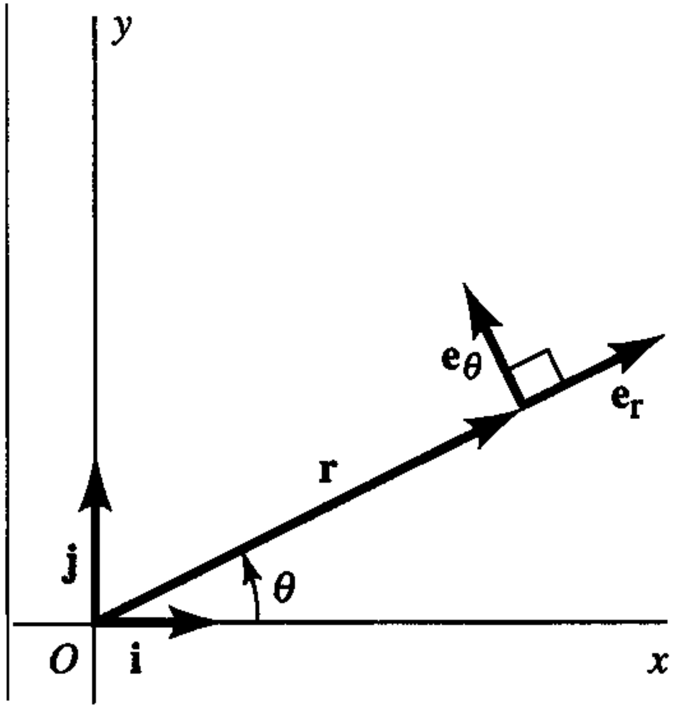
1.9 Velocity and Acceleration in Plane Polar Coordinates

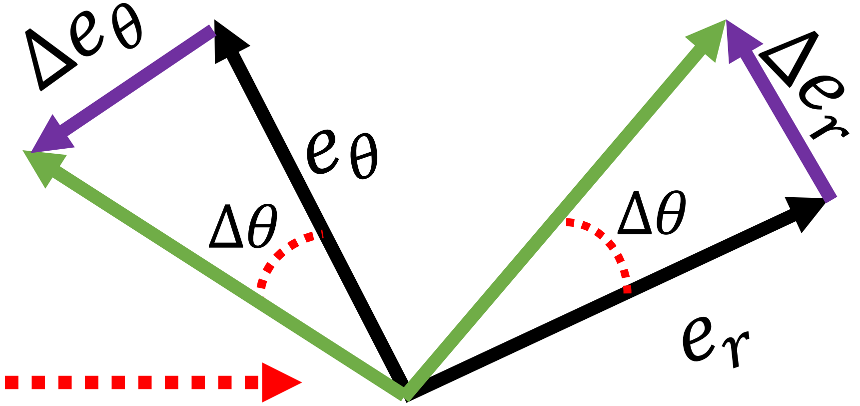
It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector e_r:

