**University of Basrah**

**College of Engineering**

**Materials Engineering Department**

# Mathematics I

*1st Year / 1st Semester*

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## Subjects:

- The Cartesian Plane and Functions
- The Limits and Continuity
- Differentiation
- Applications of Differentiation
- Integration
- Inverse Functions

- The Cartesian Plane and Functions

## The Real Number System

Real Numbers
(all numbers are real)

### Rational Numbers (Q)

**Integers Numbers (Z)**

**Whole Numbers (W)**

**Natural Numbers (N)**

……., -3, -2, -1, 0, 1, 2, 3, ………

### Irrational Numbers ($\bar{Q}$)

Any number that is not rational

Example:

$\pi$ = 3.1415….

e = 2.7182….

$\sqrt{3}$= 1.732….

- The fractions numbers are Rational Numbers such as $\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{7}{3}, \frac{9}{5}, \frac{4}{5}, \frac{7}{3}, \ldots\ldots$

- The decimal fractions numbers such as 3.54, 0.64, 75.432, 0.643

  - Not all decimal can be Rational Numbers.

  - The non-terminating (لانهائي) and non-recurring (غير متكرر) decimal be irrational Numbers.

❖ Example 1 complete using one of the symbols Q or $\bar{Q}$:

1. 8 is ..Q....
   non-terminating ✗
   non-recurring ✗

2. $\frac{7}{6}$ is ..Q.... (1.16666........)
   non-terminating √
   non-recurring ✗

3. $\sqrt{3}$ is ..$\bar{Q}$.... (1.73205........)
   non-terminating √
   non-recurring √

4. $\sqrt{9}$ is ..Q.... (3)
   non-terminating ✗
   non-recurring ✗

5. -5 is ..Q....
   non-terminating ✗
   non-recurring ✗

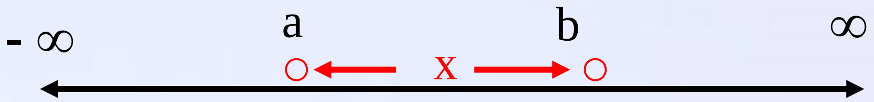
6. $\sqrt[3]{8}$ is ..Q.... (2)
   non-terminating ✗
   non-recurring ✗

# Point Sets and Intervals

A set of points (real numbers) located on the real axis is called a one-dimensional point set.

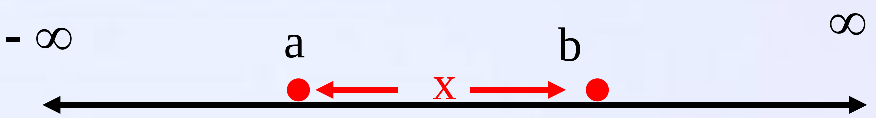
The Interval is a set of all real numbers between two points on the real number line.

1. Open interval



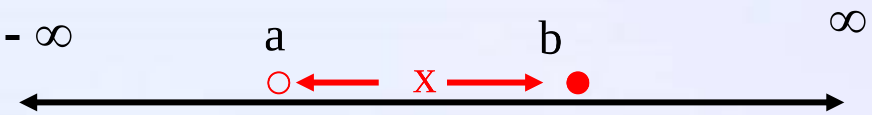

The set a < x < b is denoted by (a, b).

2. Close interval




The set a ≤ x ≤ b is denoted by [a, b].

3. Half- open interval

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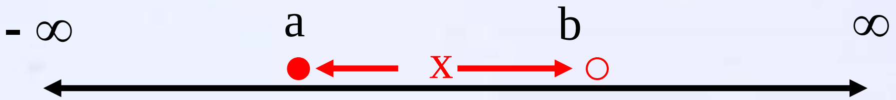



The set a ≤ x < b is denoted by [a, b).

The set consisting of all elements which belong to A or B (or both) is often called the union of A and B, denoted by $A \cup B$.

The set consisting of all elements which are contained in both A and B is called the intersection of A and B, denoted by $A \cap B$.

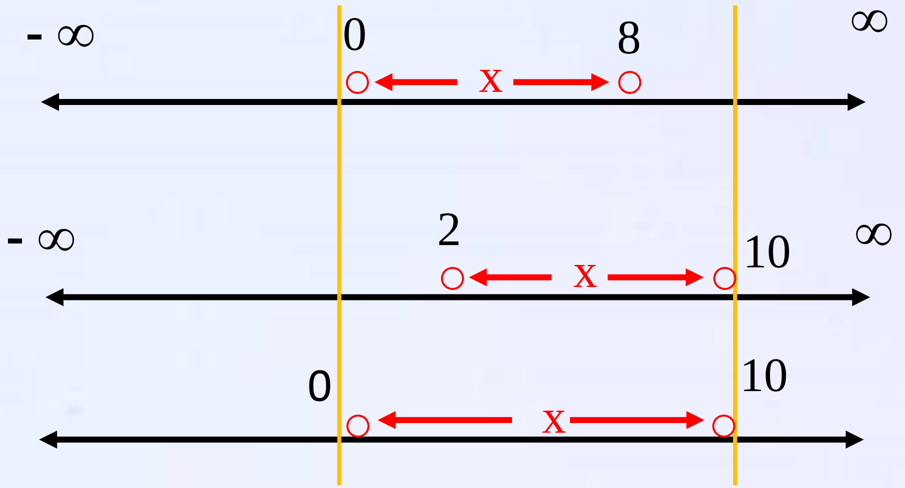
❖ Example 2 Find the interval of x: $\{x: 0 < x < 8\} \cup \{x: 2 < x < 10\}$

Solve:-

$\{x: 0 < x < 8\}$ or $(0, 8)$

$\{x: 2 < x < 10\}$ or $(2, 10)$

$(0,8) \cup (2, 10)$




The interval of x is $\{0 < x < 10\}$ or $(0, 10)$

**Inequalities**

if a, b & c are real numbers, then

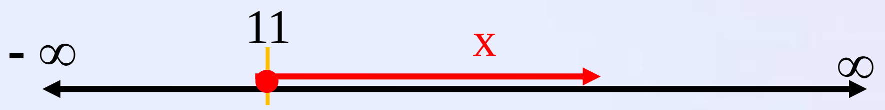
1) $a < b \Rightarrow a+c < b+c$
2) $a < b \Rightarrow a-c < b-c$
3) $a < b$ & $c > 0 \Rightarrow a*c < b*c$
4) $a < b \Rightarrow -a > -b$
5) $a > 0 \Rightarrow 1/a > 0$
6) If a & b are both positive or negative numbers, then
7) $a < b \Rightarrow 1/a > 1/b$

❖ Example 3 Solve the following inequalities solution set:

1. $4x+3 \le 5x-8$

Solve:

$4x-5x \le -8-3 \longrightarrow -x \le -11 \longrightarrow x \ge 11$

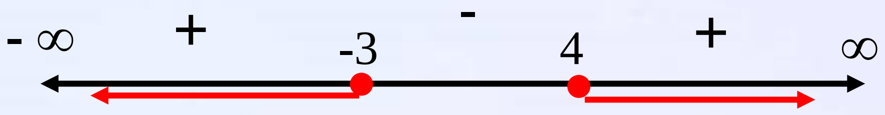



The solution set is $[11, \infty)$

2. $x^2-x \ge 12$

Solve:

$x^2-x-12 \ge 0 \longrightarrow (x-4)(x+3) \ge 0$




$x \le -3$ or $x \ge 4$

The solution set is $(-\infty, -3] \cup [4, \infty)$

Assignment:

❖ Find the interval of:-

1. $x > 10$    2. $4 < x \leq 11$    3. $x \leq 3$ or $x > 5$

❖ Solve the following inequalities solution set:

1. $x^2 + 9 > 6x$    2. $2x^2 > x + 6$    3. $x^2 - 3x > 10$    4. $\frac{(x-4)(x+1)}{x-3} < 0$

- Absolute value :

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

Example 4

$|-5| = 5,$ $\quad |+2| = 2,$ $\quad \left|-\frac{4}{7}\right| = \frac{4}{7},$ $\quad |-\sqrt{3}| = \sqrt{3},$ $\quad |0| = 0$

Absolute value properties:

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4. $|a + b| \leq |a| + |b|$

if $a$ is any positive number, then:

1. $|x| = a$ if and only if $x = \pm a$
2. $|x| < a$ if and only if $-a < x < a$
3. $|x| > a$ if and only if $x > a$ or $x < -a$
4. $|x| \leq a$ if and only if $-a \leq x \leq a$
5. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

❖ Example 5  Solving an equations with absolute values:

*1.* $|2x - 3| = 7$

**Solution:**

$$2x - 3 = \pm 7 \longrightarrow 2x - 3 = 7 \longrightarrow 2x = 10 \longrightarrow x = 5$$

$$\phantom{2x - 3 = \pm 7} \longrightarrow 2x - 3 = -7 \longrightarrow 2x = -4 \longrightarrow x = -2$$

2. Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$

**Solution:**

$$-1 < 5 - \frac{2}{x} < 1 \longrightarrow -6 < -\frac{2}{x} < -4 \longrightarrow \left[-3 < -\frac{1}{x} < -2\right] \times -1 \longrightarrow 3 > \frac{1}{x} > 2$$

$$\frac{1}{3} < x < \frac{1}{2}$$

The inequality reverses:
1. Multiplying by a negative Number.
2. Taking reciprocals.

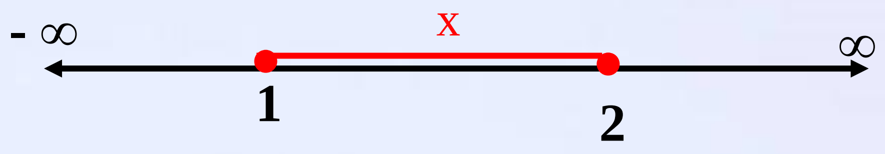
❖ Example 6  Solve the inequality and show the solution set on the real line:

1. $|2x - 3| \leq 1$

**Solution:**

$$-1 \leq 2x - 3 \leq 1 \longrightarrow 2 \leq 2x \leq 4 \longrightarrow 1 \leq x \leq 2$$

The solution set is the closed interval [1, 2]




2. $|2x - 3| \geq 1$

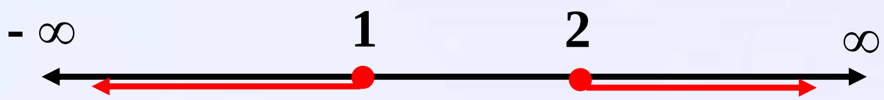
**Solution:**

$$2x - 3 \geq 1 \quad or \quad 2x - 3 \leq -1$$

$$x - \frac{3}{2} \geq \frac{1}{2} \quad or \quad x - \frac{3}{2} \leq -\frac{1}{2}$$

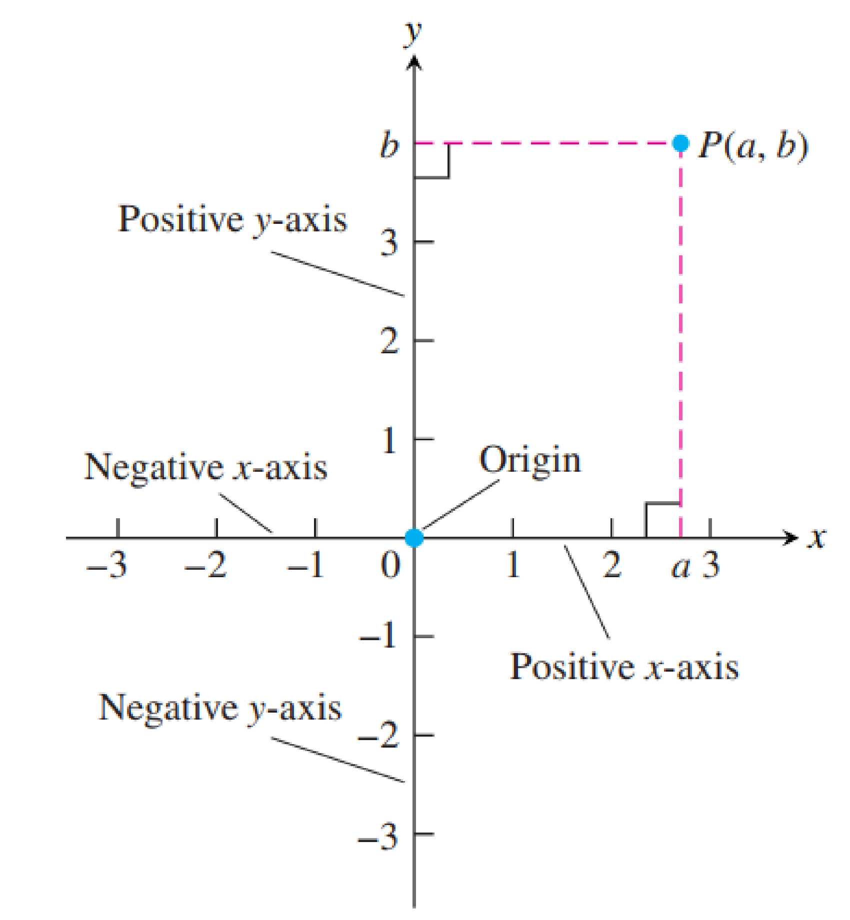
$$x \geq 2 \qquad\qquad x \leq 1$$

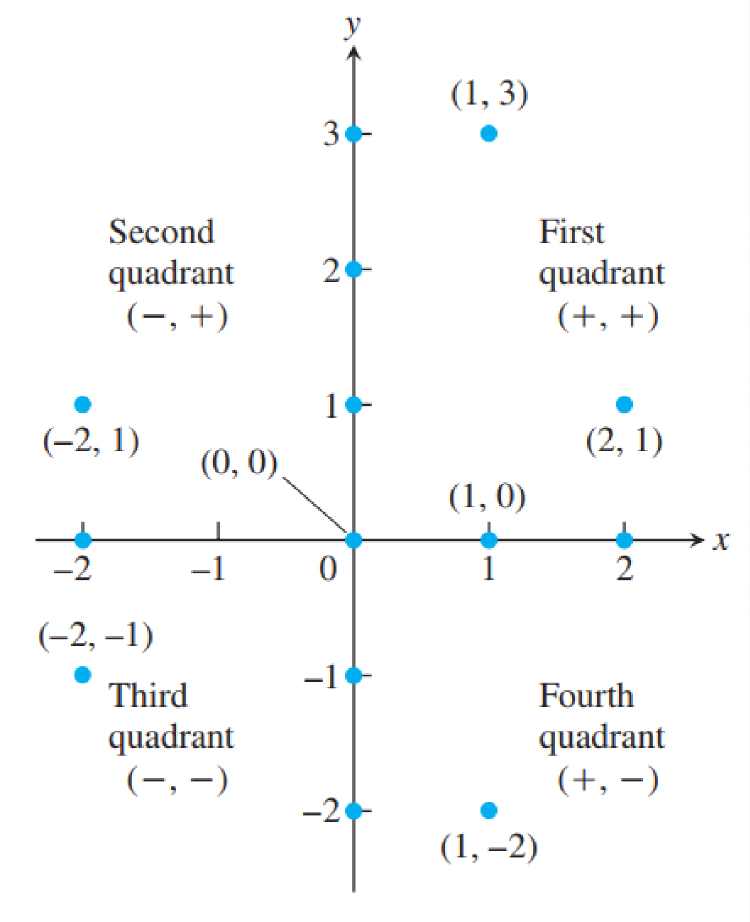
The solution set is (- ∞, 1]U[2, ∞)



- Analytical Geometry

 - If two perpendicular lines that intersected at the O- point, then these lines called the coordinate axes.




- This coordinate system is called the Rectangular system of Cartesian system.

- Slope of the line

The slope tells us the direction (uphill, downhill) and steepness of a line

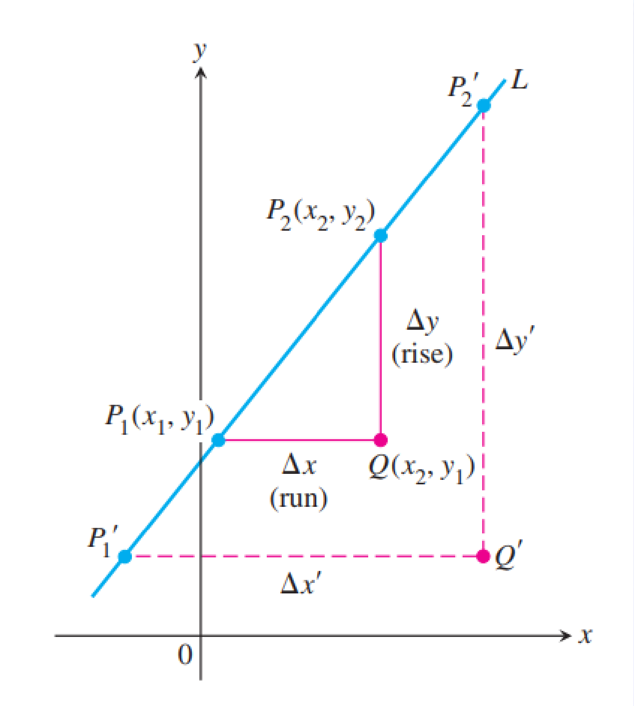
$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

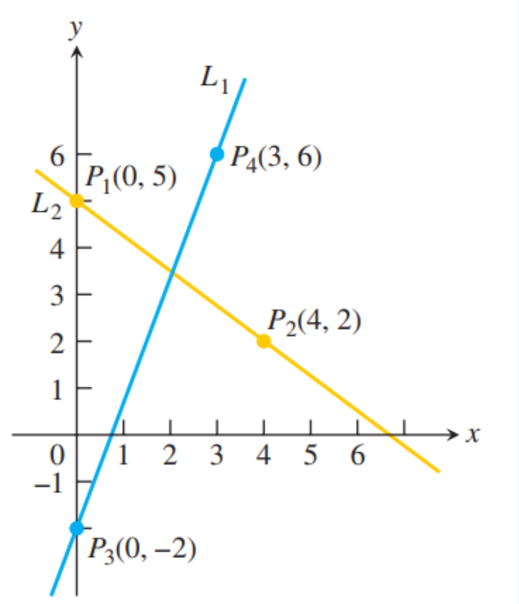
The slope of $L_1$ is $m = \frac{\Delta y}{\Delta x} = \frac{6-(-2)}{3-0} = \frac{8}{3}$

That is, y increases 8 units every time x increases 3 units.

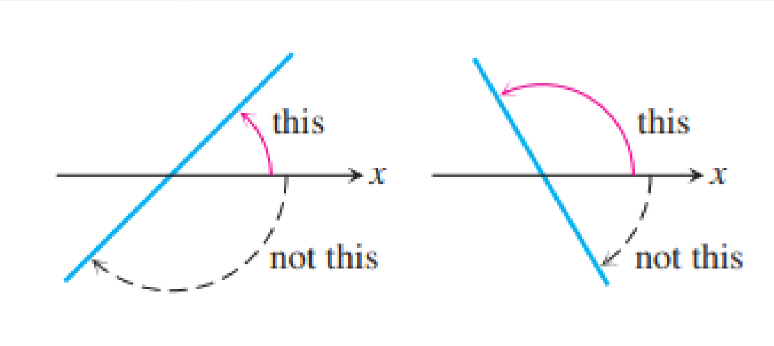
The slope of $L_2$ is $m = \frac{\Delta y}{\Delta x} = \frac{2-5}{4-0} = \frac{-3}{4}$

That is, y decreases 3 units every time x increases 4 units.



- The **angle of inclination (∅)** define the direction and steepness of a line with an angle. The angle of inclination of a line is:




- ✓ The smallest counterclockwise angle from the *x*-axis to the line.
- ✓ Horizontal line is 0°
- ✓ Vertical line is 90°.
- ✓ $0 \leq \emptyset < 180$
- ✓ $m = tan\emptyset$

- **The point-slope equation**

If the line that passes through the point and has slope *m*, the point-slope equation use:

$$y = y_1 + m(x - x_1)$$

❖ Example 7 Write an equation for the line Through the point (2, 3) with slope -3/2.

Solution:

$x_1 = 2$, $y_1 = 3$, m = -3/2

$$y = y_1 + m(x - x_1)$$

$$y = 3 + (-\frac{3}{2})(x - 2)$$

$$y = 3 - \frac{3}{2}x + 3$$

$$y = -\frac{3}{2}x + 6$$

❖ Example 8 Write an equation for the line through the points $P_1$ (-2, -1) and $P_2$ (3, 4).

Solution:

$x_1 = -2$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 4$

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - (-1)}{3 - (-2)} = \frac{5}{5} = 1$$

- Slope-Intercept equation

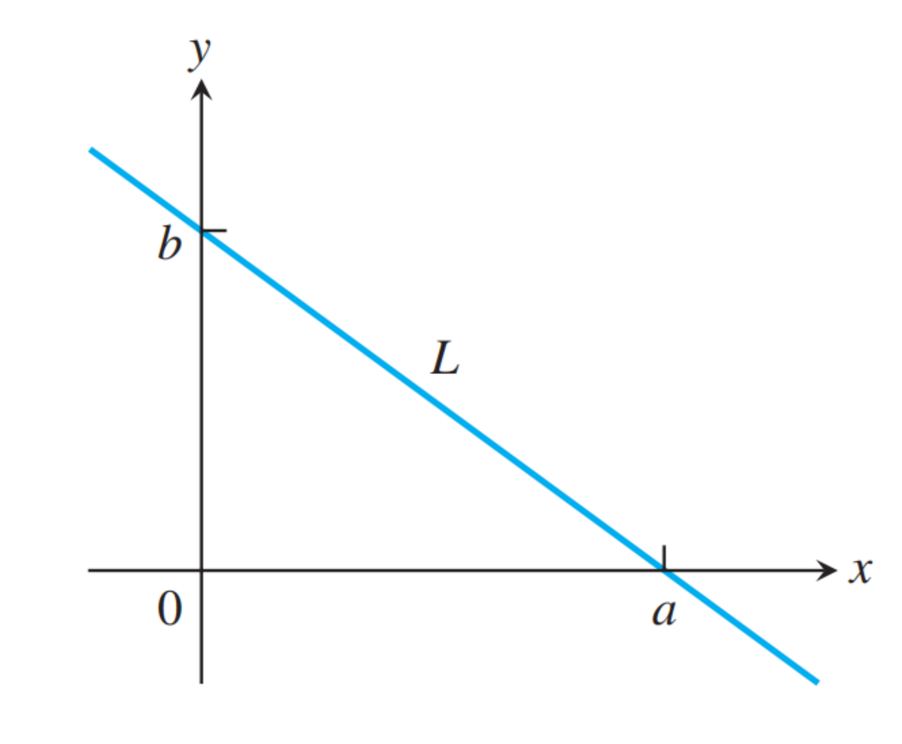
$$y = mx + b$$

m : slope of the line
b : intercept of the line with y-axis

- General linear equation

$$Ax + By + c = 0$$

Where A,B and C are constants and A and B not both 0.



