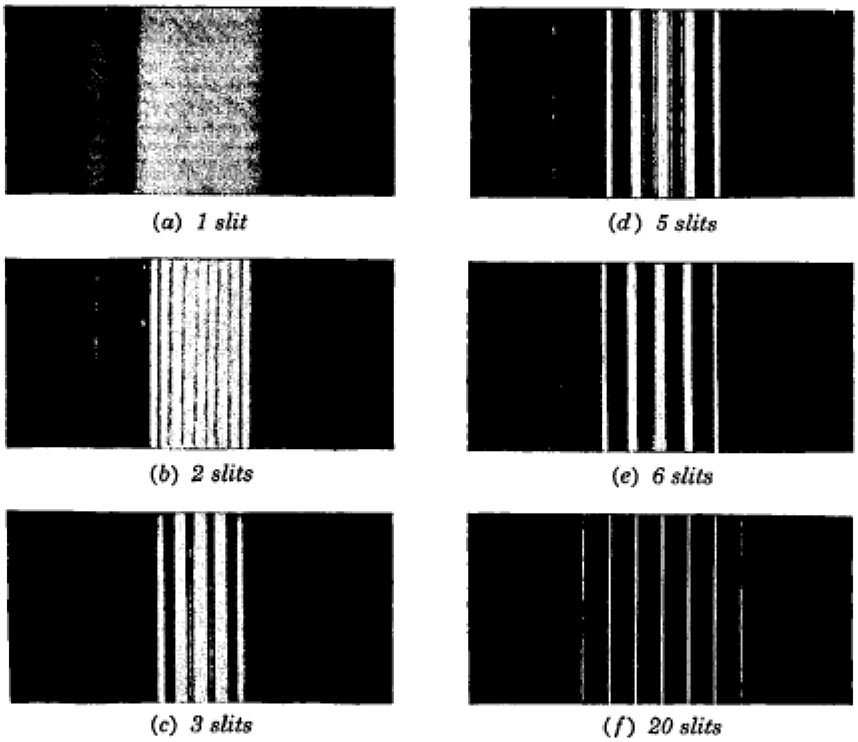
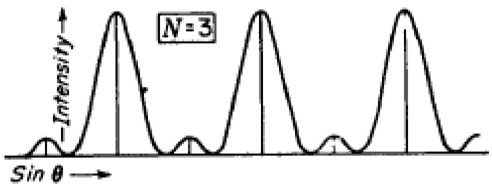


Figure 17A Fraunhofer diffraction patterns for gratings containing different numbers .of slits



.Figure 17B Principal and secondary maxima from a grating of three slits

Intensity Distribution From an Ideal Grating

The procedure used for the single and double slits could be used here, performing the integration over the clear aperture of the slits, but it becomes cumbersome. Instead let us apply the more powerful method of adding the complex amplitudes. The situation is simpler than in the case of multiple reflections, because for the grating the amplitudes contributed by the individual slits are all of equal magnitude. We designate this magnitude by a and the number of slits by N. The phase will change by equal amounts δ from one slit to the next; so the resultant complex amplitude is the sum of the series

$$Ae^{i\theta} = a(1 + e^{i\delta} + e^{i2\delta} + e^{i3\delta} + \cdots + e^{i(N-1)\delta}) = a\frac{1 - e^{iN\delta}}{1 - e^{i\delta}} \qquad (17a)$$

To find the intensity, this expression must be multiplied by its complex conjugate

$$A^2 = a^2 \frac{(1 - e^{iN\delta})(1 - e^{-iN\delta})}{(1 - e^{i\delta})(1 - e^{-i\delta})} = a^2 \frac{1 - \cos N\delta}{1 - \cos \delta}$$

Using the trigonometric relation 1 - cos α = 2 sin^2 *(α /2),* we may then write

$$A^2 = a^2 \frac{\sin^2 (N\delta/2)}{\sin^2 (\delta/2)} = a^2 \frac{\sin^2 N\gamma}{\sin^2 \gamma} \qquad (17b)$$

where, as in the double slit, $\gamma=\delta/2 = (\pi /\lambda)d$ sin θ. Now the factor a^2 represents the intensity diffracted by a single slit, and after inserting its value from Eq. (15d) we finally obtain for the intensity in the Fraunhofer pattern of an ideal grating

$$I \approx A^2 = A_0{}^2 \frac{\sin^2 \beta}{\beta^2} \frac{\sin^2 N\gamma}{\sin^2 \gamma} \qquad (17c)$$

Upon substitution of $N = 2$ in this formula, it readily reduces to Eq. (16c) for the double slit.