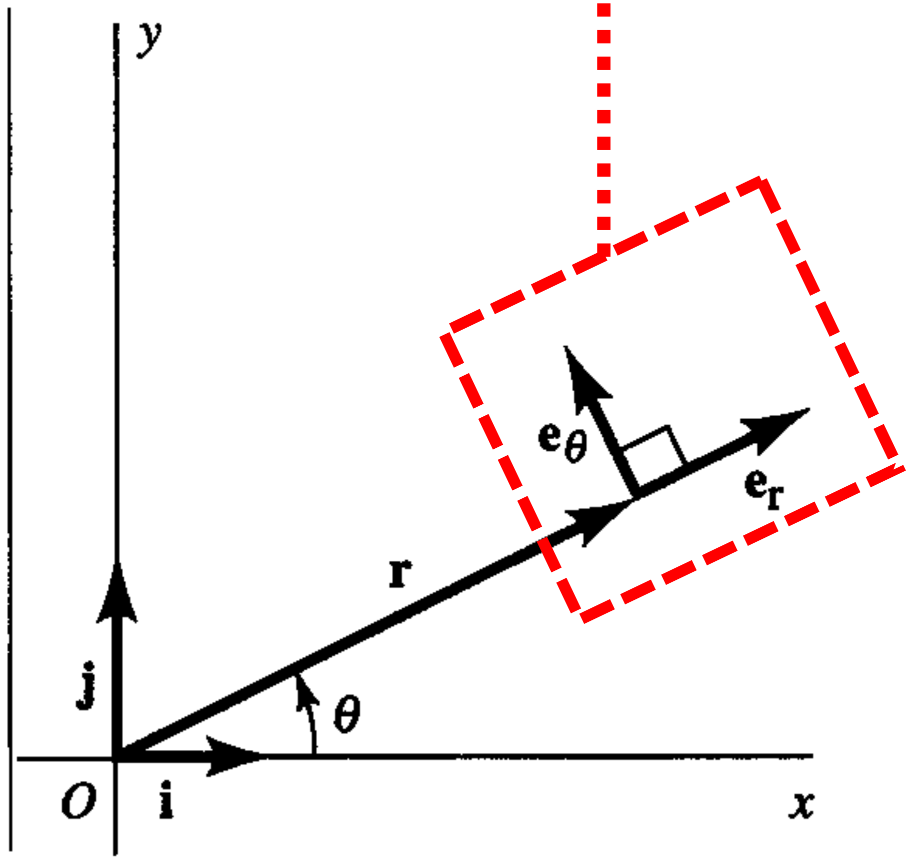
$$\boxed{\mathbf{r} = re_r}$$

As the particle moves, both r and e_r vary; thus, they are both functions of the time. Hence, if we differentiate with respect to t, we have

$$v = \frac{d\boldsymbol{r}}{dt} = \dot{r}e_r + r\frac{de_r}{dt}$$







$$\Delta e_r \cong \Delta\theta$$

$$\Delta e_r = e_\theta \Delta\theta$$

$$\boxed{\frac{de_r}{dt} = e_\theta \frac{d\theta}{dt}}$$

$$\Delta e_\theta \cong \Delta\theta$$

$$\Delta e_\theta = -e_r \Delta\theta$$

$$\boxed{\frac{de_\theta}{dt} = -e_r \frac{d\theta}{dt}}$$

$$v = \frac{dr}{dt} = \dot{r}e_r + r\boxed{\frac{de_r}{dt}}$$

$$\boxed{\frac{de_r}{dt} = e_\theta \frac{d\theta}{dt}}$$

$$\boxed{\frac{de_\theta}{dt} = -e_r \frac{d\theta}{dt}}$$

$$v = \frac{dr}{dt} = \dot{r}e_r + re_\theta \frac{d\theta}{dt}$$

$$\boxed{v = \frac{dr}{dt} = \boxed{\dot{r}e_r} + \boxed{r\dot{\theta}e_\theta}}$$

$$v_r = \dot{r}$$

Radial component

$$v_\theta = r\dot{\theta}$$

Transverse component

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$v = \dot{r} e_r + r\dot{\theta} e_\theta$$

$$a = \frac{dv}{dt} = \ddot{r} e_r + \dot{r} \frac{de_r}{dt} + (r\ddot{\theta} + \dot{r}\dot{\theta}) e_\theta + r\dot{\theta} \frac{de_\theta}{dt}$$

$$\frac{de_r}{dt} = e_\theta \frac{d\theta}{dt}$$

$$\frac{de_\theta}{dt} = -e_r \frac{d\theta}{dt}$$

$$a = \ddot{r}e_r + \dot{r}(e_\theta\dot{\theta}) + (r\ddot{\theta} + \dot{r}\dot{\theta})e_\theta + r\dot{\theta}(-e_r\dot{\theta})$$

$$a = \ddot{r}e_r + \dot{r}\dot{\theta}e_\theta + (r\ddot{\theta} + \dot{r}\dot{\theta})e_\theta - re_r\dot{\theta}^2$$