❖ Example 9    Find the slop and y-intercept of the line $8x + 5y = 20$

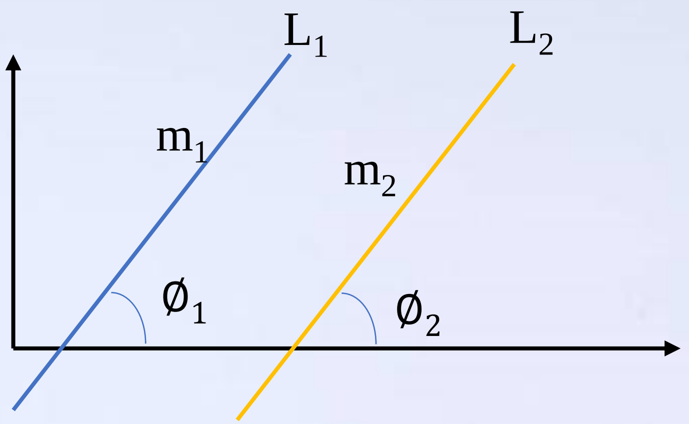
$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = \frac{-8}{5}x + 4$$

The slope m = $\frac{-8}{5}$. The y- intercept is b = 4

- Parallel and Perpendicular Lines

If two lines $L_1$ and $L_2$ are

I. Parallel lines, they have equal slopes ($m_1$=$m_2$) and equal angles of inclination ($\emptyset_1 = \emptyset_2$).

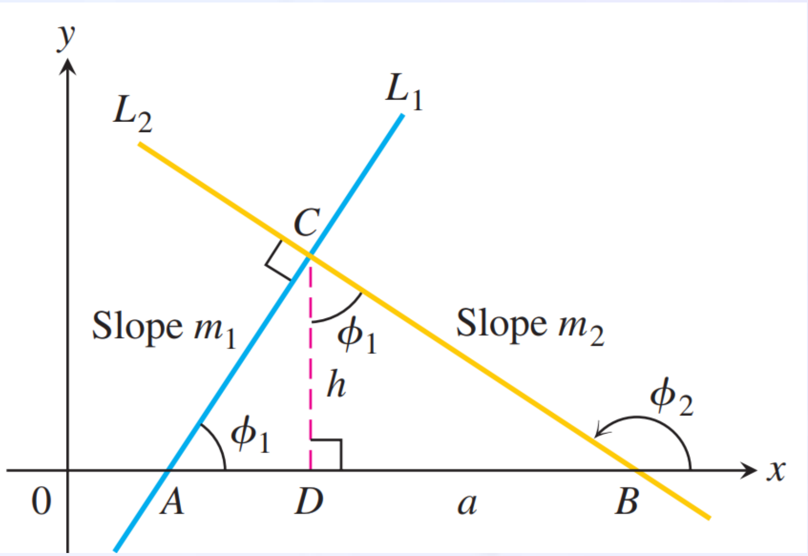



I. Perpendicular lines and nonvertical, they have each slope are the negative reciprocal of the other.

$$m_1 m_2 = -1, \qquad m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}$$

$m_1 = \frac{a}{h}$ ….. for angle ($\emptyset_1$)

$m_2 = -\frac{h}{a}$ ….for angle ($\emptyset_2$)



- Distance between two points

The distance between P $(x_1,y_1)$ and Q $(x_2,y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad = \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The mid-point M $(x_o,y_o)$ formula is:

$$x_o = \frac{x_1+x_2}{2}, yo = \frac{y_1+y_2}{2}$$

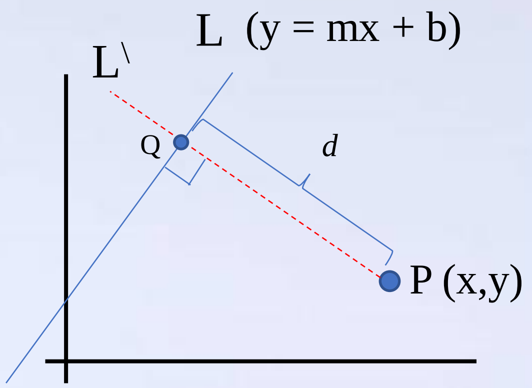
❖ Example 10 Find the distance and mid-point between P (-2,1) and Q (4,3)

Solution:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \longrightarrow d = \sqrt{(4-(-2))^2 + (3-1)^2} \longrightarrow d = \sqrt{(6)^2 + (2)^2} \longrightarrow d = 6.32$$

$$M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) \longrightarrow M = (\frac{-2+4}{2}, \frac{1+3}{2}) \longrightarrow M = \left(\frac{2}{2}, \frac{4}{2}\right) \longrightarrow M = (1,2)$$

- Distance from a point to a line




The distance (*d*) between a point (P) and a line is a perpendicular line segment (L).

1. Find the slope of the perpendicular line formed from the point (L\).

(Negative reciprocal from the given line)

2. Find the equation of the line with the shortest distance y = mx + b.

(Use the slope you found in step 1 and substitute the values of the point to find the b value)

3. Find the point of intersection of the two lines (Q) by solving the systems of two equations.

4. Find the length of the line segment by using the point of intersection from step 3 to the given point.

❖ Example 11 Find the distance from point P (-1,3) to the line (L) x + y – 5 = 0

Solution:

1. Find the slope of the line x + y – 5 = 0

y = -x + 5 (compare with y = mx + b)

m of given line (L) is -1, and m of the perpendicular line L\ =1.

2. Find the equation of the line L\.

$(y-y_1) = m\,(x-x_1) \xrightarrow{m = 1,\ P(-1,3)} (y-3) = 1\,(x-(-1))\ ) \rightarrow y-3 = x+1 \rightarrow y = x + 4$

3. Find the point of intersection of the two lines (Q).

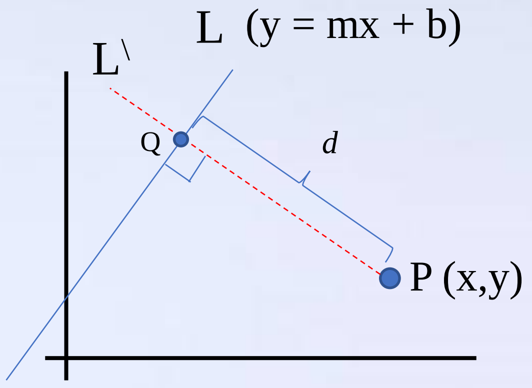
at Q x $_{\text{of L equation}}$ = x $_{\text{of L` equation}}$ and y $_{\text{of L equation}}$ = y $_{\text{of L` equation}}$

$y_L = yL_{`} \rightarrow -x+5 = x+4 \rightarrow x = \frac{1}{2}$

$find\ y\ by\ using\ \ eq\ of\ (L$ or L`) $\xrightarrow{L}$ x + y – 5 = 0 $\rightarrow y = -\frac{1}{2}+5 \rightarrow y = \frac{9}{2}$

$or\ x_L = x_{L`} \rightarrow y+5 = y-4 \rightarrow 2y = 5+4 \rightarrow y = \frac{9}{2}$

Q (1/2, 9/2)




4. Find the distance (*d*) between Q and P.

d= $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(\frac{1}{2}-(-1))^2+(\frac{9}{2}-3)^2}$

$= \sqrt{\frac{9}{4}+\frac{9}{4}} \rightarrow = \sqrt{\frac{18}{4}} \rightarrow d = 2.12$

Another solution:

From L equation x + y – 5 = 0 and general form Ax + By + C = 0

A = 1, B = 1 and C = -5. while x = -1 and y = 3, from given point P(-1,3)

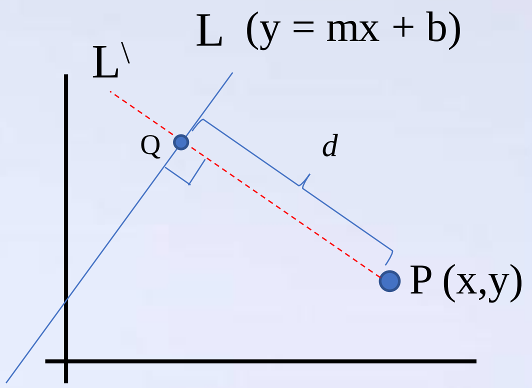
$$d = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}} \qquad d = \frac{|1(-1)+(1*3)+(-5)|}{\sqrt{1^2+1^2}}$$

$$d = \frac{|-3|}{\sqrt{2}} \qquad d = 2.12$$

❖ Example 12 Find the

i. Slope of the line -2x + 3y + 4 = 0

ii. Distance from point P (5,6) to the line -2x + 3y + 4 = 0




Solution:

i. Find the slope of the line -2x + 3y + 4 = 0

Put the equation in the form y = mx + b,

$$-2x + 3y + 4 = 0 \rightarrow y = \frac{2}{3}x - \frac{4}{3} \rightarrow m = \frac{2}{3}, b = -\frac{4}{3}$$

ii. Distance from P (5,6) to -2x + 3y + 4 = 0

1. Slope of perpendicular line (L\`) is $m = -\frac{3}{2}$

2. Find the equation of the line L\\.

$(y-y_1) = m\,(x-x_1) \xrightarrow{m = -3/2,\ P(5,6)} (y-6) = -3/2\,(x-5)$

y-6 = -3/2 x + 15/2 y = -3/2 x + 27/2

3. Find the point of intersection (Q).

$\frac{2}{3}x - \frac{4}{3} = -\frac{3}{2}x + 13.5$ ($y_{\text{of L equation}} = y_{\text{of L\` equation}}$)

$$\frac{2}{3}x + \frac{3}{2}x = \frac{4}{3} + \frac{27}{2} \rightarrow \frac{13}{6}x = \frac{89}{6} \rightarrow x = \frac{89}{13}$$

$$y = \left(-\frac{3}{2}\right)\left(\frac{89}{13}\right) + \frac{27}{2} \rightarrow y = -\frac{267}{26} + \frac{27}{2} \rightarrow y = -\frac{267}{26} + \frac{27}{2} \rightarrow y = \frac{168}{52} \rightarrow y = \frac{42}{13}$$

4. Find the distance ($d$) between Q and P.

P (5,6) and Q ($\frac{89}{13}$ , $\frac{42}{13}$)

$$d = \sqrt{(\frac{89}{13} - 5)^2 + (\frac{42}{13} - 6)^2}$$

$$= \sqrt{(\frac{89}{13} - \frac{65}{13})^2 + (\frac{42}{13} - \frac{78}{13})^2}$$

$$= \sqrt{\frac{576}{169} + \frac{1296}{169}} \qquad = \sqrt{\frac{1872}{169}}$$

$$d = 3.328$$

Another solution:

From L equation -2x + 3y + 4 = 0 and general form Ax + By + C = 0

A = -2, B = 3 and C = 4. while x = 5 and y = 6, from given point P(5,6)

$$d = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}} \qquad d = \frac{|-2(5)+(3*6)+(4)|}{\sqrt{(-2)^2+(3)^2}}$$

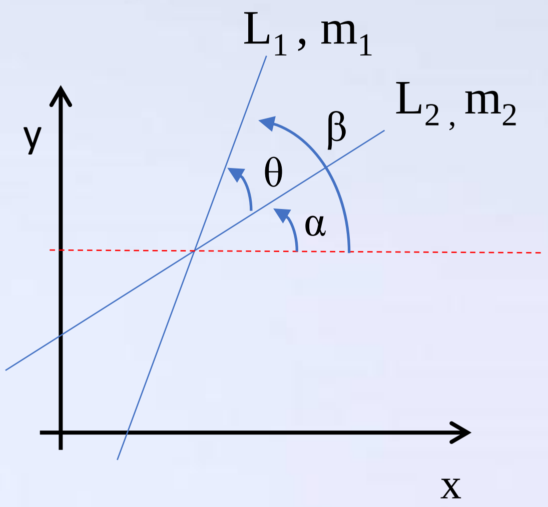
$$d = \frac{|12|}{\sqrt{13}} \qquad d = 3.328$$

- Angles between two lines

$$\vartheta = \beta - \alpha$$

$$tan\partial = \tan(\beta - \alpha)$$

$$tan\partial = \frac{tan\beta - tan\alpha}{1 + tan\beta . tan\alpha}$$




Where: $m_1$=tan β and $m_2$ = tan α

$$tan\partial = \frac{m_1 - m_2}{1 + m_1 m_2}$$

# Assignment:

1. By using different solutions, Find the

i. Slope of the line 2x + 3y = 5

ii. Distance from point P (-1,0) to the line 2x + 3y = 5

2. By using different solutions, Find the

i. Slope of the line 3x - 4y = -8

ii. Distance from point P (3, -2) to the line 3x - 4y = -8

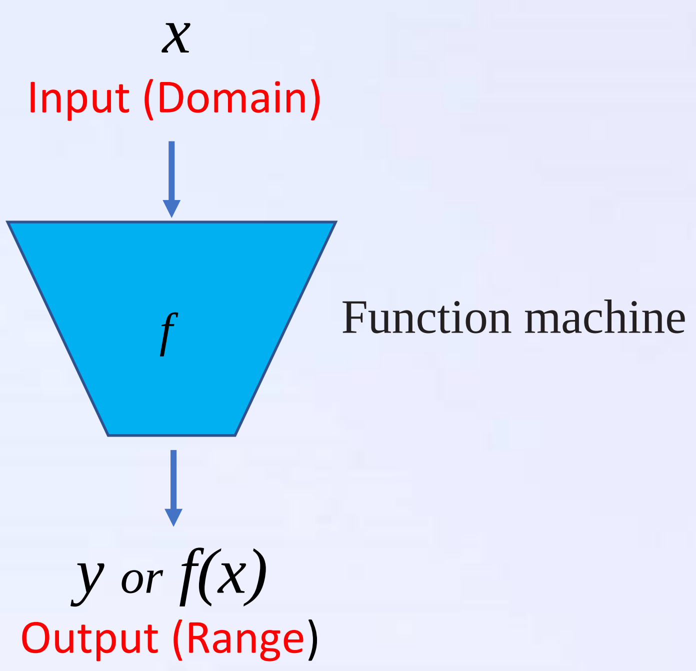
# Functions

The value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

Domain (D) : is the set of all possible inputs (x values)

Range : is the set of all possible outputs (y values)

A function from a set $D$ to a set $Y$ is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$



❖ Example 12 Verify the domains and ranges of these functions.

1. $y = x^2$ ,  2. $y = \frac{1}{x}$,  3. $y = \sqrt{1 - x^2}$

Solution:

1. $y = x^2$

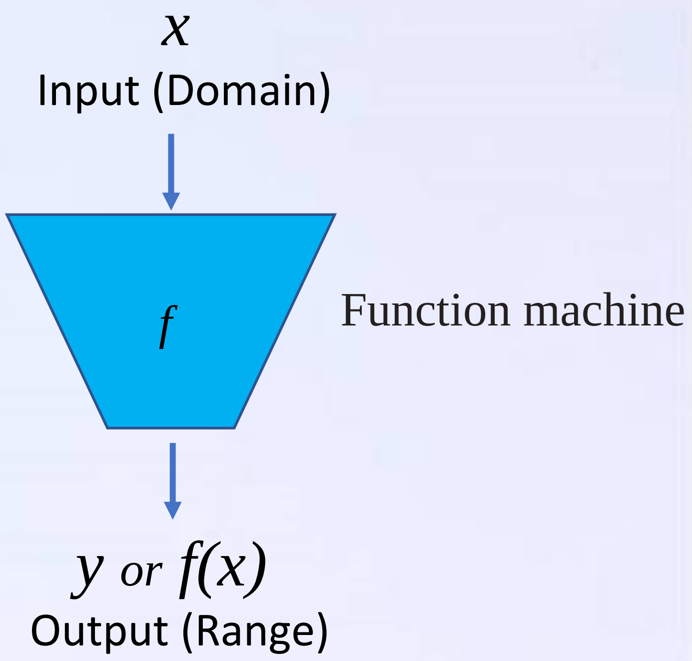
The formula gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = (\sqrt{y})^2$, for $y \geq 0$.

Or

$x = \sqrt{y}$

$y \geq 0$ So the range is $[0, \infty)$.

Note: The value under even root must be positive or zero.



2. $y = \frac{1}{x}$

Solution:

This formula gives a real y-value for every x except x = 0.

The domain and range are (-∞, 0)∪(0, ∞).

Or

$x = \frac{1}{y}$

$y \neq 0$   So the range is (-∞, 0)∪(0, ∞).

Note: The denominator must not be equal to zero

3. $y = \sqrt{1 - x^2}$

Solution:

$1 - x^2 \geq 0 \longrightarrow -1 \leq -x^2 \longrightarrow x^2 \leq 1$

$-1 \leq x \leq 1$

The domain is [-1, 1].

and range is  [0, 1].

Or

$y = \sqrt{1 - x^2} \longrightarrow x^2 = 1 - y^2 \longrightarrow x = \pm\sqrt{1 - y^2}$

$1 - y^2 \geq 0 \longrightarrow -y^2 \geq -1 \longrightarrow y^2 \leq 1$

$-1 \leq y \leq 1$

Because the $x^2$ , the y should be positive

$0 \leq y \leq 1$

- Graphs of Functions

Another way to visualize a function is its graph. If $f$ is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for $f$. In set notation, the graph is

$$\{(x, f(x))\ | x \in D\}$$

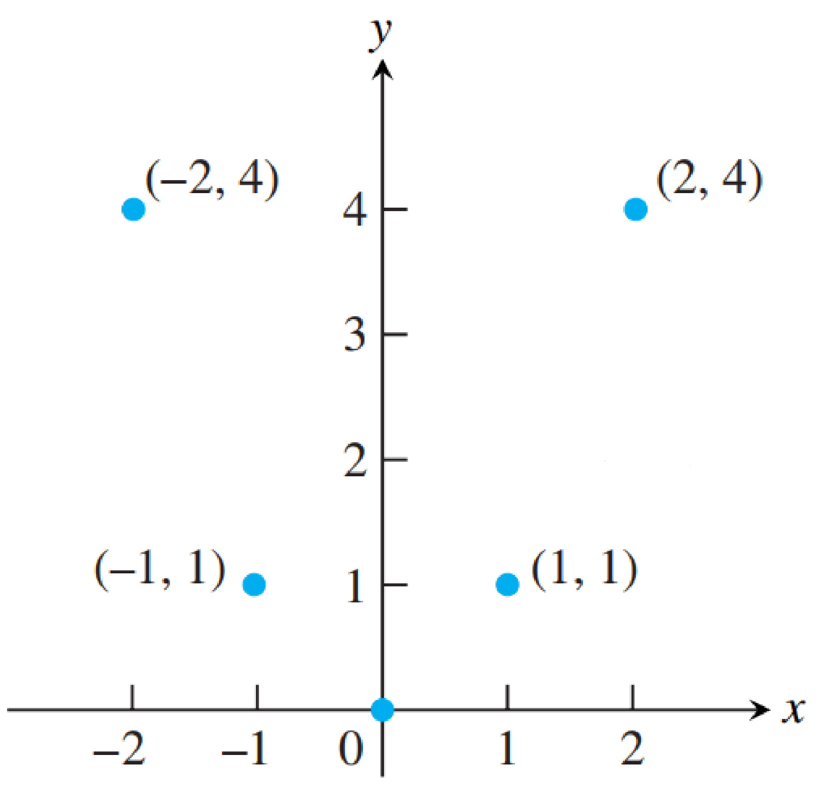
❖ Example 13 Sketching a graph of the function $y = x^2$ over the interval [-2, 2]

Solution:

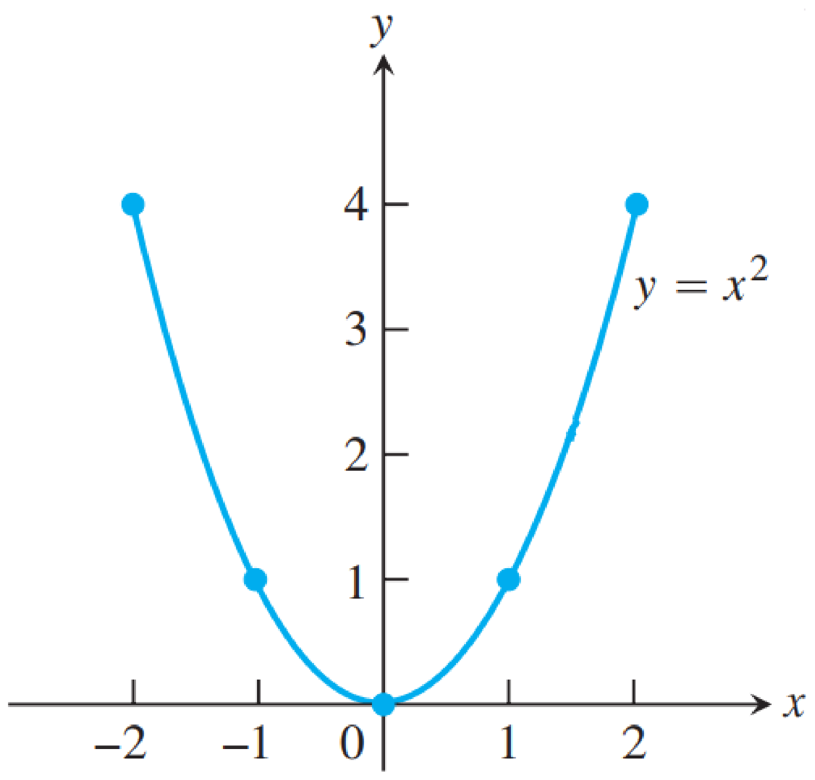
1. Make a table of *xy*-pairs that satisfy the function rule, in this case the equation $y = x^2$

| x | y = $x^2$ |
|---|---|
| -2 | 4 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

2. Plot the points (x, y) whose coordinates appear in the table.



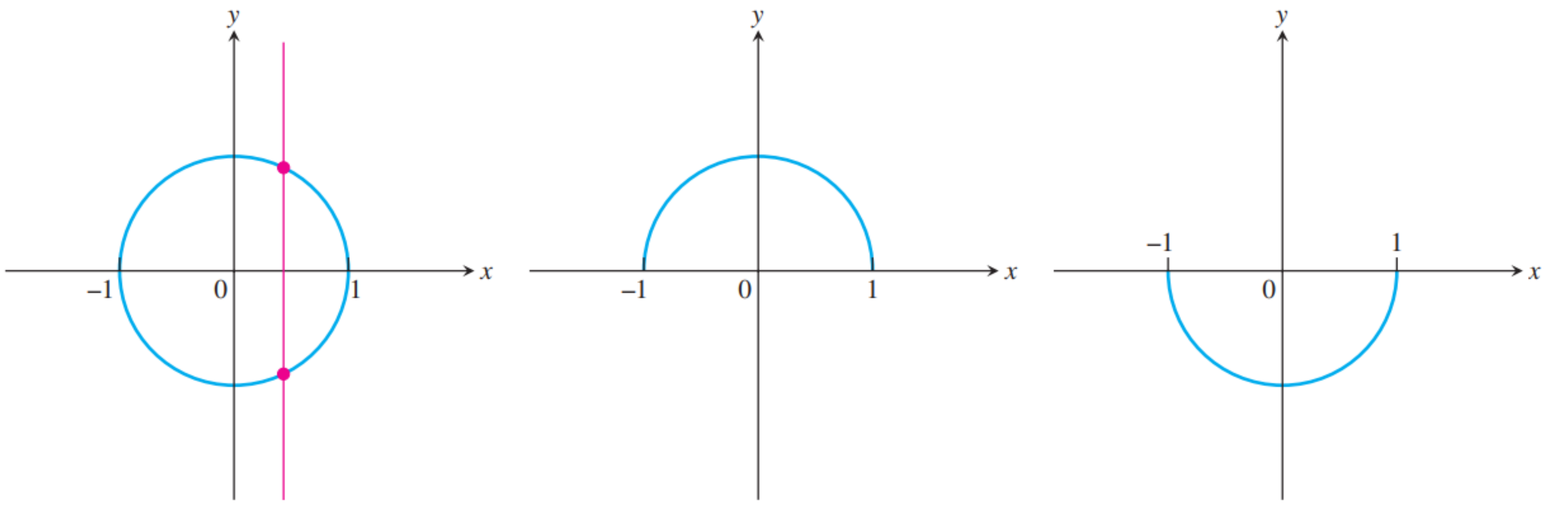

3. Draw a curve through the plotted points. Label the curve with its equation



- The Vertical Line Test

Not every curve you draw is the graph of a function. A function $f$ can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once.