PRINCIPAL MAXIMA

The new factor (sin^2 Nγ)/(sin$^2\gamma$) may be said to represent the *interference* term for N slits. It possesses maximum values equal to N^2 for $\gamma = 0, \pi, 2\pi, \ldots,$ Although the quotient becomes indeterminate at these values, this result can be obtained by noting that

$$\lim_{\gamma \to m\pi} \frac{\sin N\gamma}{\sin \gamma} = \lim_{\gamma \to m\pi} \frac{N \cos N\gamma}{\cos \gamma} = \pm N$$

These maxima correspond in position to those of the double slit, since for the above values of γ

$$d \sin \theta = 0, \lambda, 2\lambda, 3\lambda, \ldots = m\lambda \qquad \textit{Principal maxima} \qquad (17d)$$

They are more intense, however, in the ratio of the square of the number of slits. The relative intensities of the different orders *m* are in all cases governed by the single slit diffraction envelope $(\sin^2 \beta)/\beta^2$, Hence the relation between β and γ in terms of slit width and slit separation [Eq. (l6d)] remains unchanged, as does the condition for missing orders [Eq. (16h)].

Minima and Secondary Maxima

To find the minima of the function $(\sin^2 N\gamma)/(\sin^2\gamma)$, we note that the numerator becomes zero more often than the denominator, and this occurs at the values $N\gamma = 0$, , π , 2π , ... or, in general, $p\pi$. In the special cases when $p = 0, N, 2N, \ldots$, γ will be *0,* 0, , π , 2π... ; so for these values the denominator will also vanish, and we have the principal maxima described above. The other values of *p* give zero intensity, since for these the denominator does not vanish at the same time. Hence the condition for a minimum is $\gamma = p\pi/N$, excluding those values of *p* for which $p = mN$, *m* being the order. These values of γ correspond to path differences

$$d \sin\theta = \frac{\lambda}{N}, \frac{2\lambda}{N}, \frac{3\lambda}{N}, \ldots, \frac{(N-1)\lambda}{N}, \frac{(N+1)\lambda}{N}, \ldots \qquad \textit{Minima} \qquad (17e)$$

omitting the values 0, $N\lambda/N$, $2N\lambda/N$, ... , for which $d \sin\theta) = m\lambda$ and which according to Eq. (l7d) represent principal maxima. Between two adjacent principal maxima there will hence be $N - 1$ points of zero intensity. The two minima on either side of a principal maximum are separated by twice the distance of the others.

Between the other minima the intensity rises again, but the secondary maxima thus produced are of much smaller intensity than the principal maxima. Figure 17C shows a plot for six slits of the quantities $\sin^2 N\gamma$ and $\sin^2\gamma$, and also of their quotient, which gives the intensity distribution in the interference pattern. The intensity of the principal maxima is N^2 or 36, so that the lower figure is drawn to a smaller scale. The intensities of the secondary maxima are also shown. These secondary maxima are not of equal intensity but fall off as we go out on either side of each principal maximum. Nor are they in general equally spaced, the lack of equality being due to the fact that the maxima are not quite symmetrical. This lack of symmetry is greatest for the secondary maxima immediately adjacent to the principal maxima, and is such that the secondary maxima are slightly shifted toward the adjacent principal maximum.

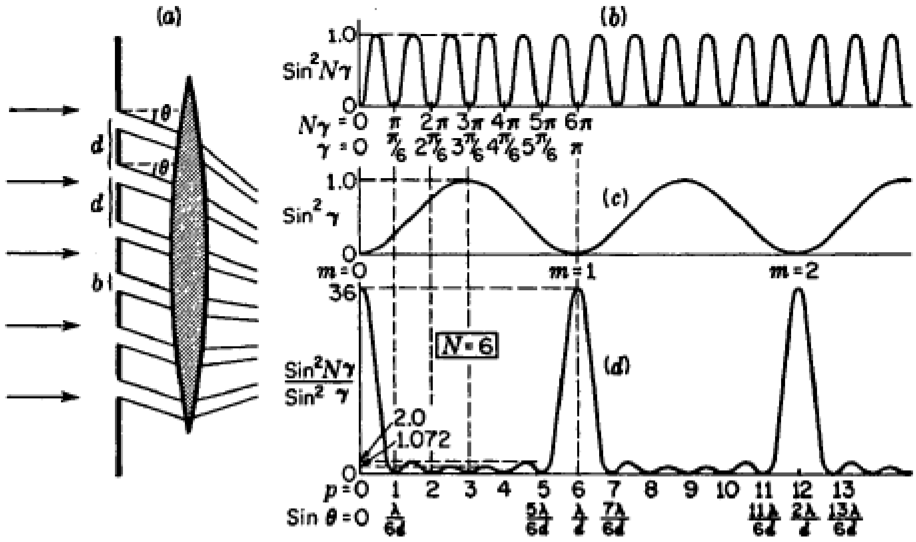


Figure 17C Fraunhofer diffraction by a grating of six very narrow slits and details of the intensity pattern.