$$\boxed{a = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta}$$

The acceleration vector

The radial component

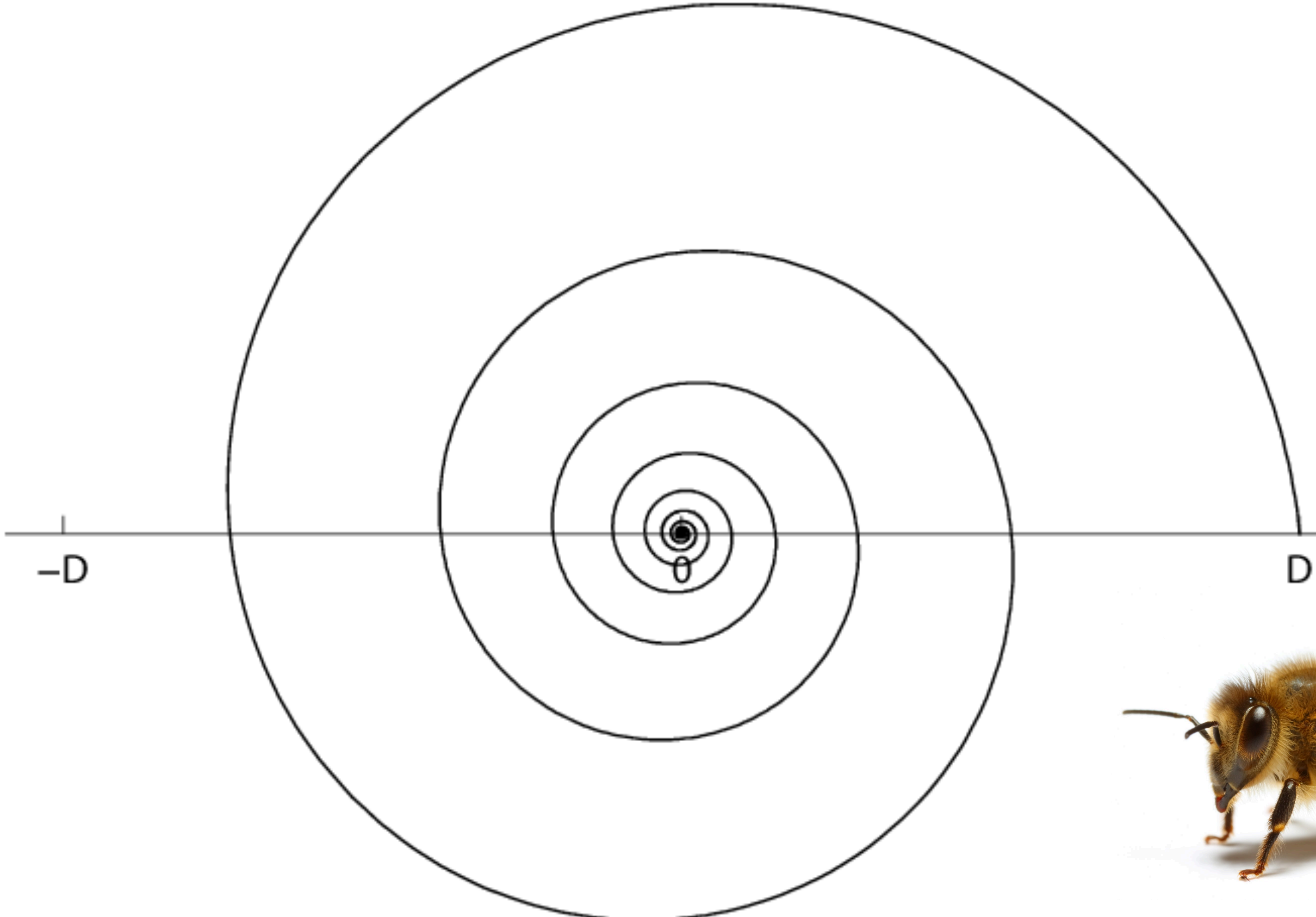
The transverse component

$$a_r = (\ddot{r} - r\dot{\theta}^2)$$

$$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Example 1.9.1

A honeybee hones in on its hive in a spiral path in such a way that the radial distance decreases at a constant rate, $r = b - ct$, while the angular speed increases at a constant rate, $\dot{\theta} = kt$. Find the speed as a function of time.



$$r = b - ct$$

$$\dot{\theta} = kt$$

$$v = \frac{dr}{dt} = \dot{r}e_r \quad + \quad r\dot{\theta}e_\theta$$

$$r = b - ct \qquad \dot{\theta} = kt$$

$$\dot{r} = -c$$

$$v = \frac{dr}{dt} = -ce_r \quad + \quad (b - ct)kte_\theta$$

$$|v| = [c^2 \quad + \quad (b - ct)^2k^2t^2]^{1/2}$$

Notice $|v|$=c $\quad$ $when\ t = 0$ $\Rightarrow$ $r = b$

$when\ t = b/c$ $\Rightarrow$ $r = 0$

Example 1.9.2

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratically with time $r = bt^2, \theta = \omega t$, where b and t are constants. Find the acceleration of the bug.

Known

$r = bt^2, \qquad \theta = \omega t$

We have $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega, \ddot{\theta} = 0$