- ✓ A circle cannot be the graph of a function since some vertical lines intersect the circle twice.
- ✓ If a is in the domain of a **function $f$**, then the vertical line will intersect the graph of $f$ in the single point (a, $f$(a)).



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

- Piecewise-Defined Functions

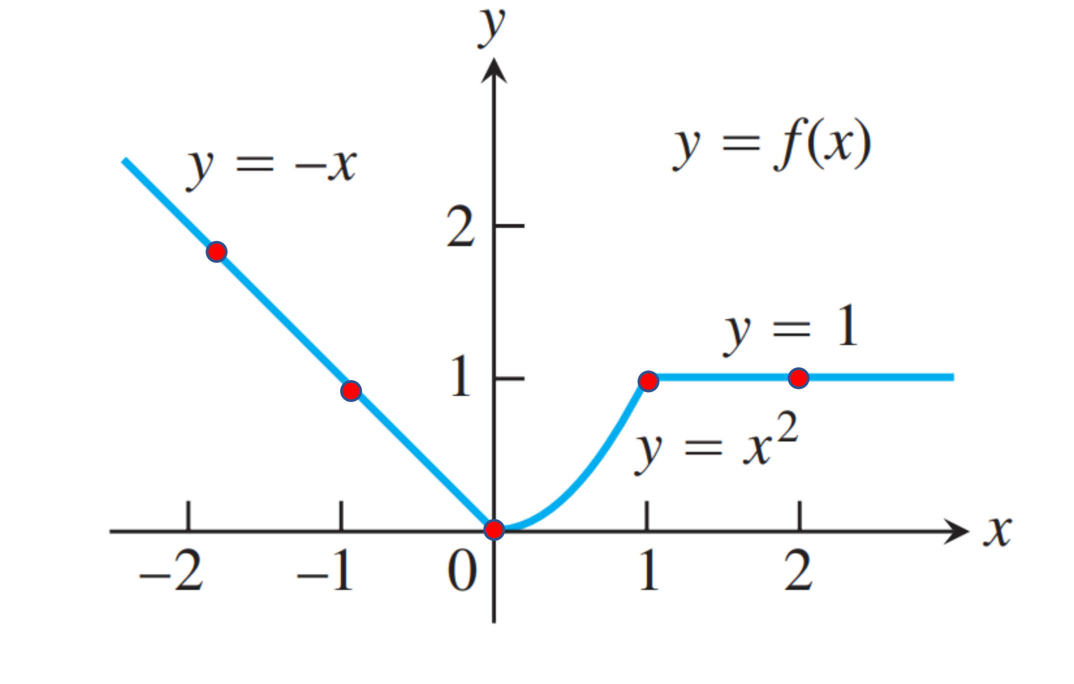
Sometimes a function is described by using different formulas on different parts of its domain.

❖ Example 14 graphing the piecewise-defined function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

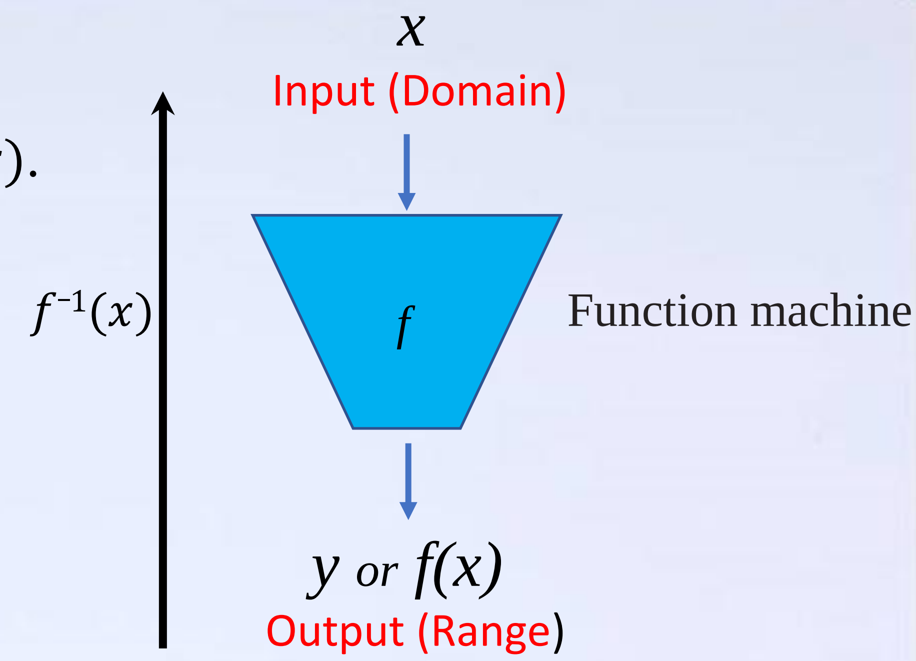
Solution:

| x | y = -x | y = x² | 1 |
|---|---|---|---|
| -2 | 2 | | |
| -1 | 1 | | |
| 0 | | 0 | |
| 1 | | 1 | |
| 2 | | | 1 |

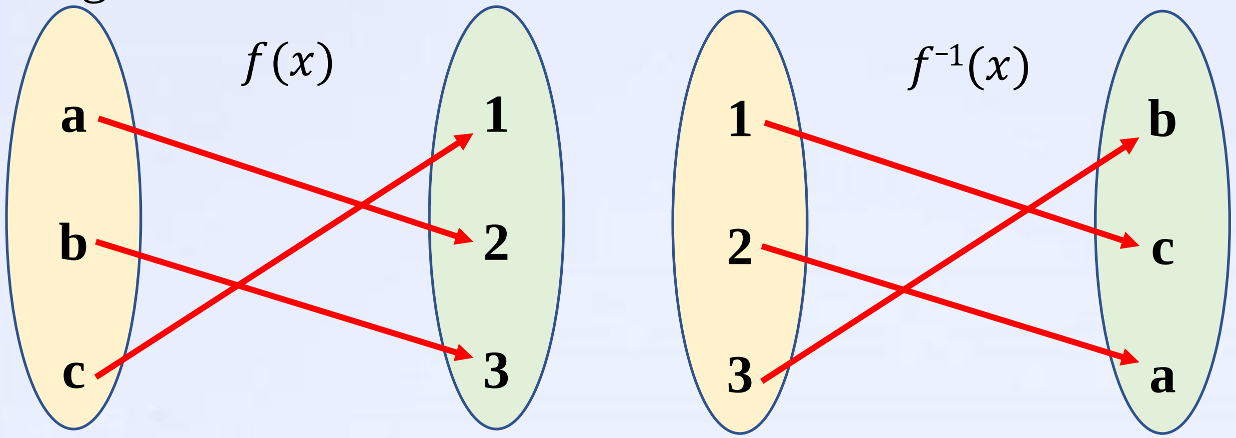


- Inverse functions

An inverse function $f^{-1}(x)$ is defined as the inverse function of $f(x)$.




In the figure below, the ordered pairs are reversed the original function to its reverse.

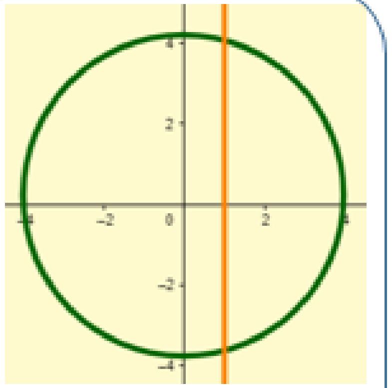



- An inverse function reverses the inputs and outputs.
- To find the inverse formula of a function, write it in the form of y and x, switch y and x, and then solve for y.
- Some functions have no inverse function, as a function cannot have **multiple outputs (one to one)**.
- $f^{-1}(x)$ graph is reflecting the $f(x)$ graph across the line y = x.

# Vertical and Horizontal Line Tests

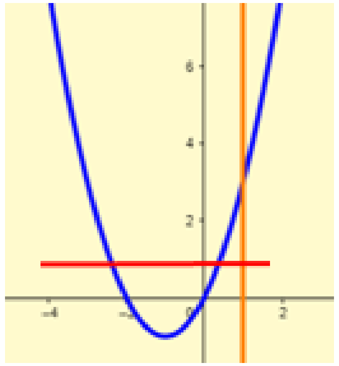
Pass Vertical Line Test: Any vertical line drawn will intersect the graph at only one point.

Pass Horizontal Line Test: Any horizontal line drawn will intersect the graph at only one point.



Fail Vertical Line Test

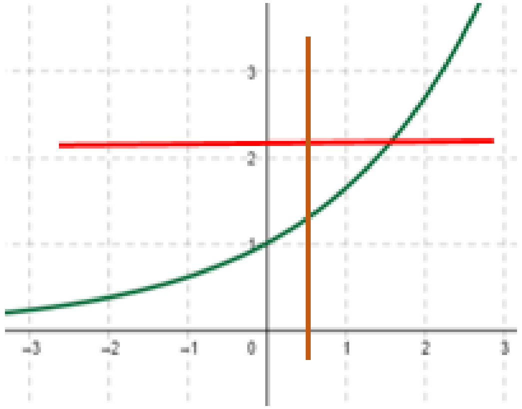
Not a Function



Pass Vertical Line Test

Fail Horizontal Line Test

Not a One-to-One Function



Pass Vertical Line Test

Pass Horizontal Line Test

A One-to-One Function

An invertible function that has an inverse

❖ Example 15 What is the inverse of the following function?

$$f(x) = \frac{3x-7}{4x+3}$$

Solution:

1. Write in form x and y $\quad y = \frac{3x-7}{4x+3}$
2. switch y and x $\quad x = \frac{3y-7}{4y+3}$
3. Solve

$3y - 7 = 4yx + 3x$

$3y - 4yx = 3x + 7$

$(3 - 4x)y = 3x + 7$

$$y = \frac{3x+7}{3-4x}$$

$f^{-1}(x) = \frac{3x+7}{3-4x}$

❖ Example 16 What is the inverse of the following function with graphs?

$$f(x) = \frac{1}{2}x - 5$$

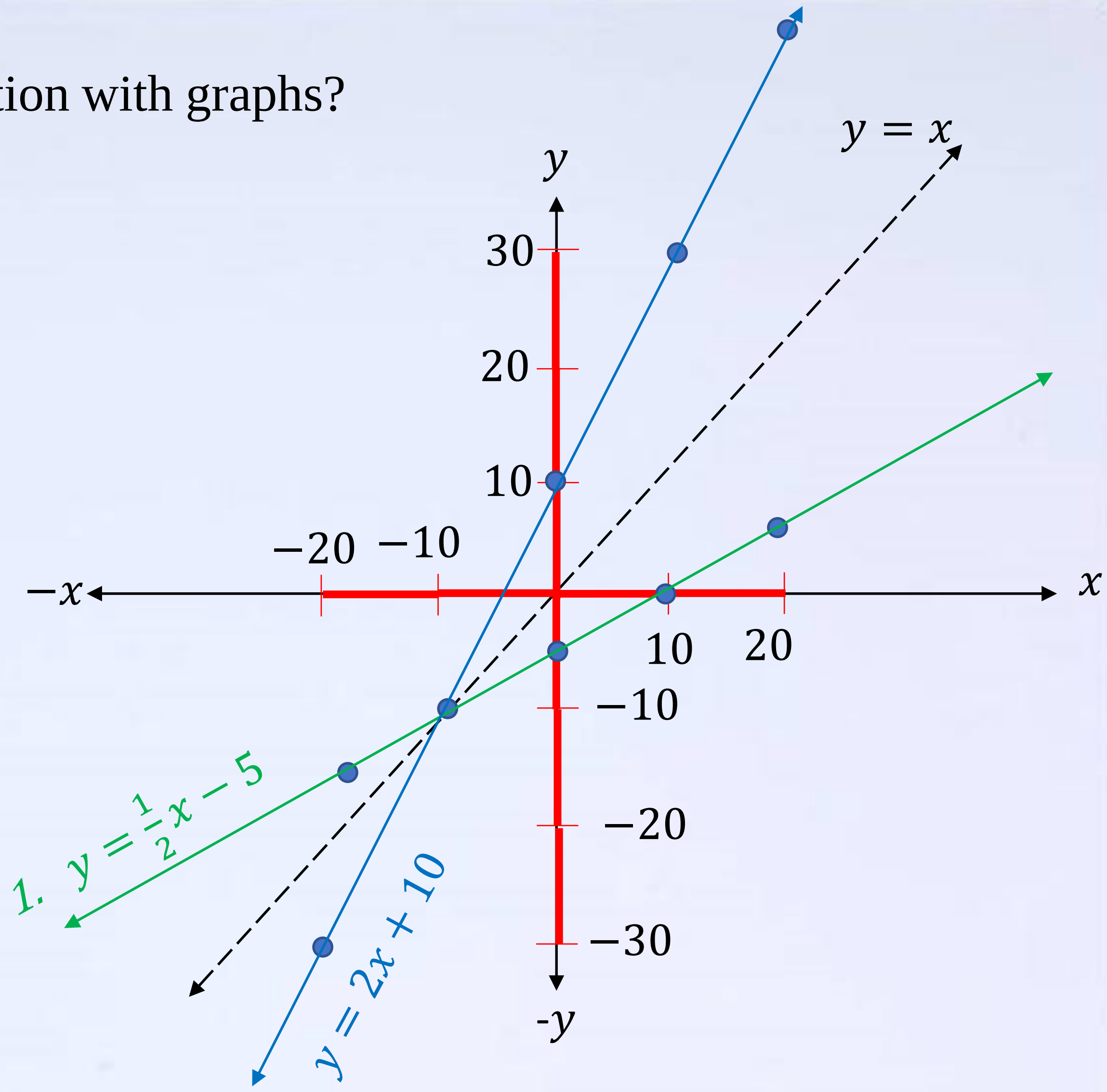
Solution:

1. Write in form x and y → $y = \frac{1}{2}x - 5$
2. switch y and x → $x = \frac{1}{2}y - 5$
3. Solve

$$x = \frac{1}{2}y - 5 \rightarrow \frac{1}{2}y = x + 5 \rightarrow y = 2x + 10$$

$$f^{-1}(x) = 2x + 10$$

| x | $y = \frac{1}{2}x - 5$ | $y = 2x + 10$ |
|---|---|---|
| -20 | -15 | -30 |
| -10 | -10 | -10 |
| 0 | -5 | 10 |
| 10 | 0 | 30 |
| 20 | 5 | 50 |



❖ Example 16 In each case, find $f^{-1}(x)$ and identify the domain and range of $f^{-1}$. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

$$f(x) = 2x - 9$$

Solution:

1. Write in form x and y → $y = 2x - 9$
2. switch y and x → $x = 2y - 9$
3. Solve

$2y = x + 9$

$y = \dfrac{x+9}{2}$

$f^{-1}(x) = \dfrac{x+9}{2}$

Domain of $f^{-1}(x)\ is\ (-\infty, \infty)$

Range of $f^{-1}(x)\ is\ (-\infty, \infty)$

As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

to find $f(f^{-1}(x))$, Substitute $f^{-1}(x)$ into the $f(x)$

$f(f^{-1}(x)) = 2(\dfrac{x+9}{2}) - 9$

$f(f^{-1}(x)) = x + 9 - 9$

$f(f^{-1}(x)) = x$

- *Types of functions*

1) Linear functions: $y = mx + b$

2) Power functions: $y = x^a$

3) Polynomial Functions: $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$

4) Trigonometric functions: sin, cos, tan, sec, csc & cot.

5) Exponential functions: $y = a^x$

6) Logarithmic functions: $y = \log_a x$

7) Rational functions: it is the ratio of two polynomial, $f(x) = \frac{g(x)}{q(x)}$

8) Algebraic Functions: An algebraic function is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions.

9) Transcendental Functions: These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.

- *Even and odd functions:*

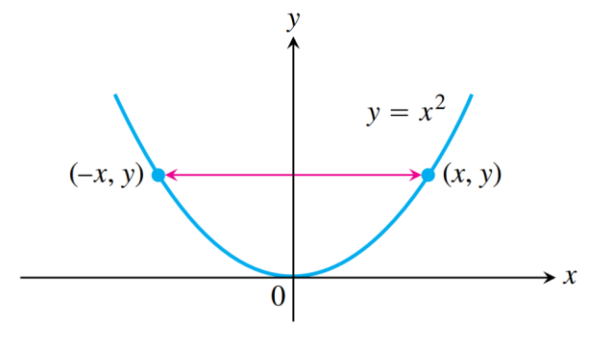
If $f(x)= f(-x)$ then, the function is even. It is symmetry about the y-axis.

If $f(x)= -f(x)$ then, the function is odd. It is symmetry about the origin.

❖ Example 17 check the symmetry of function:

1. $f(x)= x^2$, 2. $f(x)= 5+x^2$ , 3. $f(x)= x$, 4. $f(x)= x+1$

Solution:




1. $f(-x)= (-x)^2$ $f(-x)=x^2$ $f(-x)=f(x)$

So the function is even, it has symmetry about the y-axis.

2. $f(-x)= 5+(-x)^2$ $f(-x)= 5+x^2$ $f(-x) = f(x)$

So the function is even, it has symmetry about the y-axis.

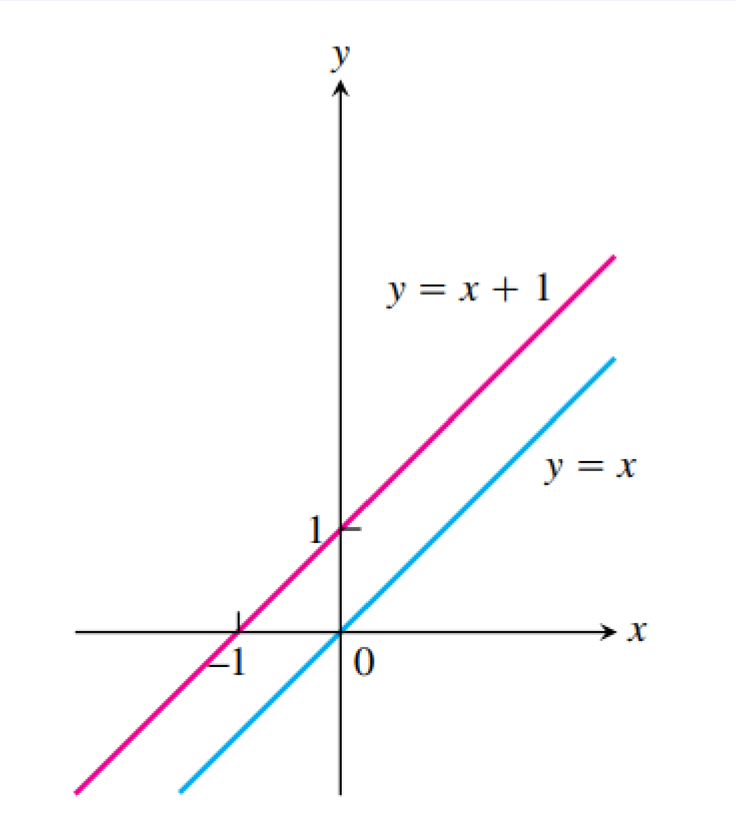
3. $f(-x) = -x$ $\qquad f(-x) \neq f(x)$

So the function is odd, it has symmetry about the origin.