These features of the secondary maxima show a strong resemblance to those of the secondary maxima in the *single-slit* pattern. Comparison of the central part of the intensity pattern in Fig. *17C(d)* with Fig. 150 for the single slit will emphasize this resemblance. As. the number of slits is increased, the number of secondary maxima is also increased, since it is equal to N - 2. At the same time the resemblance of any principal maximum and its adjacent secondary maxima to the single-slit pattern increases. In Fig. 170 is shown the interference curve for $N = 20$, corresponding to the last photograph shown in Fig.17A. In this case there are 18secondary maxima between each pair of principal maxima, but only those fairly close to the principal maxima appear with appreciable intensity, and even these are not sufficiently strong to show in the photograph. The agreement with the single-slit pattern is here practically complete.

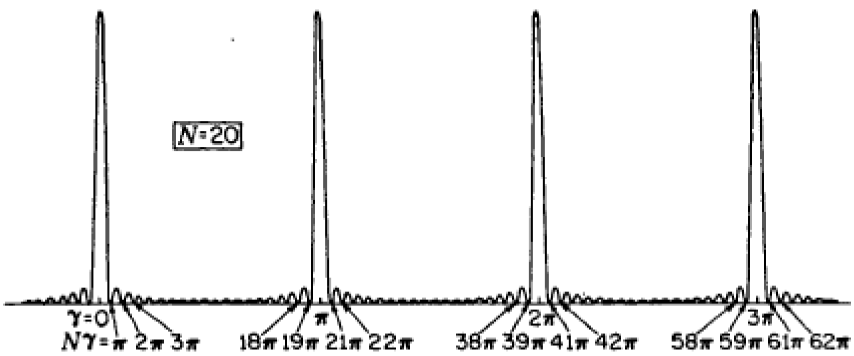


Figure 17D Intensity pattern for 20 narrow slits.

Resolving Power of the Diffraction Grating

The diffraction grating is useful for measuring wavelengths accurately. Like the prism, the diffraction grating can be used to separate white light into its wavelength components. Of the two devices, a grating with very small slit separation is more precise if one wants to distinguish two closely spaced wavelengths.
For two nearly equal wavelengths λ_1 and λ_2 between which a diffraction grating can just barely distinguish, the resolving power R of the grating is defined as

$$R \equiv \frac{\lambda}{\lambda_2 - \lambda_1} = \frac{\lambda}{\Delta\lambda} \qquad (38.11)$$

Resolving power

where $\lambda = (\lambda_1 + \lambda_2)/2$ and $\Delta\lambda = \lambda_2 - \lambda_1$. Thus, a grating that has a high resolving
power can distinguish small differences in wavelength. If N slits of the grating are illuminated, it can be shown that the resolving power in the mth-order diffraction is

$$R = Nm \qquad (38.12)$$

Resolving power of a grating

Thus, resolving power increases with increasing order number and with increasing number of illuminated slits.
Note that $R = 0$ for $m = 0$; this signifies that all wavelengths are indistinguishable for the zeroth-order maximum. However, consider the second-order diffraction pattern ($m = 2$) of a grating that has 5 000 rulings illuminated by the light source. The resolving power of such a grating in second order is R =5 000 x 2 = 10 000. Therefore, for a mean wavelength of, for example, 600 nm, the minimum wavelength separation between two spectral lines that can be just resolved is $\Delta\lambda = \lambda/R = 6 \times 10^{-2}$ nm. For the thir dorder principal maximum, R = 15 000 and $\Delta\lambda = 4 \times 10^{-2}$ nm, and so on.

Example : **Resolving Sodium Spectral Lines**

When a gaseous element is raised to a very high temperature, the atoms emit radiation having discrete wavelengths. The set of wavelengths for a given element is called its *atomic spectrum* . Two strong components in the atomic spectrum of sodium have wavelengths of 589 nm and 589.59 nm.

(A) What resolving power must a grating have if these wavelengths are to be distinguished?

Solution Using Equation 38.11,