$$\boxed{a = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta}$$

$$a = (2b - bt^2\omega^2)e_r + (0 + 2(2bt)\omega)e_\theta$$

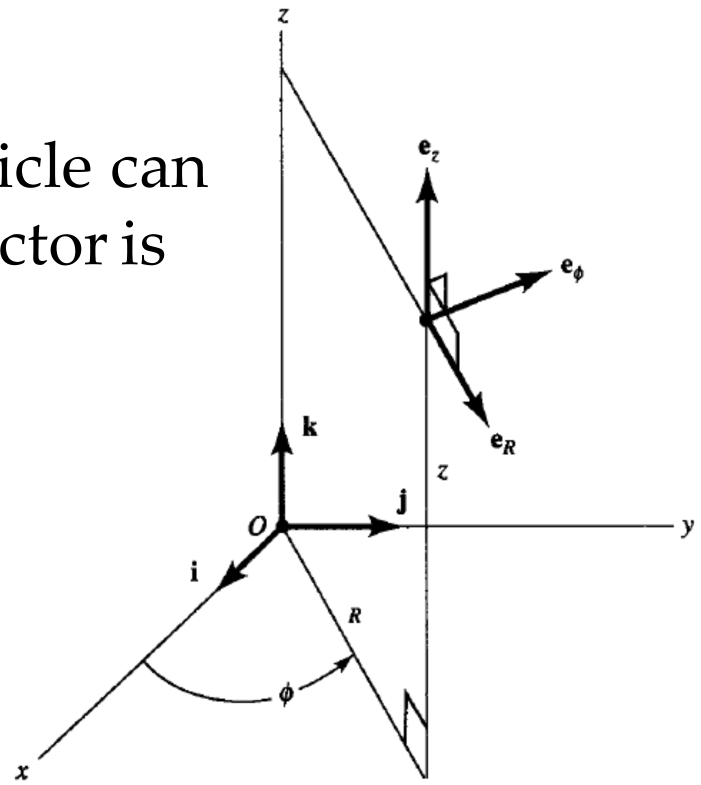
$$a = b(2 - t^2\omega^2)e_r + 4bt\omega e_\theta$$

1.10 Velocity and Acceleration in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as

$$r = Re_R + ze_z$$



where e_R is a unit radial vector in the xy plane.
e_Z is the unit vector in the z −direction.

A third unit vector e_ϕ is needed so that the three vectors $e_R e_\phi e_Z$ constitute a right-handed triad.

The velocity and acceleration vectors are found by differentiating, as before. This again involves derivatives of the unit vectors. An argument similar to that used for the plane case shows that

$k = e_z$

HWs

$$\frac{de_R}{dt} = e_\phi \dot{\phi} \qquad \frac{de_\phi}{dt} = -e_R \dot{\phi}$$

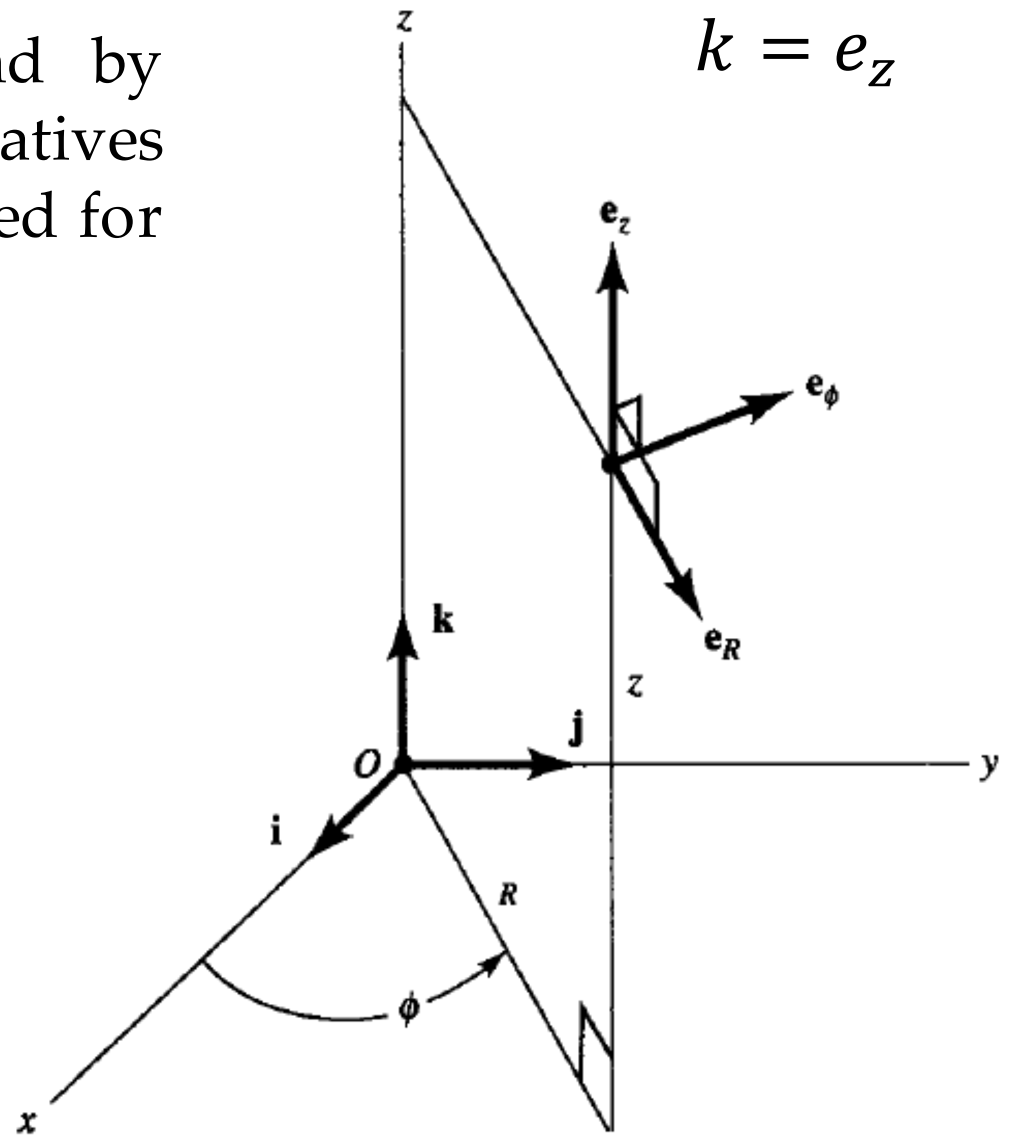
$$r = Re_R + ze_z \quad \Rightarrow \quad v = \dot{R}e_R + R\frac{de_R}{dt} + \dot{z}e_z$$