4. $f(-x) = (-x)+1 \qquad f(-x) = -x+1 \qquad f(-x) \neq f(x)$

$-f(x) = -(x+1) \qquad -f(x) = -x-1 \qquad -f(x) \neq f(x)$

So the function is not even nor odd also.



- *Asymptotes*

1) An asymptote is a straight line which acts as a boundary for the graph of a function.
2) When a function has an asymptote (and not all functions have them) the function gets closer and closer to the asymptote as the input value to the function approaches either a specific value a or positive or negative infinity.
3) The functions most likely to have asymptotes are rational functions.
4) **Vertical asymptotes** occur when the following condition is met:

The denominator of the simplified rational function is equal to 0. Remember, the simplified rational function has cancelled any factors common to both the numerator and denominator.

5) **Horizontal asymptotes** occur when either one of the following conditions is met (you should notice that both conditions cannot be true for the same function).

i. The degree of the numerator is less than the degree of the denominator. In this case the asymptote is the horizontal line $y = 0$.

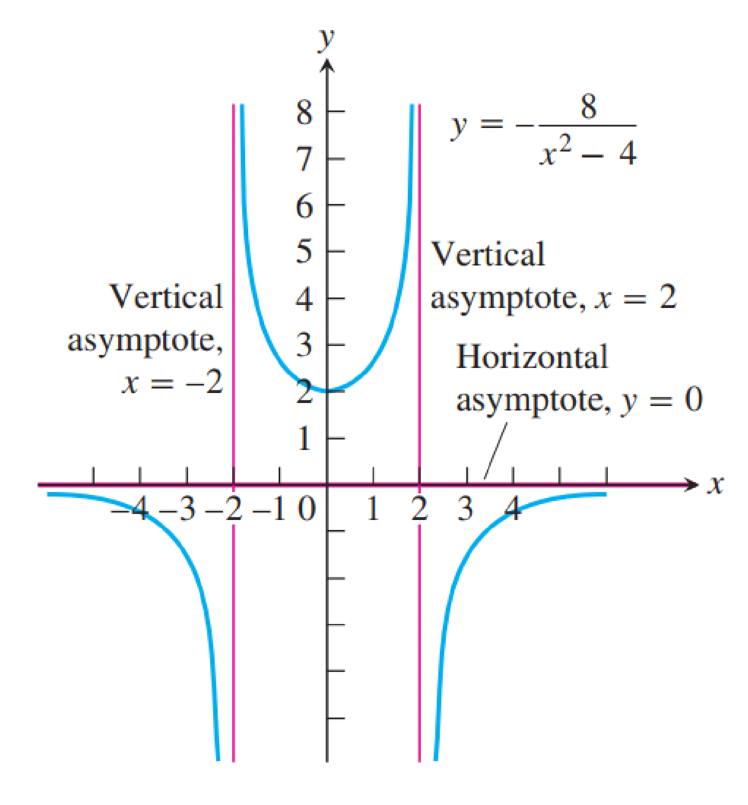
ii. The degree of the numerator is equal to the degree of the denominator. In this case the asymptote is the horizontal line $y = a/b$ where a is the leading coefficient in the numerator and b is the leading coefficient in the denominator.

$$y = \frac{ax - 4}{bx + 2}$$

- When the degree of the numerator is greater than the degree of the denominator there is no horizontal asymptote.

❖ Example 18 Find the horizontal and vertical asymptotes of

$$f(x) = -\frac{8}{x^2 - 4}$$




Solution:

Vertical asymptote

Is a denominator equal to 0?

$x^2 - 4 = 0 \longrightarrow x^2 = 4 \longrightarrow x = \mp 2$

Horizontal asymptote if

Is a The degree of the numerator less than the degree of the denominator ? *(y=0)*

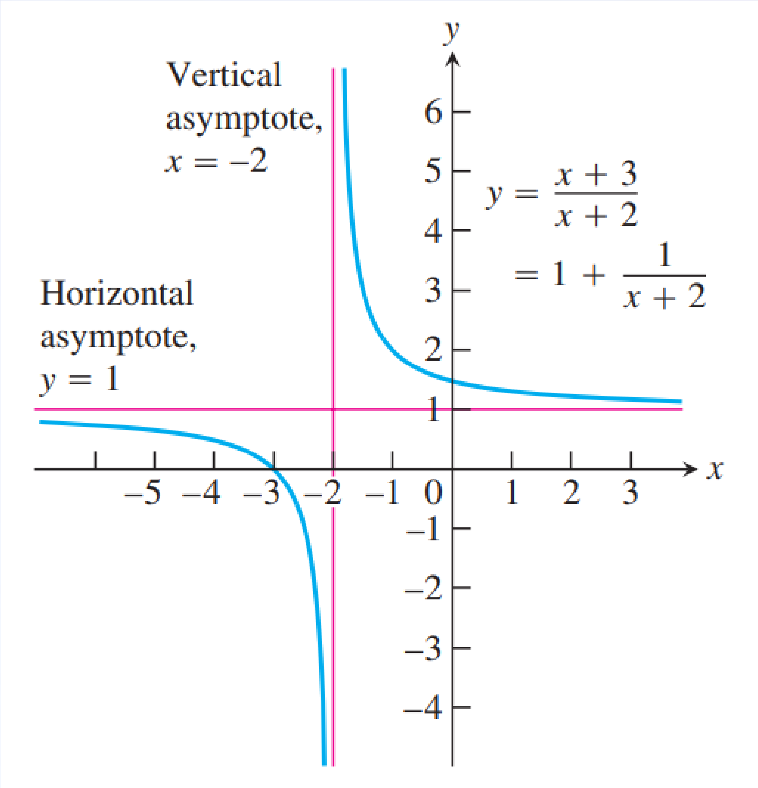
Is a The degree of the numerator equal to the degree of the denominator ? *(y=a/b)*

The degree of the numerator less ($x^0$) and the degree of the denominator ($x^1$)

The degree of the numerator less than the degree of the denominator and *y=0.*

❖ Example 19 Find the horizontal and vertical asymptotes of

$$f(x) = \frac{x+3}{x+2}$$




Solution:

Vertical asymptote

Is a denominator equal to 0?

$x + 2 = 0 \longrightarrow x = -2$

Horizontal asymptote

The degree of the numerator less ($x^1$) and the degree of the denominator ($x^1$).

If the degree of the numerator equal to the degree of the denominator *then y=a/b*

a is the leading coefficient in the numerator (*ax+3*) and b is the leading coefficient in the denominator (*bx+2*).

*a =1 and b =1* *y=1/1* *y=1.*

❖ Example 20 Find the asymptotes of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

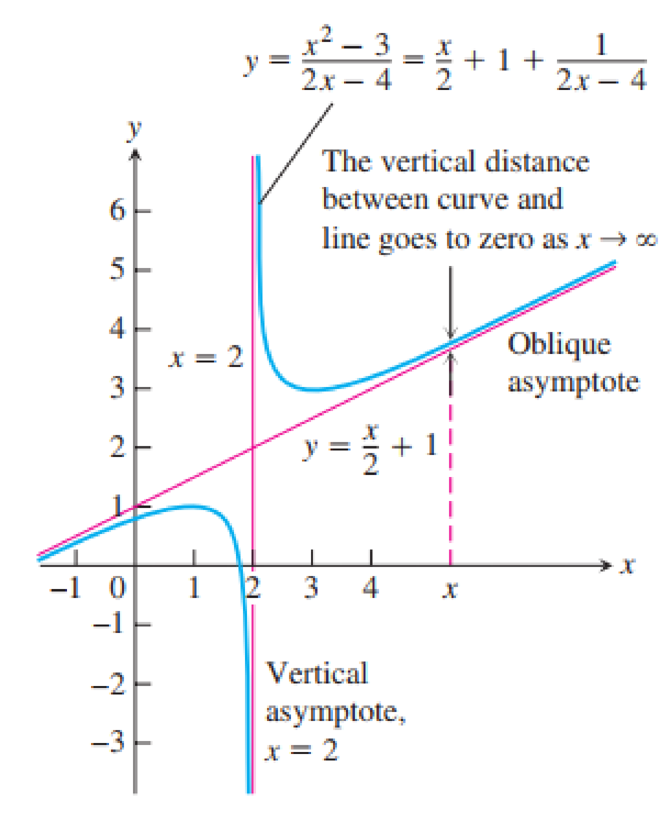
If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an oblique (slanted) asymptote. We find an equation for the asymptote by dividing numerator by denominator to express $f$ as a linear function plus a remainder that goes to zero as x$\rightarrow \pm\infty$.

Solution:

$$\frac{x^2}{2x} = \frac{x}{2}$$

$$\begin{array}{r|l} & \frac{x}{2} + 1 \\ \hline 2x - 4 & x^2 - 3 \\ & x^2 - 2x \\ \hline & \quad 2x - 3 \\ & \quad 2x - 4 \\ \hline & \qquad 1 \end{array}$$

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}$$




Vertical asymptote

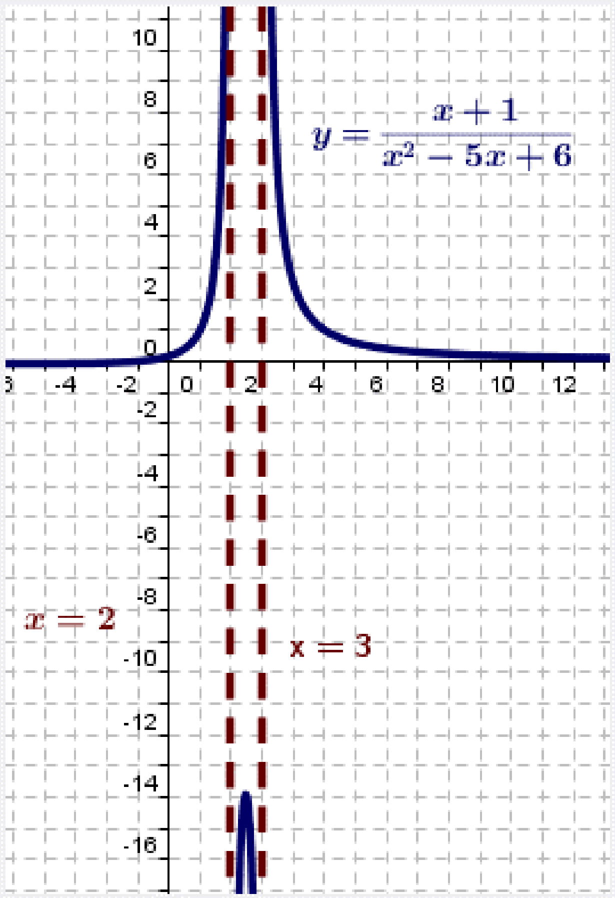
$$2x - 4 = 0 \longrightarrow x = 2$$

Horizontal asymptote None

Oblique asymptote $y = \frac{x}{2} + 1$

❖ Example 21 Find the horizontal and vertical asymptotes of

$$f(x) = \frac{x+1}{x^2 - 5x + 6}$$




Solution:

Vertical asymptote

Is a denominator equal to 0?

$x^2 - 5x + 6 = 0 \longrightarrow (x-3)(x-2) \longrightarrow x = 2, x = 3$

Horizontal asymptote if

The degree of the numerator less ($x^1$) and the degree of the denominator ($x^2$)

Therefore, the Horizontal asymptote at $y=0$.

❖ Example 22 Find the horizontal and vertical asymptotes of

$$f(x) = \frac{2x^2 - x + 1}{x - 2}$$

Solution:

Vertical asymptote

Is a denominator equal to 0?

$x - 2 = 0 \longrightarrow x = 2$

Horizontal asymptote if

The degree of the numerator greater ($x^2$) and the degree of the denominator ($x^1$)

Therefore, there is no Horizontal asymptote. But If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an oblique (slanted) asymptote

$$\begin{array}{r|l} & 2x + 3 \\ \hline x - 2 & 2x^2 - x + 1 \\ & \underline{2x^2 - 4x} \\ & \quad 3x + 1 \\ & \quad \underline{3x - 6} \\ & \qquad 7 \end{array}$$

$$f(x) = \underbrace{2x + 3}_{\text{linear}} + \underbrace{\frac{7}{x - 2}}_{\text{remainder}}$$

Oblique asymptote $y = 2x + 3$

## • *Trigonometric Functions*

The trigonometric functions are important because they are periodic, or repeating, and therefore model many naturally occurring periodic processes.

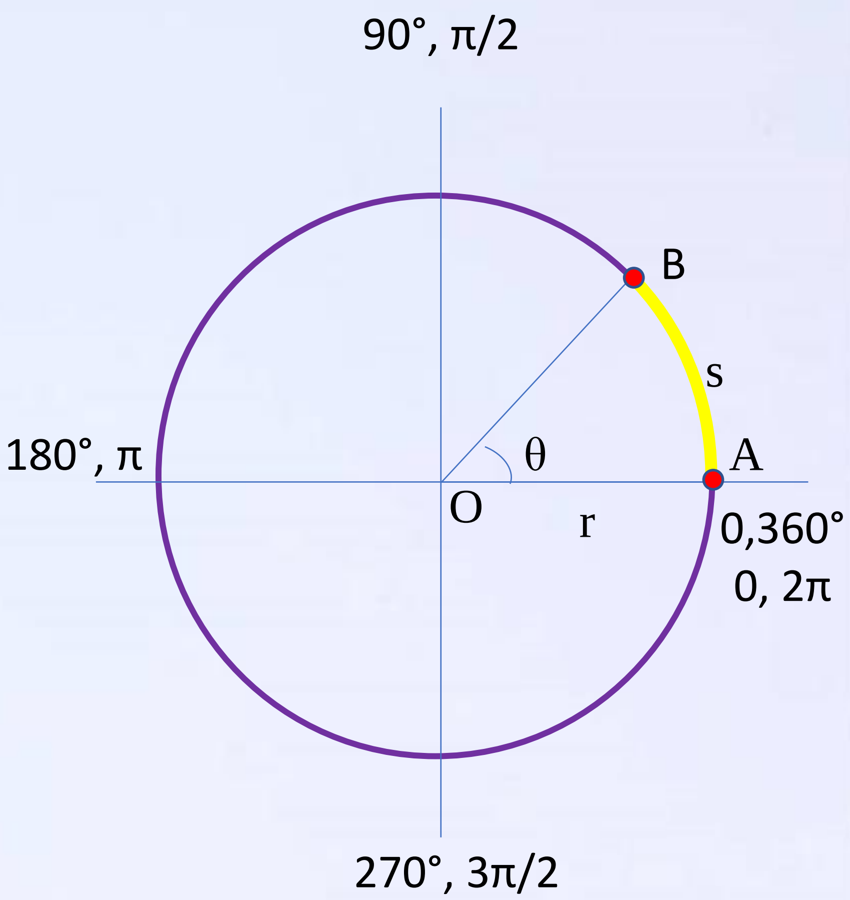
- Radian Measure

The radian measure the length of arc cut (AB) from a circle of radius r when the subtending angle producing the arc is measured in radians.

$$\boxed{s = r\theta}$$

Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by

| Degree | radian |
|---|---|
| 360° | 2π |
| 270° | 3π/2 |
| 180° | π |
| 90° | π/2 |



❖ Example 23 convert the angles:

1. *From degree to radian,*

60° and 150°

2. *From radian to degree*

π/4 and 2π/3

Solution:

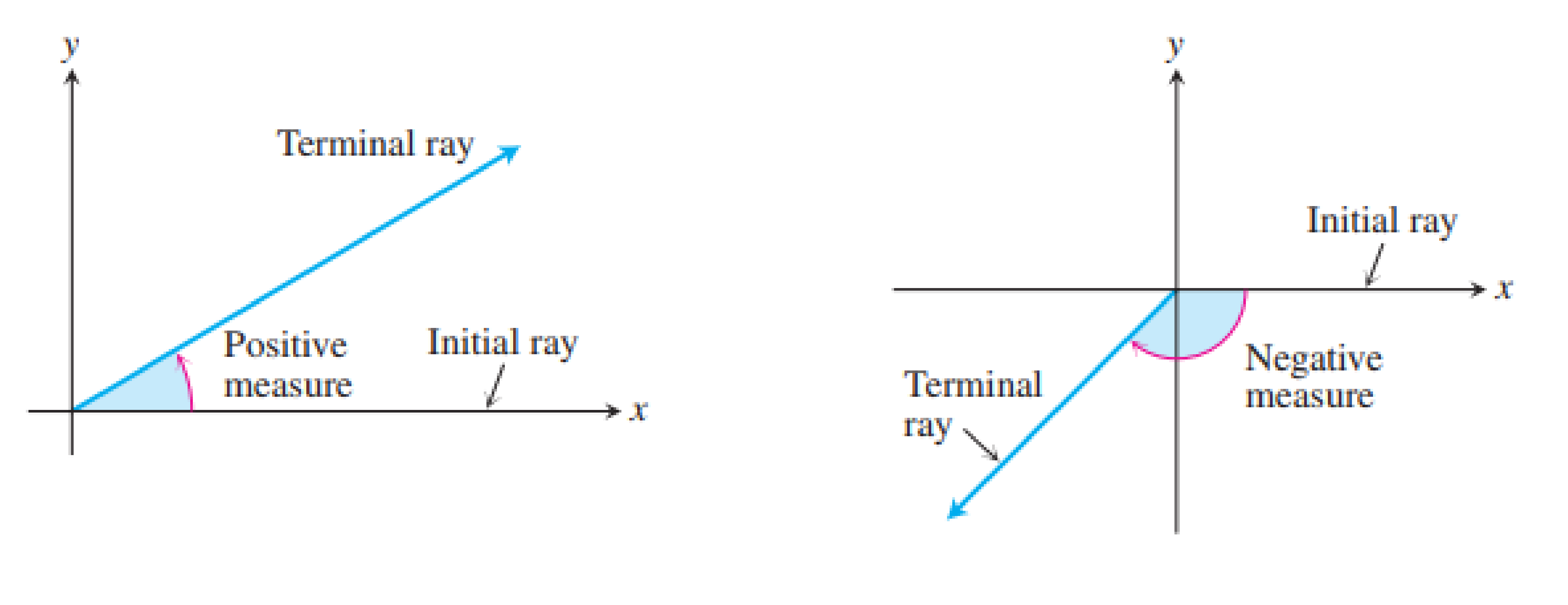
$$60° \times \frac{\pi}{180} = \frac{\pi}{3}$$

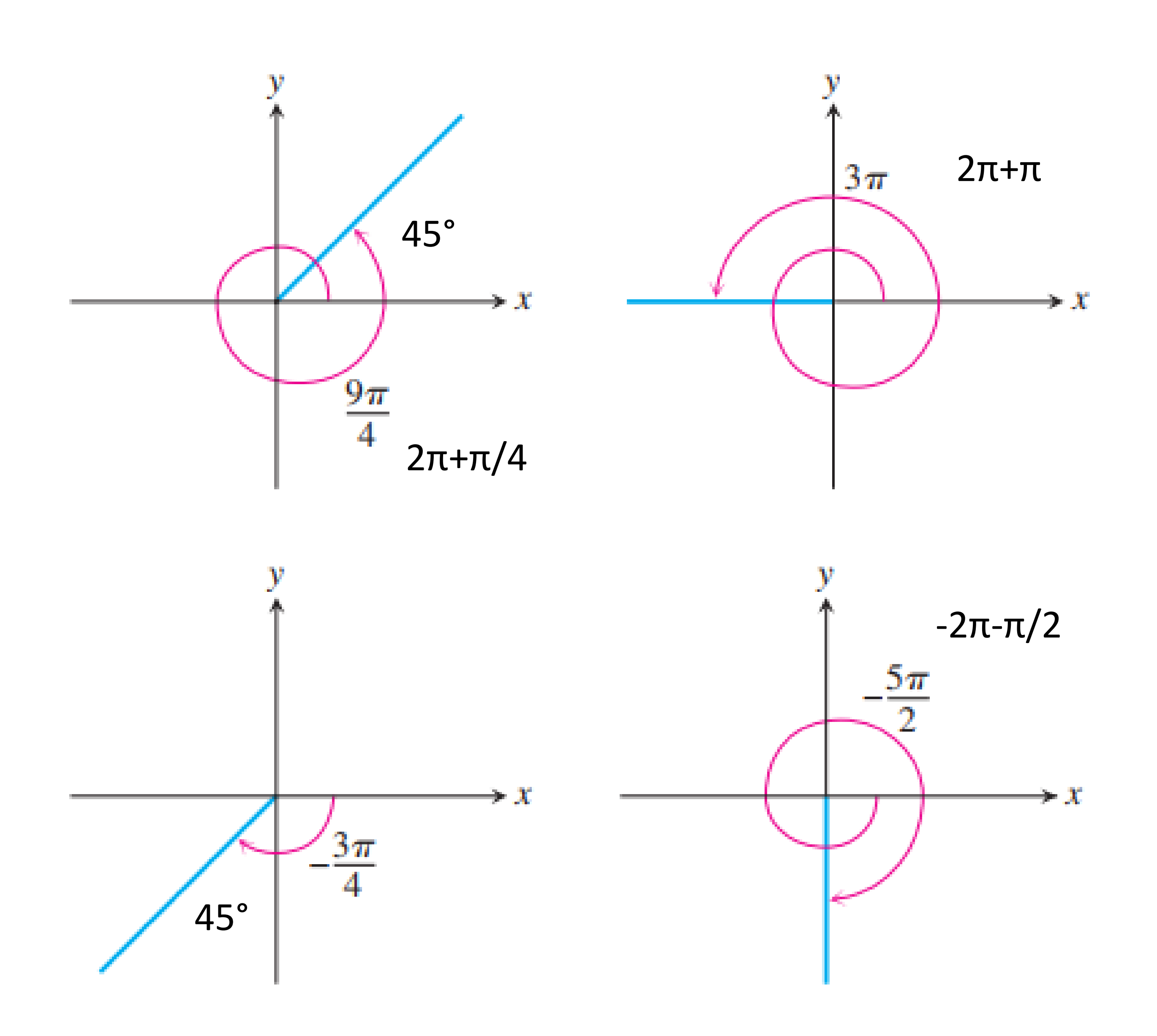
$$150° \times \frac{\pi}{180} = \frac{5\pi}{6}$$

$$\frac{\pi}{4} \times \frac{180}{\pi} = 45°$$

$$\frac{2\pi}{3} \times \frac{180}{\pi} = 120°$$

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis. Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.





y
x
45°
$\frac{9\pi}{4}$
2π+π/4
y
x
3π
2π+π
y
x
$-\frac{3\pi}{4}$
45°
y
x
-2π-π/2
$-\frac{5\pi}{2}$

- *The Six Basic Trigonometric Functions*

sine: $\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$

cosecant: $\csc\theta = \frac{hyp}{opp} = \frac{r}{y}$

$\csc\theta = \frac{1}{\sin\theta}$

cosine: $\cos\theta = \frac{adj}{hyp} = \frac{x}{r}$

secant: $\sec\theta = \frac{hyp}{adj} = \frac{r}{x}$

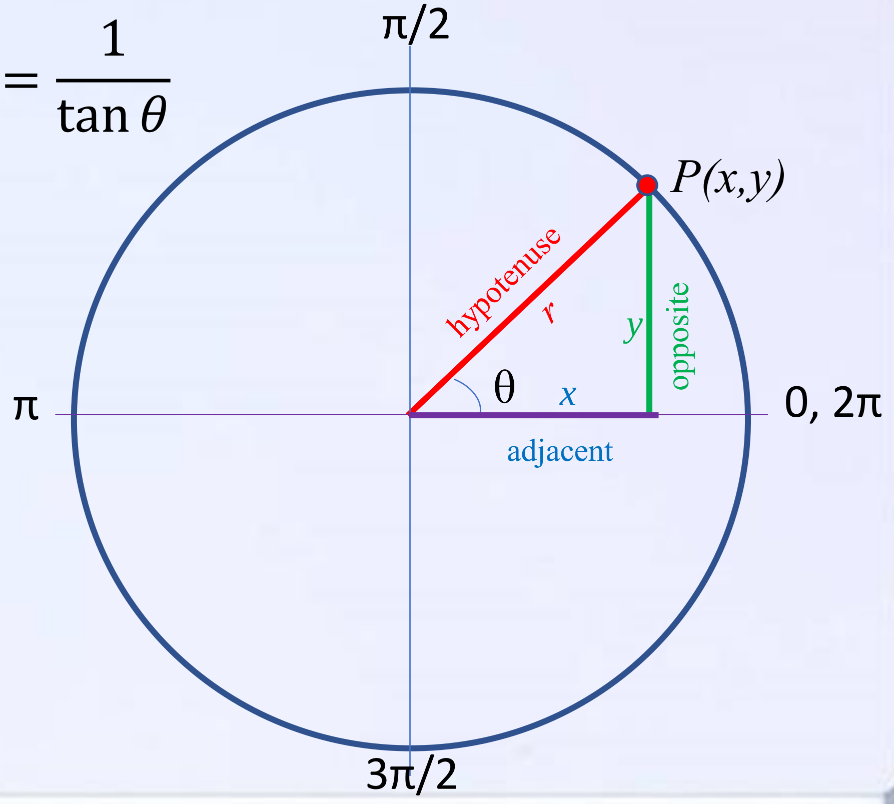
$sec\,\theta = \frac{1}{\cos\theta}$

tangent: $\tan\theta = \frac{opp}{adj} = \frac{y}{x}$

cotangent: $\cot\theta = \frac{adj}{opp} = \frac{x}{y}$

$\cot\theta = \frac{1}{\tan\theta}$

$$\tan\theta = \frac{sin\theta}{cos\theta}$$



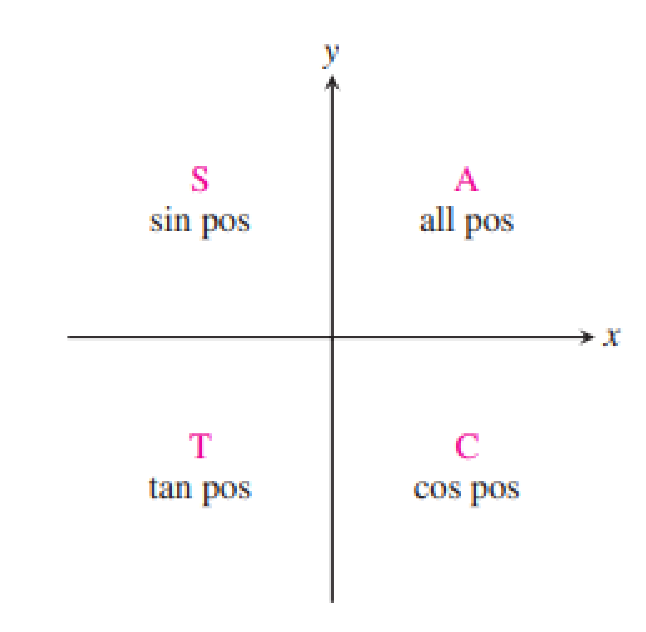

As you can see, tan θ and sec θ are not defined if $x = 0$. This means they are not defined if θ is π/2, 3π/2.

Similarly, cot θ and csc are not defined for values of θ for which $y = 0$, namely $\theta = 0$, π, 2π.

| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta°$ | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° | 210° | 225° | 240° | 270° | 300° | 315° | 330° | 360° |
| **sin** $\theta$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $0$ |
| **cos** $\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| tan$\theta$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | - | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | - | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | $0$ |

A S T C

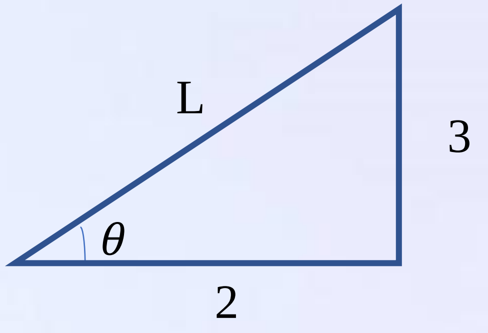
The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.



❖ Example 24 If $tan\theta = \frac{3}{2}$ and $0 < \theta < \frac{\pi}{2}$, find the five other trigonometric functions of $\theta$.

Solution:

From we construct the right triangle of height 3 (opposite) and base 2 (adjacent) .




The length of the hypotenuse

$L = \sqrt{2^2 + 3^2}$

$L = \sqrt{4 + 9}$

$L = \sqrt{13}$

From the triangle we write the values of the other five trigonometric functions:

$$\sin\theta = \frac{3}{\sqrt{13}} \qquad \csc\theta = \frac{\sqrt{13}}{3}$$

$$\cos\theta = \frac{2}{\sqrt{13}} \qquad \sec\theta = \frac{\sqrt{13}}{2}$$

$$\cot\theta = \frac{2}{3}$$

- Periodicity of the Trigonometric Functions

When an angle of measure $\theta$ and an angle of measure $\theta + 2\pi$ are in standard position (*xy*-plane), their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \qquad \tan(\theta + 2\pi) = \tan\theta$

$\csc(\theta + 2\pi) = \csc\theta \qquad \sec(\theta + 2\pi) = \sec\theta \qquad \cot(\theta + 2\pi) = \cot\theta$

Similarly

$\sin(\theta - 2\pi) = \sin\theta \qquad \cos(\theta - 2\pi) = \cos\theta \qquad \tan(\theta - 2\pi) = \tan\theta$

$\csc(\theta - 2\pi) = \csc\theta \qquad \sec(\theta - 2\pi) = \sec\theta \qquad \cot(\theta - 2\pi) = \cot\theta$

That's why describe this repeating behavior by saying that the six basic trigonometric functions are ***periodic***.

❖ Example 25

$\sin 30 = 0.5$ $\sin(30 + 2\pi) = 0.5$ $\sin(30 - 2\pi) = 0.5$

$\sec 120 = -2$ $\sec(120 + 2\pi) = -2$ $\sec(120 - 2\pi) = -2$

$\tan 230 = 1.19$ $\tan(230 + 2\pi) = 1.19$ $\tan(230 - 2\pi) = 1.19$

$\cos 300 = 0.5$ $\cos(300 + 2\pi) = 0.5$ $\cos(300 - 2\pi) = 0.5$

- Even-Odd Properties

- A function $f$ is even if $f(-\theta) = f(\theta)$ for all $\theta$ in the domain of $f$
- A function $f$ is odd if $f(-\theta) = -f(\theta)$ for all $\theta$ in the domain of $f$

| | |
|---|---|
| $sin(-\theta) = -sin(\theta)$ | $sin(-30) = -0.5 = -sin(30) \neq sin(30)$ |
| $cos(-\theta) = cos(\theta)$ | $cos(-60) = 0.5 = cos(60)$ |
| $tan(-\theta) = -tan(\theta)$ | $tan(-45) = -1 = -tan(-45)$ |
| $csc(-\theta) = -csc(\theta)$ | $csc(-120) = -1.15 = -csc(120)$ |
| $sec(-\theta) = sec(\theta)$ | $sec(-150) = -1.15 = sec(150)$ |
| $cot(-\theta) = -cot(\theta)$ | $cot(-280) = 0.17 = -cot(280)$ |

- Cosine and secant are even functions
- The other functions are odd functions

- Identities

$$\sin\theta = \frac{y}{r} \rightarrow y = r sin\theta$$

$$\cos\theta = \frac{x}{r} \rightarrow x = r cos\theta$$

$$r = \sqrt{x^2 + y^2} \rightarrow r = \sqrt{(rcos\theta)^2 + (rsin\theta)^2}$$

When $r = 1$,

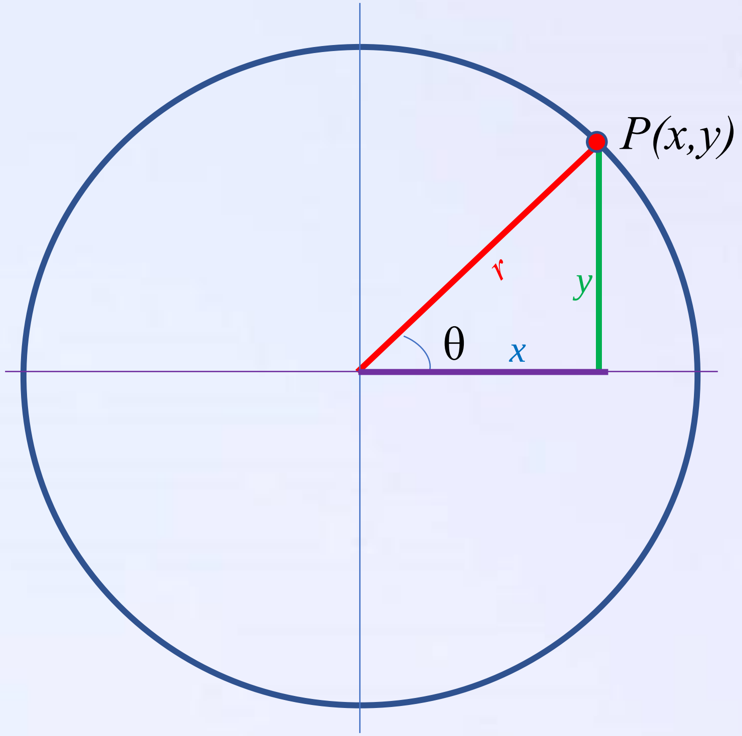
$$\boxed{cos^2\theta + sin^2\theta = 1 \quad \ldots 1}$$

Dividing Eq.1 by $cos^2\theta$ and give

$$\boxed{1 + tan^2\theta = sec^2\theta}$$

Dividing Eq.1 by $sin^2\theta$ and give

$$\boxed{1 + cot^2\theta = csc^2\theta}$$



- Identities

## Addition Formulas

The following formulas hold for all angles A and B

$$\boxed{\cos(A+B) = cosA\ cosB - sinA\ sinB \quad \ldots 2}$$

$$\boxed{\sin(A+B) = sinA\ cosB + cosA\ sinB \quad \ldots 3}$$

## Double-Angle Formulas

By substituting $\theta$ for both $A$ and $B$ in the addition formulas (Eq. 2 & 3, respectively) gives:

$$\boxed{\cos 2\theta = cos^2\theta - sin^2\theta}$$

$$\boxed{\sin 2\theta = 2sin\theta\ cos\theta}$$

- Identities

### Half-Angle Formulas

By combining the equations $cos^2\theta + sin^2\theta = 1$ and $\cos 2\theta = cos^2\theta - sin^2\theta$

$cos^2\theta + sin^2\theta = 1$

$+$

$cos^2\theta - sin^2\theta = \cos 2\theta = \ 2cos^2\theta = 1 + cos2\theta \longrightarrow \boxed{cos^2\theta = \frac{1+cos2\theta}{2}}$

By subtract the equations $cos^2\theta + sin^2\theta = 1$ and $\cos 2\theta = cos^2\theta - sin^2\theta$

$cos^2\theta + sin^2\theta = 1$

-

$cos^2\theta - sin^2\theta = \cos 2\theta = \quad 2sin^2\theta = 1 - cos2\theta \longrightarrow \boxed{sin^2\theta = \frac{1-cos2\theta}{2}}$

- Identities

**The Law of Cosines**

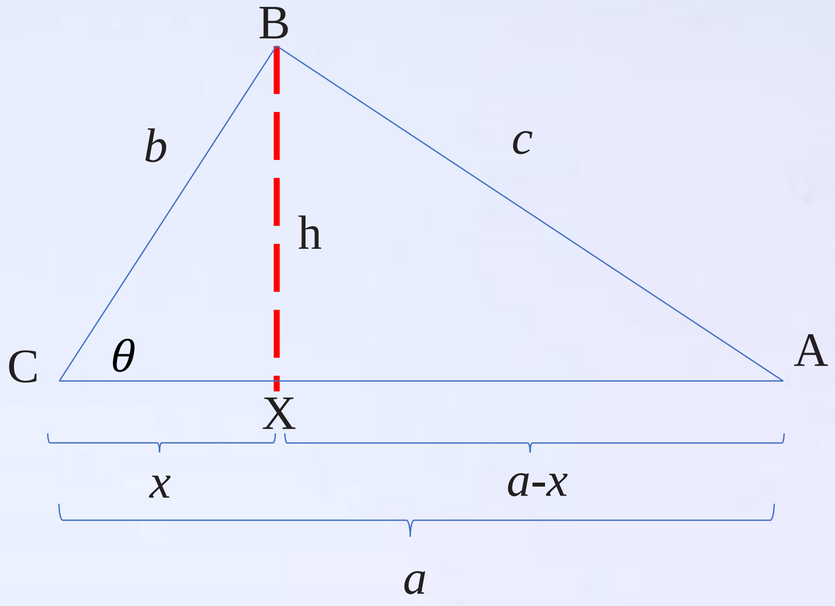
From triangle CXB

$b^2 = h^2 + x^2 \rightarrow h^2 = b^2 - x^2 \quad ...1$

$cos\theta = \frac{x}{b} \rightarrow x = bcos\theta \quad ...2$

From triangle AXB

$c^2 = h^2 + (a-x)^2 \rightarrow h^2 = c^2 - (a-x)^2 \quad ...3$




From Eq.1 and 2

$b^2 - x^2 = c^2 - (a-x)^2$

$b^2 - x^2 = c^2 - a^2 + 2ax - x^2$

$b^2 = c^2 - a^2 + 2ax$ From Eq.2

$b^2 = c^2 - a^2 + 2a(bcos\theta)$

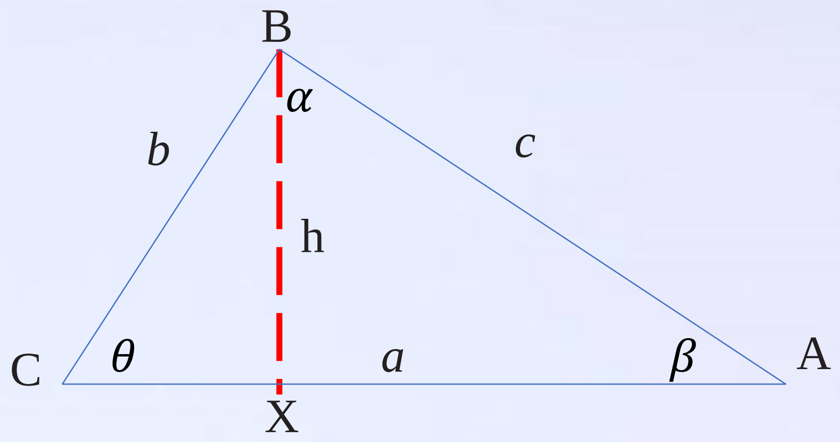
$$\boxed{c^2 = a^2 + b^2 - 2ab\,cos\theta}$$

- Identities

**The Law of Sine**

$$\sin\theta = \frac{h}{b} \rightarrow h = b\sin\theta \ldots 1$$

$$\sin\beta = \frac{h}{c} \rightarrow h = c\sin\beta \ldots 2$$




From Eq.1 and 2

$$b\sin\theta = c\sin\beta$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\theta}$$

By repeat this with different side to get

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\theta}}$$

- Identities

$$\sin(x + \frac{\pi}{2}) = \cos x$$

$$\cos(x + \frac{\pi}{2}) = -\sin x$$

$$\sin(x - \frac{\pi}{2}) = -\cos x$$

$$\cos(x - \frac{\pi}{2}) = \sin x$$

$$sin(A + B) = sinA\ cosB + cosA\ sinB$$

$$\cos(A + B) = cosA\ cosB - sinA\ sinB$$

$$\tan(A + B) = \frac{tanA + tanB}{1 - tanA\ tanB}$$

$$sin(A - B) = sinA\ cosB - cosA\ sinB$$

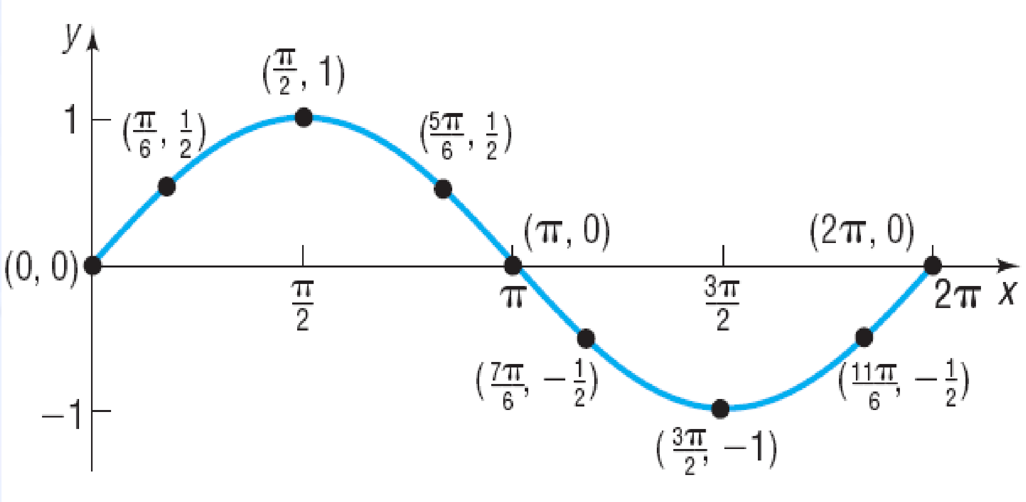
$$\cos(A - B) = cosA\ cosB + sinA\ sinB$$

$$\tan(A - B) = \frac{tanA - tanB}{1 - tanA\ tanB}$$

- Graphing the Sine Function

- *Periodicity*: Only need to graph on interval [0, 2π] (One cycle)
- Domain: All real numbers
- Range: [-1, 1]
- Odd function
- Periodic, period 2π
- x-intercepts: …, -2π, -π, 0, π, 2π …
- y-intercept: 0
- Maximum value: y = 1
- Minimum value: y = -1

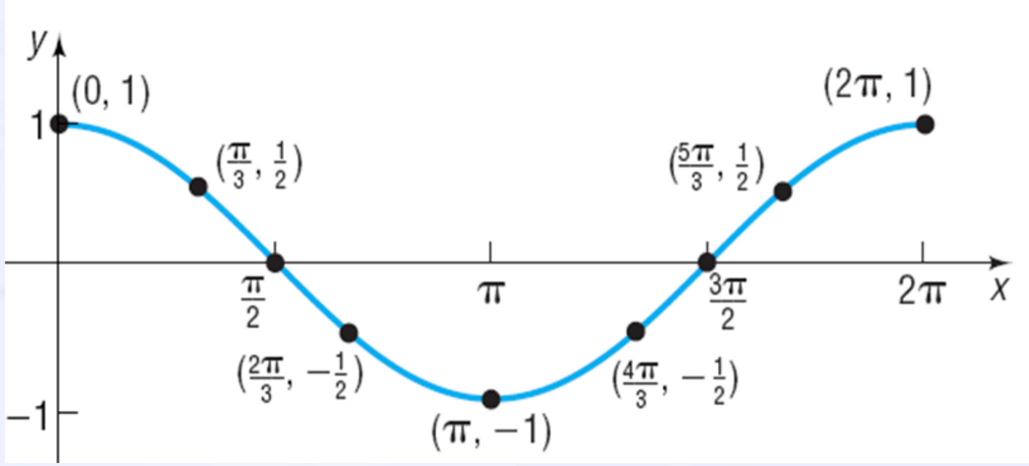
| $x$ | $y = \sin x$ | $(x, y)$ |
|---|---|---|
| $0$ | $0$ | $(0, 0)$ |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ |
| $\frac{\pi}{2}$ | $1$ | $\left(\frac{\pi}{2}, 1\right)$ |
| $\frac{5\pi}{6}$ | $\frac{1}{2}$ | $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$ |
| $\pi$ | $0$ | $(\pi, 0)$ |
| $\frac{7\pi}{6}$ | $-\frac{1}{2}$ | $\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$ |
| $\frac{3\pi}{2}$ | $-1$ | $\left(\frac{3\pi}{2}, -1\right)$ |
| $\frac{11\pi}{6}$ | $-\frac{1}{2}$ | $\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$ |
| $2\pi$ | $0$ | $(2\pi, 0)$ |



## Graphing the Cosine Function

- *Periodicity*: Only need to graph on interval [0, 2π] (One cycle)
- Domain: All real numbers
- Range: [-1, 1]
- Even function
- Periodic, period 2π
- x-intercepts: …, -3π/2, -π/2, π/2, 3π/2 …
- y-intercept: 1
- Maximum value: y = 1
- Minimum value: y = -1