The unit vector does not change in direction, so its time derivative is zero.

$$v = \dot{R}e_R + Re_\phi\dot{\phi} + \dot{z}e_z$$

HWs

$$a = (\ddot{R} - R\dot{\phi}^2)e_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})e_\phi + \ddot{z}e_z$$



Unit vectors for cylindrical coordinates.

Spherical Coordinates

$$v = e_r \dot{r} + e_\phi r \dot{\phi} \sin\theta + e_\theta r \dot{\theta}$$

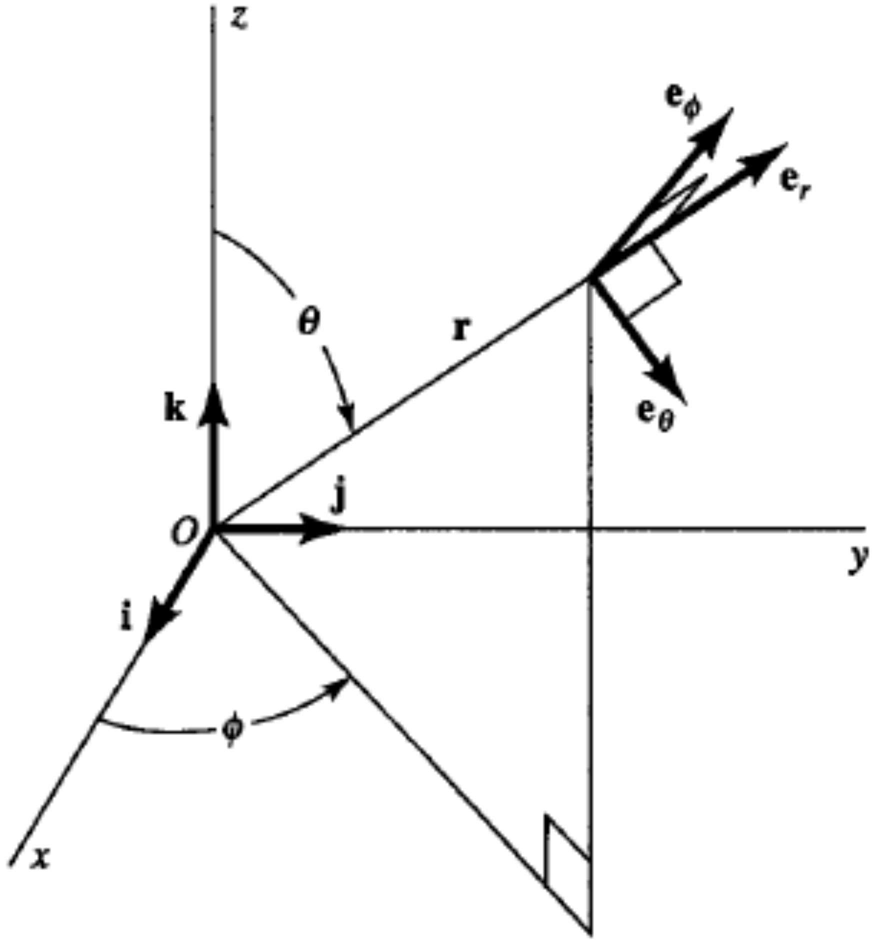
HWs

$$a = (\ddot{r} - r\dot{\phi}^2 sin^2\theta - r\dot{\theta}^2)e_r$$

HWs

$$+(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)e_\theta$$

$$+(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)e_\phi$$



Unit vectors for spherical coordinates.