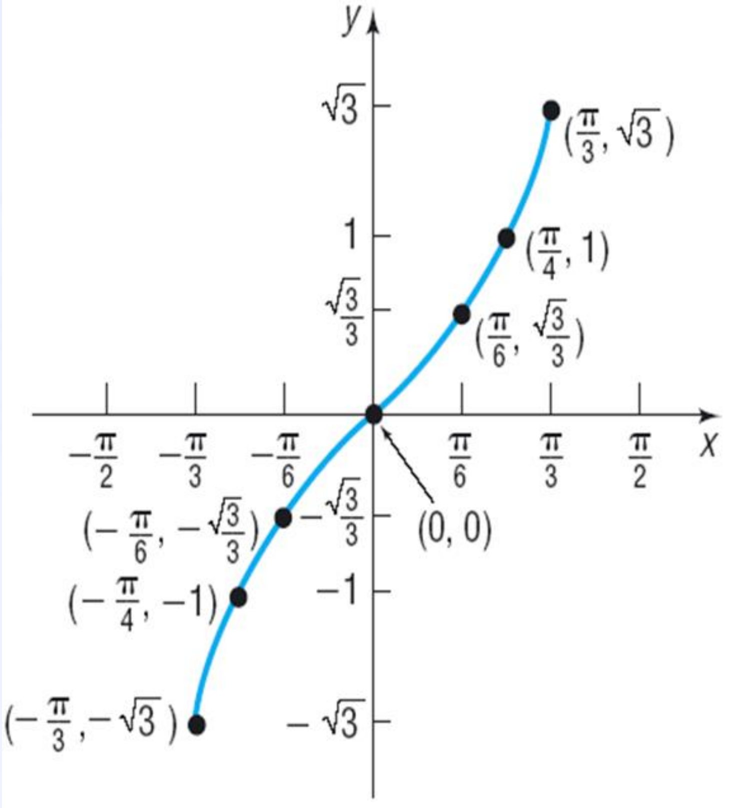
| $x$ | $y = \cos x$ | $(x, y)$ |
|---|---|---|
| $0$ | $1$ | $(0, 1)$ |
| $\frac{\pi}{3}$ | $\frac{1}{2}$ | $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ |
| $\frac{\pi}{2}$ | $0$ | $\left(\frac{\pi}{2}, 0\right)$ |
| $\frac{2\pi}{3}$ | $-\frac{1}{2}$ | $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$ |
| $\pi$ | $-1$ | $(\pi, -1)$ |
| $\frac{4\pi}{3}$ | $-\frac{1}{2}$ | $\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$ |
| $\frac{3\pi}{2}$ | $0$ | $\left(\frac{3\pi}{2}, 0\right)$ |
| $\frac{5\pi}{3}$ | $\frac{1}{2}$ | $\left(\frac{5\pi}{3}, \frac{1}{2}\right)$ |
| $2\pi$ | $1$ | $(2\pi, 1)$ |



- Graphing the Tangent Function

- *Periodicity*: Only need to graph on interval [0, π] (One cycle)
- Domain: x $\neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, ....
- Range: All real numbers
- Odd function
- Periodic, period π
- x-intercepts: …, -2π, -π, 0, π, 2π …
- y-intercept: 0

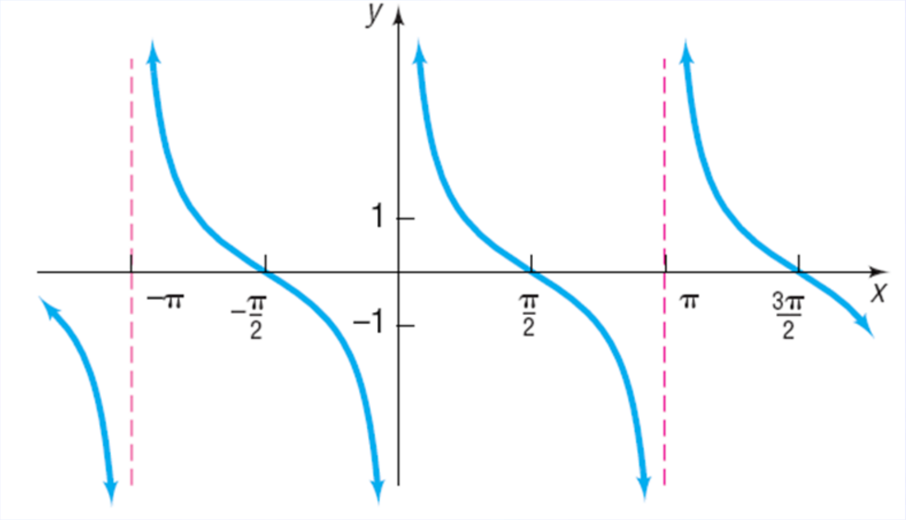
| $x$ | $y = \tan x$ | $(x, y)$ |
|---|---|---|
| $-\frac{\pi}{3}$ | $-\sqrt{3} \approx -1.73$ | $\left(-\frac{\pi}{3}, -\sqrt{3}\right)$ |
| $-\frac{\pi}{4}$ | $-1$ | $\left(-\frac{\pi}{4}, -1\right)$ |
| $-\frac{\pi}{6}$ | $-\frac{\sqrt{3}}{3} \approx -0.58$ | $\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$ |
| 0 | 0 | (0, 0) |
| $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{3} \approx 0.58$ | $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$ |
| $\frac{\pi}{4}$ | 1 | $\left(\frac{\pi}{4}, 1\right)$ |
| $\frac{\pi}{3}$ | $\sqrt{3} \approx 1.73$ | $\left(\frac{\pi}{3}, \sqrt{3}\right)$ |



## Graphing the Cotangent Function

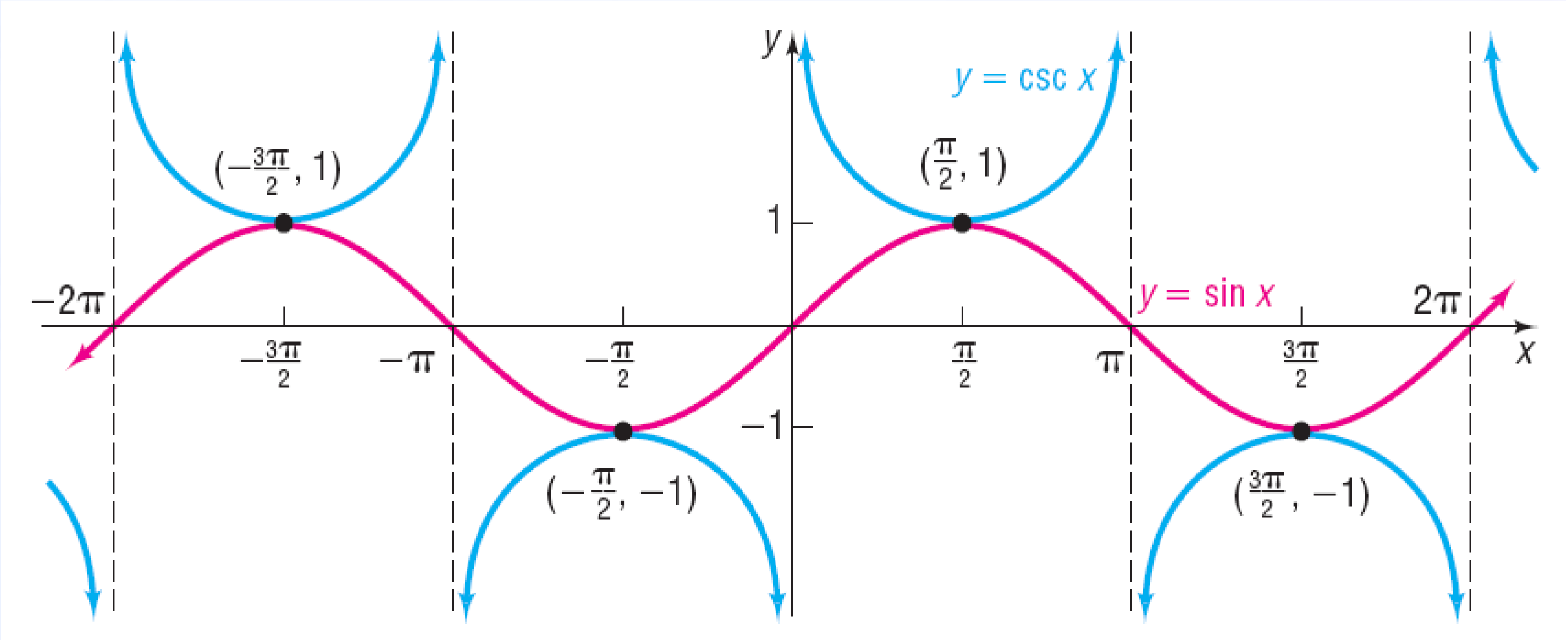
- *Periodicity*: Only need to graph on interval [0, π] (One cycle)
- Domain: x $\neq 0, \ \pm\pi, \pm 2\pi,$ ....
- Range: All real numbers
- Odd function
- Periodic, period π
- x-intercepts: …, -π/2, π/2, 3π/2 …
- y-intercept: -

| $x$ | $y = \cot x$ | $(x, y)$ |
|---|---|---|
| $\frac{\pi}{6}$ | $\sqrt{3}$ | $\left(\frac{\pi}{6}, \sqrt{3}\right)$ |
| $\frac{\pi}{4}$ | $1$ | $\left(\frac{\pi}{4}, 1\right)$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{3}$ | $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ |
| $\frac{\pi}{2}$ | $0$ | $\left(\frac{\pi}{2}, 0\right)$ |
| $\frac{2\pi}{3}$ | $-\frac{\sqrt{3}}{3}$ | $\left(\frac{2\pi}{3}, -\frac{\sqrt{3}}{3}\right)$ |
| $\frac{3\pi}{4}$ | $-1$ | $\left(\frac{3\pi}{4}, -1\right)$ |
| $\frac{5\pi}{6}$ | $-\sqrt{3}$ | $\left(\frac{5\pi}{6}, -\sqrt{3}\right)$ |



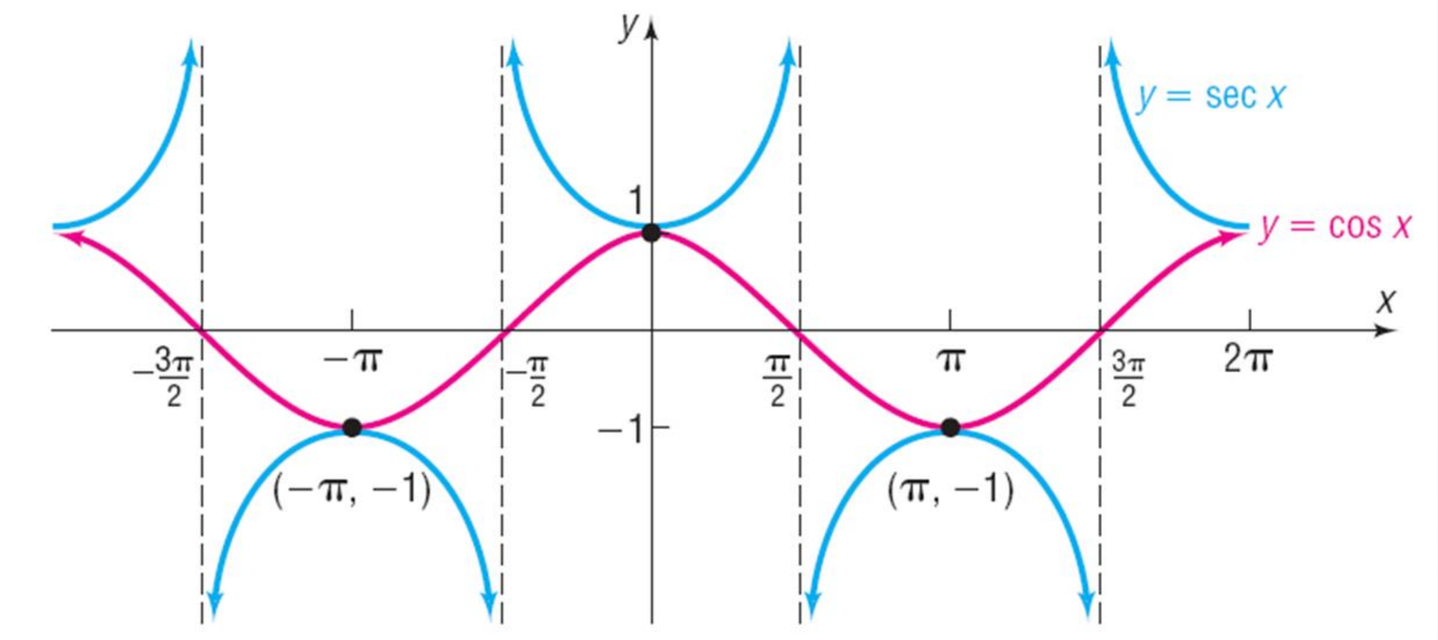
- Graphing the Cosecant Function

- *Periodicity*: Only need to graph on interval [0, 2π] (One cycle)
- Domain: x $\neq$ 0, $\pm\pi, \pm 2\pi, \ldots$.
- Range: $(-\infty, -1] \cup [1, \infty)$
- Odd function
- Periodic, period 2π

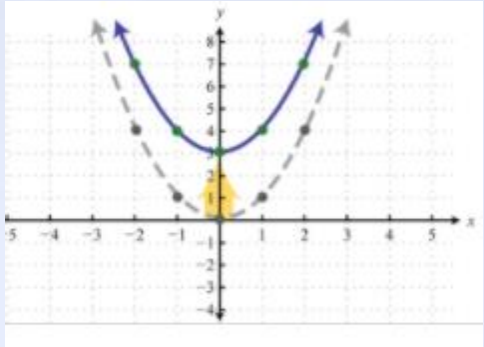


- Graphing the Secant Function

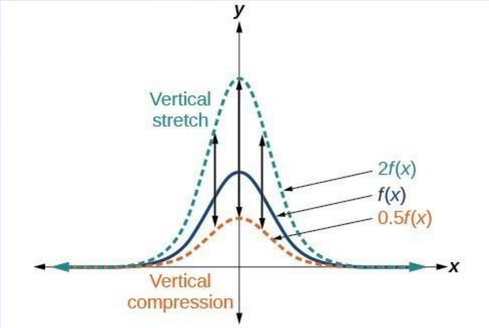
- *Periodicity*: Only need to graph on interval [0, 2π] (One cycle)
- Domain: x $\neq \pm\pi/2, \pm 3\pi/2, \ldots.$
- Range: $(-\infty, -1] \cup [1, \infty)$
- Even function
- Periodic, period 2π
- y-intercept: 1



- Transformations of Trigonometric Graphs



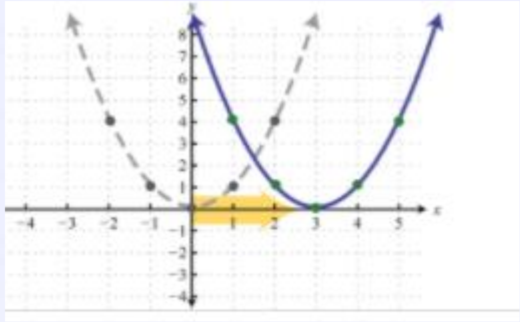
Vertical stretch or compression;
reflection about x-axis if negative

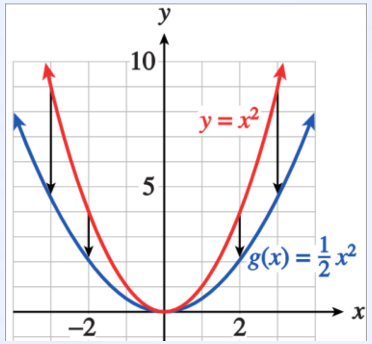



Vertical shift

$$y = af(b(x + c)) + d$$

Horizontal shift

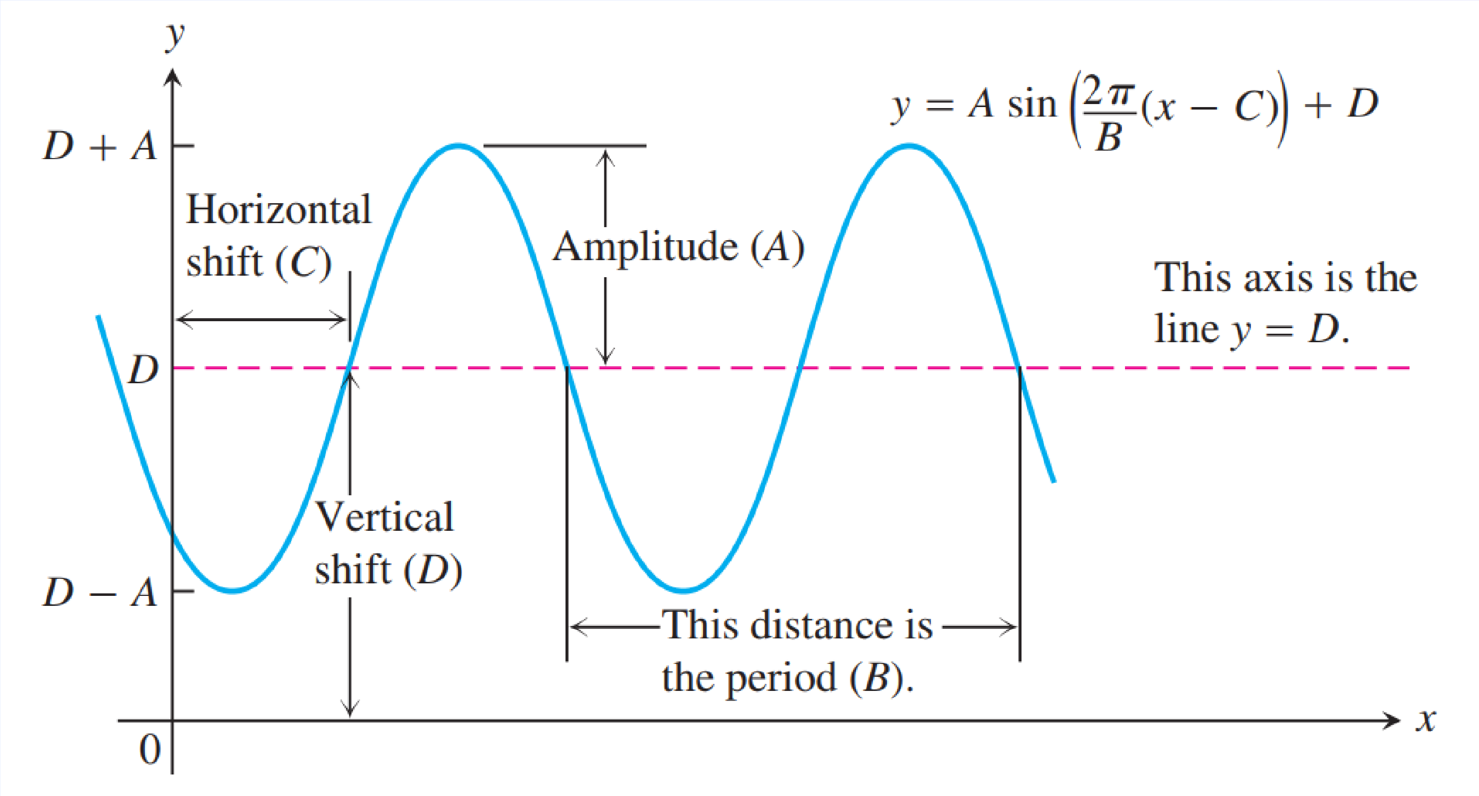





Horizontal stretch or compression;
reflection about *y*-axis if negative

- Transformations of the Graph of the Sine Functions

$$y = Asin[\frac{2\pi}{B}(x - C)] + D$$



- Transformations of the Graph of the Sine Functions

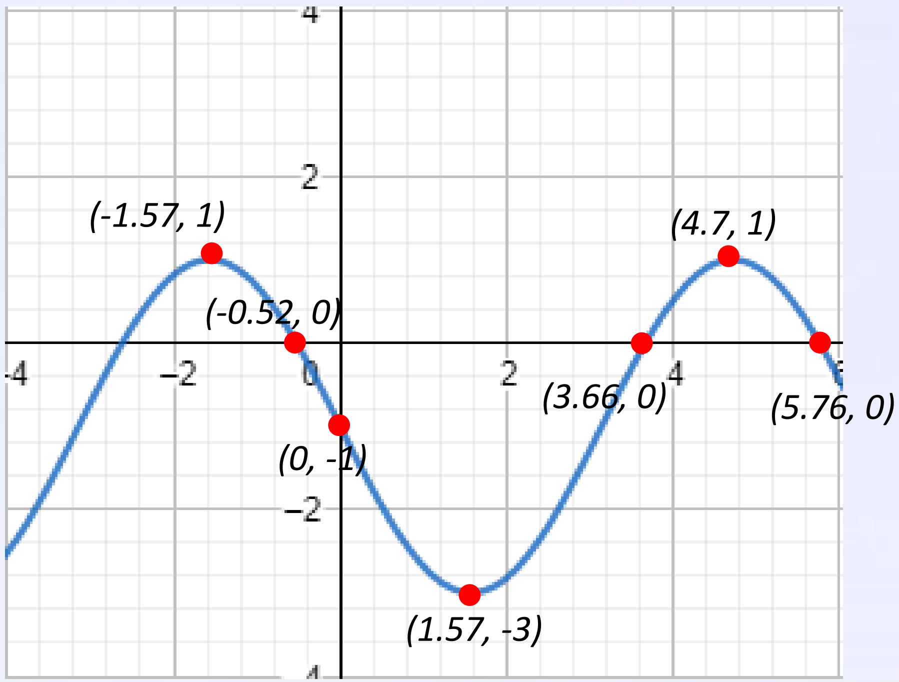
❖ Example 26 For $y = Asin[\frac{2\pi}{B}(x - C)] + D$

Identify with graph A, B, C and D for the sin function $y = 2sin(x + \pi) - 1$

Solution:

*A= 2, B= 2π, C= -π, D= -1*

| x | $y = 2sin(x + \pi)] - 1$ |
|---|---|
| -1.57 | 1 |
| -0.52 | 0 |
| 0 | -1 |
| 1.57 | -3 |
| 3.66 | 0 |
| 4.7 | 1 |
| 5.76 | 0 |



Assignment 4

1. One of sin x, cos x, and tan x is given here. Find the other two if x lies in the specified interval.

*a.* $\sin x = \frac{3}{5},\ x \in [\frac{\pi}{2}, \pi]$

*b.* $\cos x = \frac{1}{3},\ x \in [-\frac{\pi}{2}, 0]$

*c.* $\tan x = \frac{1}{2},\ x \in [\pi, \frac{3\pi}{2}]$

2. Express the given quantity in terms of sin x and cos x.

*a.* $\cos(\pi + x)$

*b.* $\sin(2\pi - x)$

*c.* $\cos(\frac{3\pi}{2} + x)$

d. Evaluate $\cos\frac{11\pi}{12}\ as\ cos(\frac{\pi}{4} + \frac{2\pi}{3})$

e. Evaluate $\sin(\frac{5\pi}{12})$

3. For $y = Asin[\frac{2\pi}{B}(x - C)] + D$ , Identify with graph A, B, C and D for the sin function

*a.* $y = \frac{1}{2} sin(\pi x - \pi) + \frac{1}{2}$

*a.* $y = -\frac{2}{\pi} sin\left(\frac{\pi}{2} t\right) + \frac{1}{\pi}$

- Limits and continuity

## Limits

If the value of f(x) can be made as close as we like to L by taking the value of x sufficiently close to a (but not equal a), then we write:

$$\lim_{x\to a} f(x) = L$$

Properties of limits:

1. If f(x)=K, then $\lim_{x\to a} f(x) = K$, where a and k are real numbers
2. $\lim_{x\to a}[f_1(x) + f_2(x)] = \lim_{x\to a} f_1(x) + \lim_{x\to a} f_2(x)$
3. $\lim_{x\to a}[f_1(x) - f_2(x)] = \lim_{x\to a} f_1(x) - \lim_{x\to a} f_2(x)$
4. $\lim_{x\to a}[f_1(x). f_2(x)] = \lim_{x\to a} f_1(x). \lim_{x\to a} f_2(x)$
5. $\lim_{x\to a} Kf(x) = K\lim_{x\to a} f(x)$
6. $\lim_{x\to a} \frac{f_1(x)}{f_2(x)} = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)}$, $\quad \lim_{x\to a} f_2(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^{r/s} = [\lim_{x\to a} f(x)]^{r/s}$, provide that $\lim_{x\to a} f(x)$ is a real number (if s is even, we assume $\lim_{x\to a} f(x) \geq 0$)

Sandwich theorem

$$h(x) \le f(x) \le g(x)$$

$$\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$$

$$\lim_{x\to a} f(x) = L$$

❖ Example 1 find the limits of *f(x):*

*1.* $8 - x^3 \le f(x) \le 8 + x^3$

$$\lim_{x\to 0} 8 - x^3 \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} 8 + x^3$$

$$8 - (0)^3 \le \lim_{x\to 0} f(x) \le 8 + (0)^3$$

$$8 \le \lim_{x\to 0} f(x) \le 8$$

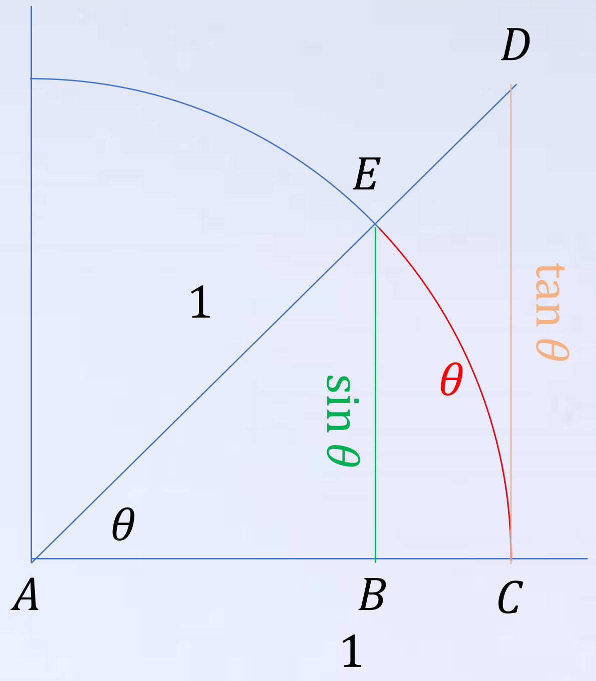
$$\lim_{x\to 0} f(x) = 8$$

## Sandwich theorem

$Arc\ length(CE) = r\theta = 1.\theta = \theta$

$From\ triangle\ ABE\ \sin\theta = \dfrac{opp}{hyp} = \dfrac{opp}{1} \rightarrow opp = \sin\theta$

$From\ triangle\ ACD\ \tan\theta = \dfrac{opp}{adj} = \dfrac{opp}{1} \rightarrow opp = \tan\theta$




$$\sin\theta \leq \theta \leq \tan\theta$$

$$\frac{1}{\sin\theta} \geq \frac{1}{\theta} \geq \frac{1}{\tan\theta} \quad (take\ the\ reciprocal)$$

$$\left(\frac{1}{\sin\theta} \geq \frac{1}{\theta} \geq \frac{1}{\tan\theta}\right) \times \sin\theta$$

$$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$$

$$\lim_{x\to 0} 1 \geq \lim_{x\to 0} \frac{\sin\theta}{\theta} \geq \lim_{x\to 0} \cos\theta$$

$$1 \geq \lim_{x\to 0} \frac{\sin\theta}{\theta} \geq \lim_{x\to 0} \cos 0$$

$$1 \geq \lim_{x\to 0} \frac{\sin\theta}{\theta} \geq 1$$

$$\boxed{\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1}$$

- Limits and continuity

There are different ways to solve the limits

First: Analysis

$$(x^2 - a^2) = (x - a).(x + a)$$

$$(x^3 - a^3) = (x - a).(x^2 + ax + a^2)$$

$$(x^3 + a^3) = (x + a).(x^2 - ax + a^2)$$

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

Second: Multiply the rational function by available denominator

$$\frac{\frac{1}{4} + \frac{1}{x}}{x + 4} * \frac{4x}{4x} = \frac{x + 4}{4x(x + 4)} \rightarrow \frac{1}{4x}$$

- Limits and continuity

Three: Use conjugate

| | | Conjugate | | |
|---|---|---|---|---|
| $x-a$ | $\times$ | $x+a$ | $\rightarrow$ | $x^2-a^2$ |
| $\sqrt{x}+a$ | $\times$ | $\sqrt{x}-a$ | $\rightarrow$ | $x-a^2$ |
| $\sqrt{x-b}-a$ | $\times$ | $\sqrt{x-b}+a$ | $\rightarrow$ | $(x-b)-a^2$ |

Four: Trigonometric functions

$1+tan^2\theta = sec^2\theta$

$1+cot^2\theta = csc^2\theta$

$\sin(2x) = 2sinx.cosx$

❖ Example 1 find the limits of the following:

1. $\lim_{x \to 2} x^2 - 4x = 2^2 - 4 * 2 \to 4 - 8 = -4$

2. $\lim_{x \to 2} \frac{x^2-4}{x^2-5x+6} = \frac{2^2-4}{2^2-5*2+6} = \frac{0}{0}$ (indeterminate quantities)

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)} \to \lim_{x \to 2} \frac{x+2}{x-3} = \frac{2+2}{2-3} = \frac{4}{-1} = -4$$

3. $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \frac{1^2+1-2}{1^2-1} = \frac{0}{0}$ (indeterminate quantities)

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} \to \lim_{x \to 1} \frac{(x+2)}{x} = \frac{(1+2)}{1} = 3$$

❖ Example 1 find the limits of the following:

4. $\lim\limits_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{(x-3)} \quad \to \quad \frac{\left(\frac{1}{3}-\frac{1}{3}\right)}{(3-3)}=\frac{0}{0}$ (indeterminate quantities)

$$\lim_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{(x-3)} \to \lim_{x\to 3}\frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{(x-3)}\times\frac{3x}{3x} \to \lim_{x\to 3}\frac{3-x}{3x(x-3)} \to \lim_{x\to 3}\frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}} \to \lim_{x\to 3}\frac{-1}{3x}=\frac{-1}{9}$$

5. $\lim\limits_{x\to 0}\frac{\left(\frac{1}{x+2}-\frac{1}{2}\right)}{x}=\frac{\left(\frac{1}{0+2}-\frac{1}{2}\right)}{0}=\frac{0}{0}$ (indeterminate quantities)

$$\lim_{x\to 0}\frac{\left(\frac{1}{x+2}-\frac{1}{2}\right)}{x}\times\frac{(x+2)(2)}{(x+2)(2)} \to \lim_{x\to 0}\frac{2-(x+2)}{2x(x+2)} \to \lim_{x\to 0}\frac{\cancel{2}-\cancel{x}-\cancel{2}}{2\cancel{x}(x+2)} \to \lim_{x\to 0}\frac{-1}{2x+4}=\frac{-1}{2(0)+4}=\frac{-1}{4}$$

❖ Example 1 find the limits of the following:

*6.* $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{0+4}-2}{0} = \frac{0}{0}$ (indeterminate quantities)

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\times\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \to \lim_{x\to 0}\frac{(x+4)-4}{x(\sqrt{x+4}+2)} \to \lim_{x\to 0}\frac{\cancel{x}+\cancel{4}-\cancel{4}}{\cancel{x}(\sqrt{x+4}+2)} \to \lim_{x\to 0}\frac{1}{(\sqrt{x+4}+2)} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$$

*7.* $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{4}-2}{4-4} = \frac{0}{0}$ (indeterminate quantities)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\times\frac{\sqrt{x}+2}{\sqrt{x}+2} \to \lim_{x\to 4}\frac{x+2\sqrt{\cancel{x}}-2\sqrt{\cancel{x}}-4}{(x-4)(\sqrt{x}+2)} \to \lim_{x\to 4}\frac{x\cancel{-}4}{(x\cancel{-}4)(\sqrt{x}+2)} \to \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

❖ Example 1 find the limits of the following:

8. $\lim_{x\to 0}\frac{\sin(5x)}{x} = \frac{\sin(5\times 0)}{0} = \frac{0}{0}$ (indeterminate quantities)

$\lim_{x\to 0}\frac{\sin(5x)}{x}\times\frac{5}{5} \to \lim_{x\to 0}\frac{\sin(5x)}{5x}.5$ $\quad(\frac{\sin(5x)}{5x} = 1)$

$\lim_{x\to 0} 5 = 5$

9. $\lim_{x\to 0}\frac{\tan x}{x} \to \lim_{x\to 0}\frac{\sin x}{\cos x}\times\frac{1}{x} \to \lim_{x\to 0}\frac{\sin x}{x}\times\frac{1}{\cos x}$ $\quad(\frac{\sin x}{x} = 1)$

$\lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$

❖ Example 1 find the limits of the following:

*10.* $\lim\limits_{x\to 0} \frac{\sin(x^2)}{x} = \frac{\sin(0^2)}{0} = \frac{0}{0}$ (indeterminate quantities)

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} \times \frac{x}{x} \to \lim_{x\to 0} \frac{\sin(x^2)}{x^2} . x \qquad (\frac{\sin(x^2)}{x^2} = 1)$$

$$\lim_{x\to 0} x = 0$$

*11.* $\lim\limits_{x\to 0} \frac{1-\cos x}{x} \to \frac{1-\cos 0}{0} = \frac{0}{0}$ (indeterminate quantities)

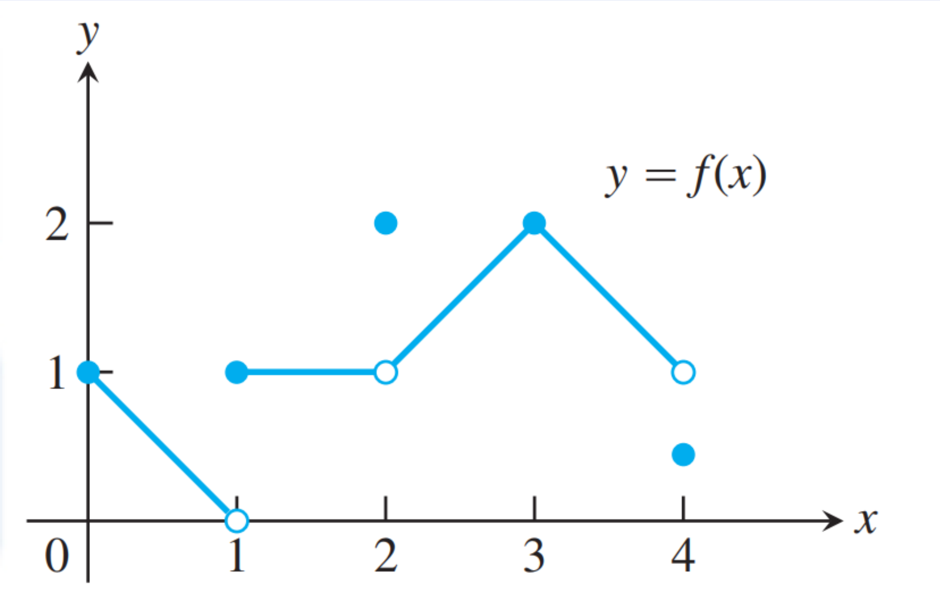
$$\lim_{x\to 0} \frac{1-\cos x}{x} \to \lim_{x\to 0} \frac{1-\cos x}{x} \times \frac{1+\cos x}{1+\cos x} \to \lim_{x\to 0} \frac{1-\cos x}{x} \times \frac{1+\cos^2 x}{x(1+\cos x)}$$

$$\lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \to \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} \to \frac{\sin x}{x} \times \frac{\sin x}{1+\cos x} = 1 + \frac{0}{1+1} = 1 \qquad (\frac{\sin x}{x} = 1)$$

## Right-hand limits and left-hand limits

A function $f(x)$ has a limit as $x$ approaches $c$ if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L \iff \lim_{x\to c^+} f(x) = L$$




❖ Example 2 find the limits of the function graphed in the figure :

At x = 0: $\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of x = 0

At x = 1: $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 0$

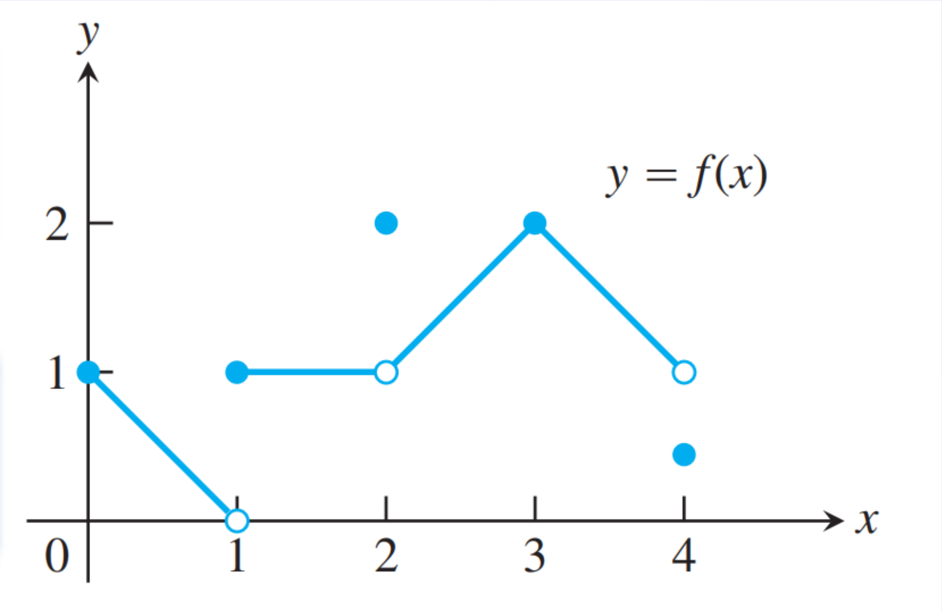
$\lim_{x\to 1} f(x)$ do not exist. The right- and left-hand limits are not equal.

At x = 2: $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 1$

At x = 3: $\lim_{x\to 3^+} f(x) = 3$, $\lim_{x\to 3^-} f(x) = 3$, $\lim_{x\to 3} f(x) = 3$

At x = 4: $\lim_{x\to 4^+} f(x)$, *and* $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of x = 4

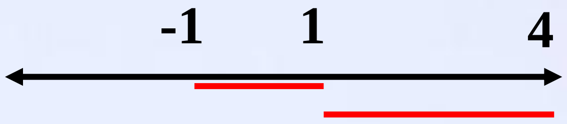
$\lim_{x\to 4^-} f(x) = 4$



❖ Example 2 check the existence of the limit of the function f(x) at x = 1:

$$f(x) = \begin{cases} 2x + 1 & -1 \le x \le 1 \\ \dfrac{x^2}{2} - 3 & 1 < x < 4 \end{cases}$$

Sol:




$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x + 1 = 2(1) + 1 = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2}{2} - 3 = \frac{(1)^2}{2} - 3 = -2.5$$

$\lim_{x \to 1} f(x)$ do not exist. The right- and left-hand limits are not equal.

- Limits involving infinity

These are the limits that include $x \to \infty$ *or* $x \to -\infty$ *and* $f(x) = \infty$ *or* $f(x) = -\infty$

Notes:

$\infty + \infty = \infty$ , $\quad -\infty - \infty = -\infty$, $\quad c \times \infty = \infty$, $\quad c \times -\infty = -\infty$, $\quad -c \times \infty = -\infty$

$-c \times -\infty = \infty$ $\quad$ *$c$ is constant and $c > 0$*

$\frac{c}{\infty} = 0$, $\quad \frac{-c}{\infty} = 0$, $\quad \frac{c}{-\infty} = 0$, $\quad \frac{-c}{-\infty} = 0$

$(\infty)^c = \infty$, $\quad (\infty)^{-c} = 0$

$\infty \times \infty = \infty$ , $\quad -\infty \times \infty = -\infty$, $\quad (-\infty) \times (-\infty) = \infty$,

Indetermined quantities:

$\frac{0}{0}$, $\quad \frac{\infty}{\infty}$, $\quad \infty - \infty$, $\quad 0 \times \infty$

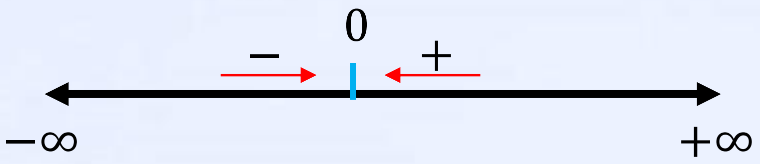
❖ Example 3 find the limits of *f(x)*:

*1.* $\lim\limits_{x\to 0} \frac{1}{x}$

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$\lim\limits_{x\to 0} f(x)$ do not exist. The right- and left-hand limits are not equal.

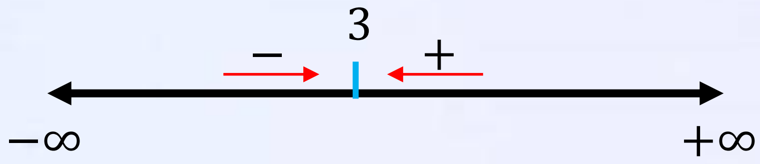



*2.* $\lim\limits_{x\to 3} \frac{x+3}{x-3}$

$$\lim_{x\to 3^+} \frac{x+3}{x-3} = \frac{+}{+} = +\infty$$

$$\lim_{x\to 3^-} \frac{x+3}{x-3} = \frac{+}{-} = -\infty$$

$\lim\limits_{x\to 3} f(x)$ do not exist. The right- and left-hand limits are not equal.

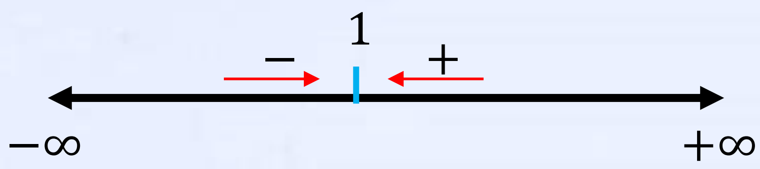


3. $\lim\limits_{x\to 1} \frac{|x+1|}{x-1}$

$$\lim_{x\to 1} \frac{|x+1|}{x-1} = \frac{+}{+} = +\infty$$

$$\lim_{x\to 1^-} \frac{|x+1|}{x-1} = \frac{+}{-} = -\infty$$

$\lim\limits_{x\to 1} f(x)$ do not exist. The right- and left-hand limits are not equal.




4. $\lim\limits_{x\to\infty} \left(\frac{5x+1}{x}\right)$

$$\lim_{x\to\infty} \frac{5x}{x} + \frac{1}{x} =$$

$$\lim_{x\to\infty} 5 + \frac{1}{x} =$$

$$5 + 0 = 5$$

5. $\lim_{x\to\infty}\left(\frac{x}{7x+4}\right)$

$$\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{7x}{x}+\frac{4}{x}} = \qquad \times\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\frac{1}{7+\frac{4}{x}} =$$

$$\frac{1}{7+0}=\frac{1}{7}$$

Note: In rational functions when $x\to\infty$, select the highest power of x in denominator to divide the function on this x power.

6. $\lim_{x\to\infty}\left(\frac{2x^2-x+3}{3x^2+5}\right)$

$$=\lim_{x\to\infty}\left(\frac{2x^2-x+3}{3x^2+5}\right) \qquad \times\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\left(\frac{\frac{2x^2}{x^2}-\frac{x}{x^2}+\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{5}{x^2}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}}\right)$$

$$=\left(\frac{2-\frac{1}{\infty}+\frac{3}{\infty^2}}{3+\frac{5}{\infty^2}}\right)==\left(\frac{2-0+0}{3+0}\right)=\frac{2}{3}$$

7. $\lim_{x\to\infty}\left(\frac{4x^2-3}{3x}\right)$

$$= \lim_{x\to\infty}\left(\frac{4x^2-3}{3x}\right) \qquad \times\frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\left(\frac{\frac{4x^2}{x}-\frac{3}{x}}{\frac{3x}{x}}\right) = \lim_{x\to\infty}\left(\frac{4x-\frac{3}{x}}{3}\right)$$

$$= \left(\frac{4(\infty)-0}{3}\right)$$

$$= \infty$$

The limit does not exist.