Example 1.10.1

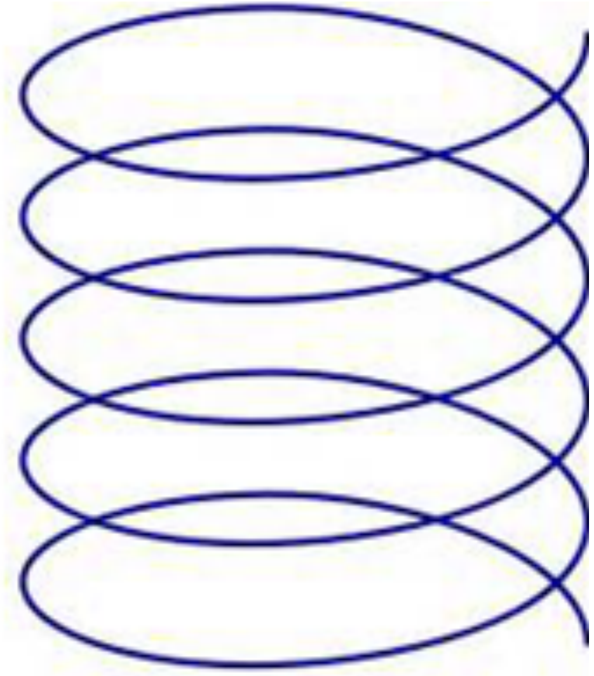
A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by $R = b,\ \phi = \omega t,\ z = ct$. Find the velocity and acceleration vectors as a function of time.

$$v = \dot{R}e_R + Re_\phi\dot{\phi} + \dot{z}e_z \qquad a = (\ddot{R} - R\dot{\phi}^2)e_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})e_\phi + \ddot{z}e_z$$

$$R = b \;\Rightarrow\; \dot{R} = \ddot{R} = 0$$

$$\phi = \omega t \;\Rightarrow\; \dot{\phi} = \omega \;\Rightarrow\; \ddot{\phi} = 0$$

$$z = ct \;\Rightarrow\; \dot{z} = c \;\Rightarrow\; \ddot{z} = 0$$



$$v = b\omega e_\phi + ce_z$$

$$a = -b\omega^2 e_R$$

Thus, in this case both velocity and acceleration are constant in magnitude, but they vary in direction because both e_ϕ and e_r change with time as the bead moves

Example 1.10.2

A wheel of radius b is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed ω_1 about its own axis, which in turn rotates with constant angular speed ω_2 about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the wheel. In particular, find the acceleration of the highest point on the wheel.

$$r = b \quad \Rightarrow \quad \dot{r} = \ddot{r} = 0$$

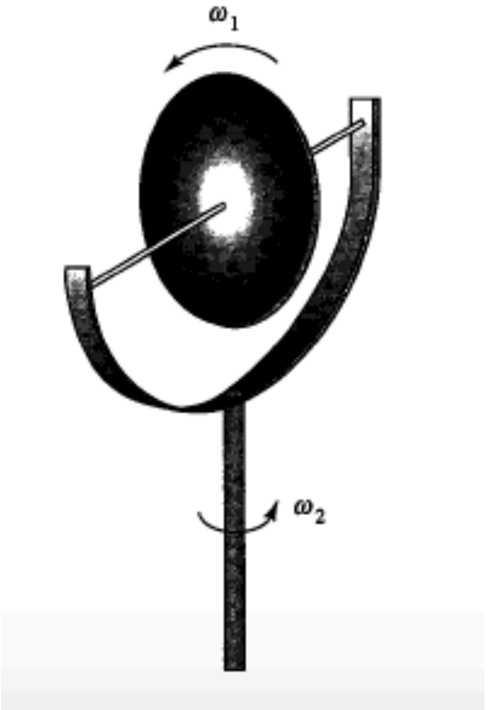
$$\theta = \omega_1 t \quad \Rightarrow \quad \dot{\theta} = \omega_1 \quad \Rightarrow \quad \ddot{\theta} = 0$$

$$\phi = \omega_2 t \quad \Rightarrow \quad \dot{\phi} = \omega_2 \quad \Rightarrow \quad \ddot{\phi} = 0$$

$$a = (\ddot{r} - r\dot{\phi}^2 sin^2\theta - r\dot{\theta}^2)e_r$$

$$+(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta)e_\theta$$

$$+(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)e_\phi$$

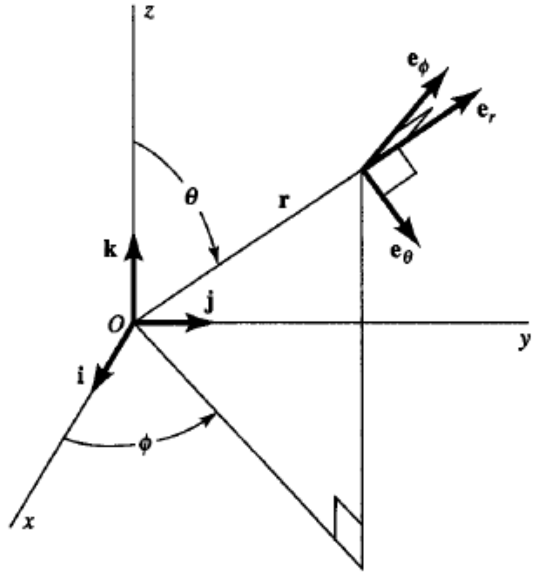


$$a = (-b\omega_2^2 \cancel{sin^2\theta} - b\omega_1^2)e_r - b\omega_2^2 \cancel{\sin\theta} \cos\theta\, e_\theta + 2b\,\omega_1\omega_2 \cos\theta\, e_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$a = -b\omega_1^2 e_r + 2b\,\omega_1\omega_2 e_\phi$$

The first term on the right side is the centripetal acceleration. The last term is a transverse acceleration normal to the plane of the wheel.



HWs

1

A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v} \cdot \mathbf{a}/va$.)

2

An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

H. W

Express the differences in $\boldsymbol{ijk}$ and the magnitude of the new vector $\boldsymbol{A-B}$ of the two vectors A=(1,0,2) and B=(0,1,1)

Answer $C = A - B = i - j + k$

1

For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

(a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$ (b) component of $\mathbf{B}$ along $\mathbf{A}$ (c) angle between $\mathbf{A}$ and $\mathbf{B}$
(d) $\mathbf{A} \times \mathbf{B}$ (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

2

Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:

(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$
(b) $3\mathbf{A} - 2\mathbf{B}$
(c) $\mathbf{A} \cdot \mathbf{B}$
(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

3

For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1

For the two vectors

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

find

(a) $\mathbf{A} - \mathbf{B}$ and $|\mathbf{A} - \mathbf{B}|$ (b) component of $\mathbf{B}$ along $\mathbf{A}$ (c) angle between $\mathbf{A}$ and $\mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ (e) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$

Answer

(a) $A - B = 3i - j - 2k$

$$|A - B| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$$

(b) $\mathrm{A.B} = \mathrm{AB}\cos\theta$

$$B\cos\theta = \frac{A.B}{|A|}$$

$$B\cos\theta = \frac{(-2+6-1)}{\sqrt{1^2+2^2+1^2}}$$

$$B\cos\theta = \frac{3}{\sqrt{6}}$$

$$(c)\ \cos\theta = \frac{A.B}{|A||B|} \Rightarrow \cos\theta = \frac{-2+6-1}{\sqrt{6}\sqrt{14}} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2\sqrt{7}}$$

$$\theta = cos^{-1}\left(\frac{\sqrt{3}}{2\sqrt{7}}\right) \Rightarrow \boxed{\theta \approx 71^{o}}$$

$$(d)\ A\times B = \begin{pmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{pmatrix} \Rightarrow A\times B = 5i + j + 7k$$

$$(e)\ A - B = 3i - j - 2k \qquad A + B = -i + 5j$$

$$(A-B)\times(A+B) = \begin{pmatrix} i & j & k \\ 3 & -1 & -2 \\ -1 & 5 & 0 \end{pmatrix} = 10i + 2j + 14k$$

2

Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:

(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$

(b) $3\mathbf{A} - 2\mathbf{B}$

(c) $\mathbf{A} \cdot \mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

Answer

(a) $A + B = i + 2j + k \Rightarrow |A + B| = \sqrt{6}$

(b) $3A - 2B = 3i + 3j - 2j - 2k \Rightarrow 3A - 2B = 3i + j - 2k$

(c) $A.B = (1)(0) + (1)(1) + (0)(1) \Rightarrow A.B = 1$

(d) $A \times B = \begin{pmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow A \times B = i\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - j\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\Downarrow$

$A \times B = i - j + k \Rightarrow |A \times B| = \sqrt{3}$

3

For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

Answer

$$A = iq + 3j + k \qquad B = iq - qj + 2k$$

$$\mathrm{A.B} = 0$$

$$(iq + 3j + k).(iq - qj + 2k) = 0 \Rightarrow q^2 - 3q + 2 = 0$$

$$(q - 2)(q - 1) = 0 \Rightarrow \boxed{\begin{array}{l} q = 2 \\ q = 1 \end{array}}$$

Extra HW

For what values of a are the vectors $\mathbf{A} = 2a\mathbf{i} - 2\mathbf{j} + a\mathbf{k}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 2\mathbf{k}$ perpendicular?

Answer

$$a = 0, a = 1$$

Extra HW

Two position vectors are expressed in Cartesian coordinates as $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Find the magnitude of the vector from point A to point B, the angle θ between $\mathbf{A}$ and $\mathbf{B}$, and the component of $\mathbf{B}$ in the direction of $\mathbf{A}$.

Answer

$$B - A = (4i + 2j - 3k) - (i + 2j - 2k) \Rightarrow B - A = 3i - k$$

$$|B - A| = \sqrt{9 + 1} = \sqrt{10}$$

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{(i + 2j - 2k).(4i + 2j - 3k)}{3\sqrt{29}} = 0.867$$

$$\boxed{\theta = 30^o}$$

HW

Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of $30°$.

Answer

$$M = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad M = \begin{pmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$A_{x'} = \left(\frac{\sqrt{3}}{2}\right)(2) + \left(\frac{1}{2}\right)(3) + (0)(-1) \quad \Rightarrow \quad A_{x'} = \frac{2\sqrt{3}+3}{2} = 3.232$$

$$A_{y'} = \left(-\frac{1}{2}\right)(2) + \left(\frac{\sqrt{3}}{2}\right)(3) + (0)(-1) \quad \Rightarrow \quad A_{y'} = \frac{-2+3\sqrt{3}}{2} = 1.598$$

$$A_{z'} = (0)(2) + (0)(3) + (1)(-1) \quad \Rightarrow \quad A_{z'} = -1$$

$$\boxed{A = 3.232\, i' + 1.598\, j' - k'}$$

HWs

1

A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \cos \omega t + \mathbf{j}2b \sin \omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

2

A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

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Answer

$$v = \frac{dr}{dt} = -ib\omega\sin\omega t + j2b\omega\cos\omega t$$

$$|v| = (b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t)^{1/2}$$

$t = 0 \quad \Rightarrow \quad |v| = 2b\omega$

$t = \dfrac{\pi}{2\omega} \quad \Rightarrow \quad |v| = b\omega$

2

A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

$$v = \frac{dr}{dt} = ib\omega \cos \omega t - jb\omega \sin \omega t + 2kct$$

$$a = \frac{dv}{dt} = -ib\omega^2 \sin \omega t - jb\omega^2 \cos \omega t + 2kc$$

$$|a| = (b^2\omega^4 \sin^2 \omega t + b^2\omega^4 \cos^2 \omega t + 4c^2)^{1/2}$$

$$|a| = (b^2\omega^4 + 4c^2)^{1/2}$$

Constant

HWs

1

A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v} \cdot \mathbf{a}/va$.)

2

An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

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$$\boxed{v = \dot{r}e_r + r\dot{\theta}e_\theta}$$

$$\boxed{a = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta}$$

$$\dot{r} = bke^{kt} \Rightarrow \ddot{r} = bk^2e^{kt}$$

$$\dot{\theta} = c \Rightarrow \ddot{\theta} = 0$$

$$\boxed{v = bke^{kt}e_r + bce^{kt}e_\theta}$$

$$\boxed{a = b(k^2 - c^2)e^{kt}e_r + 2bcke^{kt}e_\theta}$$

$$v.a = b^2k(k^2 - c^2)e^{2kt} + 2b^2c^2ke^{2kt}$$

$$|v| = be^{kt}(k^2 + c^2)^{1/2}$$

$$|a| = be^{kt}((k^2 - c^2)^{\frac{1}{2}} + 4c^2k^2)^{1/2}$$

$$\cos\theta = \frac{v.a}{|v||a|} \quad \Longrightarrow \quad \cos\theta = \frac{k}{(k^2 + c^2)^{1/2}}$$

Constant

2

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$$v = e_r \dot{r} + e_\phi r \dot{\phi} \sin\theta + e_\theta r \dot{\theta}$$

$$\dot{r} = 0 \qquad \dot{\phi} = \omega \qquad \dot{\theta} = \frac{\pi}{2}[-\omega \sin 4\omega t]$$

$$|\bar{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$