8. $\lim_{x\to\infty}\left(\frac{5x+3}{2x^2-1}\right)$

$$= \lim_{x\to\infty}\left(\frac{5x+3}{2x^2-1}\right) \qquad \times\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\left(\frac{\frac{5x}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{5}{x}+\frac{3}{x^2}}{2-\frac{1}{x^2}}\right) = \left(\frac{\frac{5}{\infty}+\frac{3}{\infty^2}}{2-\frac{1}{\infty^2}}\right)$$

$$= \left(\frac{0+0}{2-0}\right) = \frac{0}{2} = 0$$

**Notes for rational functions**

1. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)} = 0 \qquad if\ \deg(f) < \deg(g)$
2. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}\ is\ finite \qquad if\ \deg(f) = \deg(g)$
3. $\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{g(x)}\ is\ infinite \qquad if\ \deg(f) > \deg(g)$

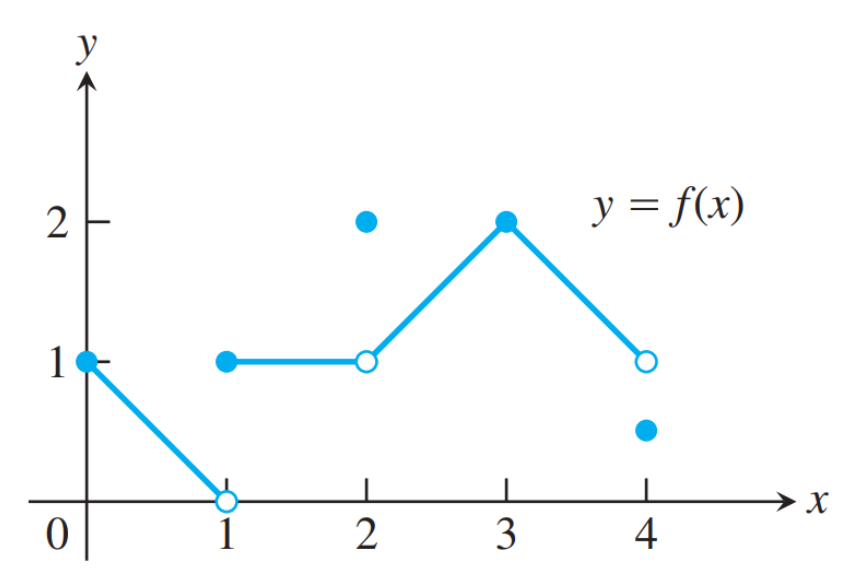
## Continuity

**Continuity Test**

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

**1.** $f(c)$ exists (*c* lies in the domain of *f*)
**2.** $\lim_{x \to c} f(x)$ exists (*f* has a limit as $x \to c$)
**3.** $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example 4 Find the points at which the function *f* in Figure is continuous and the points at which *f* is discontinuous.




At x = 0

$f(0) = 1$

$\lim_{x \to 0+} f(x) = 1$

$\lim_{x \to 0+} f(x) = f(x) = 1$ , Function is continuous at x = 0

At x = 1

$f(1) = 1$

$\lim_{x \to 1^-} f(x) = 0, \qquad \lim_{x \to 1^+} f(x) = 1$

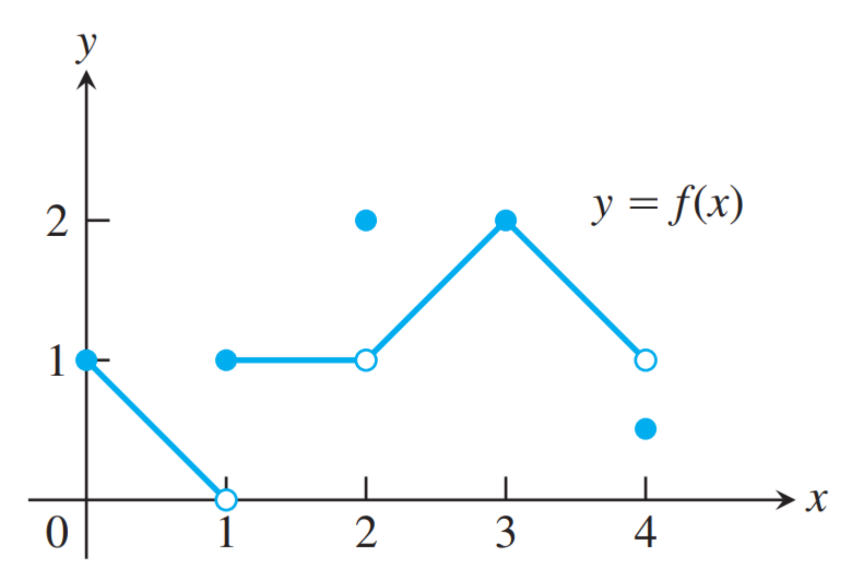
$\lim_{x \to 1} f(x) \, dose\ not\ exist$, Function is discontinuous at x = 1

At x = 2

$f(2) = 2$

$\lim_{x \to 2^-} f(x) = 1, \qquad \lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x)\ is\ exist$, but $\lim_{x \to 2} f(x) \neq f(x)$, Function is discontinuous at x = 2



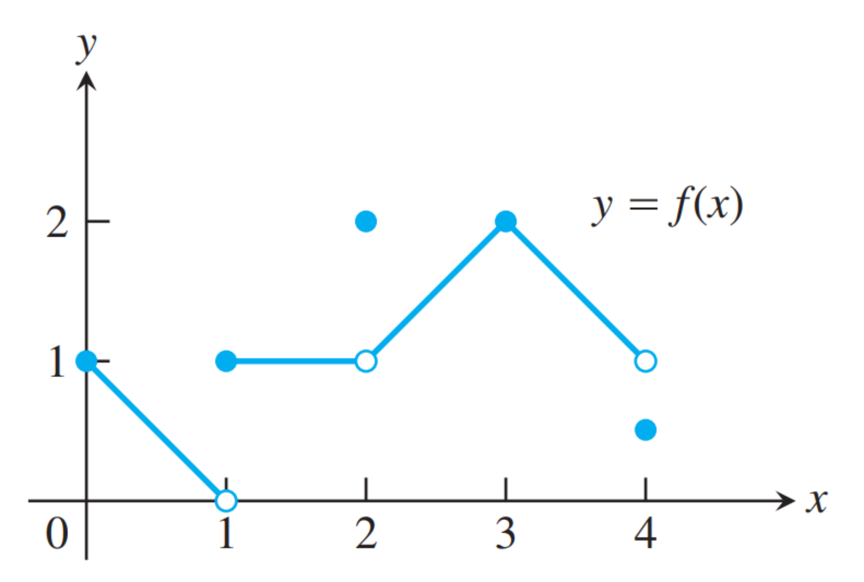
At x = 3

$f(3) = 2$

$\lim_{x \to 3^-} f(x) = 2,$ $\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x)$ *is exist*, and $\lim_{x \to 3} f(x) = f(x),$

Function is continuous at x = 3




At x = 4

$f(4) = \frac{1}{2}$

$\lim_{x \to 4^-} f(x) = 1,$

$\lim_{x \to 4^-} f(x) \neq f(x),$ Function is discontinuous at x = 4

❖ Example 5 determine whether the following functions are continuous at x = 2:

*1.* $f(x) = \frac{x^2-4}{x-2}$

Sol.

$f(2)\ is\ not\ found\ (2 \notin D)$, Function is discontinuous at x = 2

*2.* $f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$

Sol.

$f(2) = 3$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

$\lim_{x\to 2} f(x) \neq f(2)$, Function is discontinuous at x = 2

3. $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{x} \neq 2 \\ 4 & \text{x} = 2 \end{cases}$

Sol.

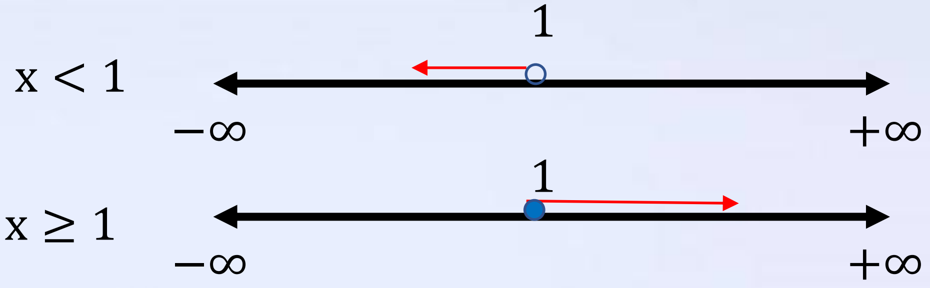
$f(2) = 4$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

$\lim_{x\to 2} f(x) = f(2)$, Function is continuous at x = 2

❖ Example 6  determine whether the following functions are continuous at x = 1:

*2.* $f(x) = \begin{cases} x^2 & x < 1 \\ \frac{x}{2} & x \geq 1 \end{cases}$




Sol.

$$f(1) = \frac{x}{2} = \frac{1}{2}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x}{2} = \frac{1}{2}$$

$\lim_{x \to 1} f(x)\ is\ not\ found,$ Function is discontinuous at x = 1

- Differentiation

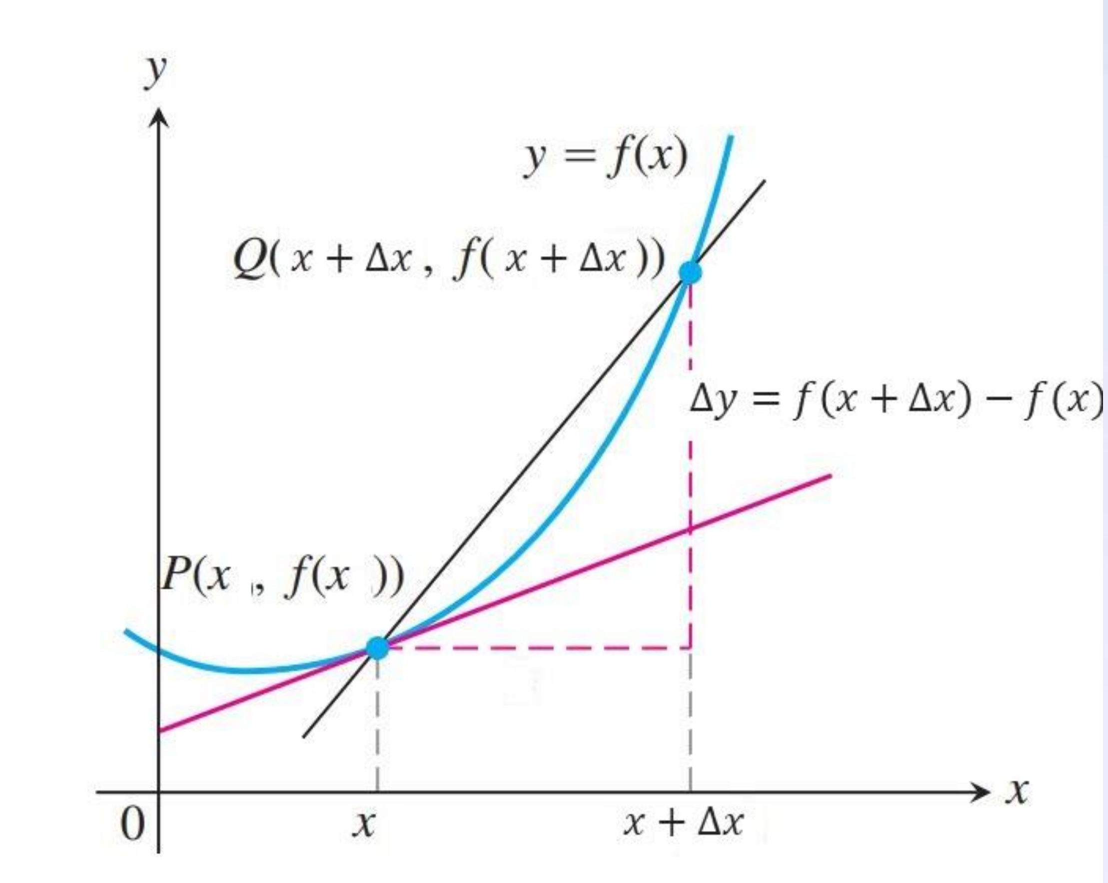
The derivative measures the rate at which a function changes.

If $y = f(x)$

$\therefore \Delta y = f(x + \Delta x) - f(x)$

So, slop of secant PQ $= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

As $Q \to P$, then slope of secant P$Q$ will equal to the slope of tangent of the curve y $= f(x)$ at P $\to Q$ or $\Delta x \to 0$.




$$f^{\backslash}(x) = \frac{dy}{dx} = y^{\backslash} = \frac{df}{dx} = \frac{d}{dx} f(x) = D_x f(x)$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This call definition of derivative of function *f(x)*.

❖ Example 10 find the derivative of the functions using the definition of derivative.

1. $f(x) = x^2$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x$$

2. $f(x) = \sqrt{x}$ for x > 0.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)} - \sqrt{x}}{\Delta x} \times \frac{\sqrt{(x+\Delta x)} + \sqrt{x}}{\sqrt{(x+\Delta x)} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{(x+\Delta x)} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{(\sqrt{(x+\Delta x)} + \sqrt{x})}$$

$$= \frac{1}{(\sqrt{(x+0)} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

## Differentiation Rules

1. If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

RULE 1 Derivative of a Constant Function

$\frac{d}{dx}(8) = 0,$ $\frac{d}{dx}(\sqrt{5}) = 0,$ $\frac{d}{dx}(\pi) = 0,$

2. If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

RULE 2 Power Rule for Positive Integers

$\frac{d}{dx}(x) = 1,$ $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},$ $\frac{d}{dx}(x^4) = 4x^3,$

## Differentiation Rules

3. If $u$ is a differentiable function of $x$, and c is a constant, then

RULE 3 Constant Multiple Rule

$$\frac{d}{dx}cu = c\frac{du}{dx}$$

$\frac{d}{dx}(3x) = 3 \times 1 = 3,\qquad \frac{d}{dx}(3x^2) = 3\frac{d}{dx}x^2 = 3 \times 2x = 6x,$

4. If $u$ and $v$ are differentiable functions of $x$, then their sum $u+v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points

RULE 4 Derivative Sum Rule

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$y = x^4 + 12x \rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \rightarrow \frac{dy}{dx} = 4x^3 + 12,$

## Differentiation Rules

5. If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

RULE 5 Derivative Product Rule

Example 11 find the derivative of $y = \frac{1}{x}(x^2 + \frac{1}{x})$

Sol: $u = \frac{1}{x}$ , $\frac{du}{dx} = -\frac{1}{x^2}$

$v = x^2 + \frac{1}{x}$, $\frac{dv}{dx} = 2x - \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} = \left(\frac{1}{x}\right)\left(2x - \frac{1}{x^2}\right) + (x^2 + \frac{1}{x})(-\frac{1}{x^2})$$

$$\frac{dy}{dx} = 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3} = 1 - \frac{2}{x^3}$$

## Differentiation Rules

6. If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$ , then the quotient $u / v$ is differentiable at $x$, and

RULE 6 Derivative Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 12 find the derivative of $y = \frac{t^2-1}{t^2+1}$

Sol: $u = t^2 - 1,\quad \frac{du}{dt} = 2t$

$v = t^2 + 1,\quad \frac{dv}{dt} = 2t$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} = \frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}$$

$$\frac{dy}{dt} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$\frac{dy}{dt} = \frac{4t}{(t^2+1)^2}$$

Example 13 find the horizontal tangents of $y = x^4 - 2x^2 + 2$

Sol: The horizontal tangents, if any, occur where the slope $\frac{dy}{dx}$ is zero.

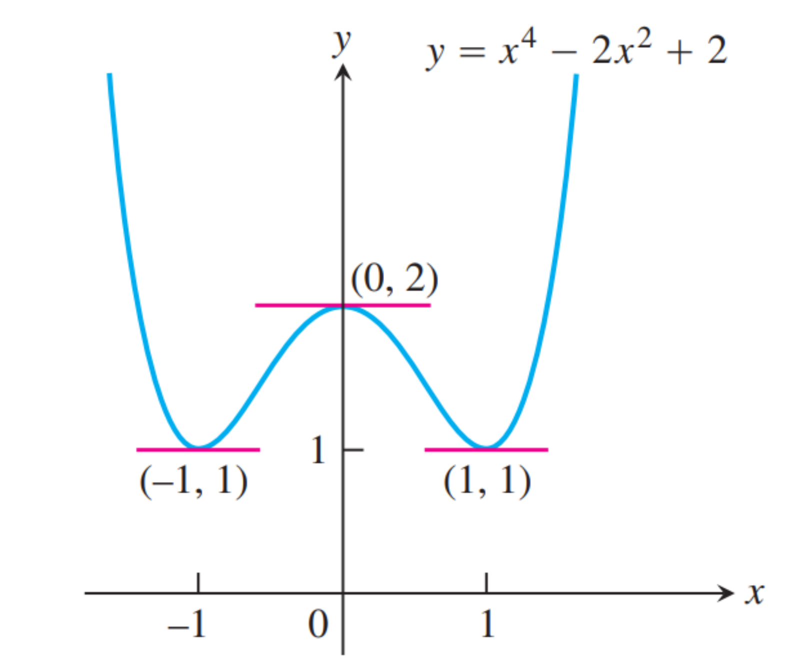
$$\frac{dy}{dx} = \frac{d}{dx}x^4 - \frac{d}{dx}2x^2 + \frac{d}{dx}2$$

$$\frac{dy}{dx} = 4x^3 - 4x + 0$$

$$\frac{dy}{dx} = 0 = 4x^3 - 4x$$

$$4x(x^2 - 1) = 0$$

$$x = 0, 1, -1$$




| x | $y = x^4 - 2x^2 + 2$ |
|---|---|
| 0 | 2 |
| 1 | 1 |
| -1 | 1 |

Example 14 find the derivative of

1. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Sol:

$$\frac{dy}{dt} = \frac{d}{dt}x^3 + \frac{d}{dt}\frac{4}{3}x^2 - \frac{d}{dt}5x + \frac{d}{dt}1$$

$$\frac{dy}{dt} = 3x^2 + \frac{4}{3} \times 2x - 5 + 0$$

$$\frac{dy}{dt} = 3x^2 + \frac{8}{3}x - 5$$

2. $y = (x^2 + 1)(x^3 + 3)$

Sol:

$$u = x^2 + 1, \qquad \frac{du}{dx} = 2x$$

$$v = x^3 + 3, \qquad \frac{dv}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x$$

**Or**

$$y = (x^2 + 1)(x^3 + 3) = x^5 + 3x^2 + x^3 + 3$$

$$y = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

- ## The Chain Rule

If $f(u)$is differentiable at the point $u = g(x)$and $g(x)$ is differentiable at $x$, then the composite function

$(f \circ g)(x) = f(g(x))$ is differentiable at x.

If $y = f(u)$ *and* $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Where $dy/du$ is evaluated at $u = g(x)$.

Example 15 find the derivative of

1. $y = 9x^4 + 6x^2 + 1$

Sol:

$9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$

$y = u^2$ and $u = 3x^2 + 1$

$$\frac{dy}{du} = 2u, \qquad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2u)(6x) = 2(3x^2 + 1) \times 6x$$

$$= 36x^3 + 12x$$

2. $y = \frac{(\sqrt[3]{x^2+2})^2-1}{(\sqrt[3]{x^2+2})^2+1}$

Sol:

$y = \frac{(u)^2-1}{(u)^2+1}$ and $u = \sqrt[3]{x^2+2}$

$$\frac{dy}{du} = \frac{2u(u^2+1) - 2u(u^2-1)}{(u^2+1)^2}$$

$$= \frac{2u^3 + 2u - 2u^3 + 2u}{(u+1)^2}$$

$$\frac{dy}{du} = \frac{4u}{(u+1)^2}$$

$$\frac{du}{dx} = (x^2+2)^{\frac{1}{3}} = \frac{1}{3}(x^2+2)^{\frac{-2}{3}} \times 2x$$

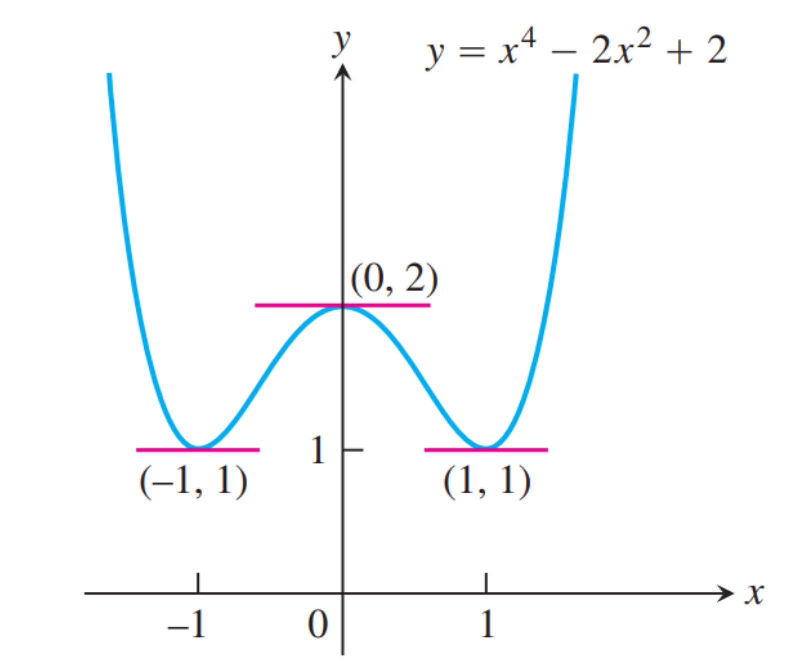
$$\frac{du}{dx} = \frac{2x}{3(x^2+2)^{\frac{2}{3}}} = \frac{2x}{3(\sqrt[3]{x^2+2})^2} = \frac{2x}{3u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{4u}{(u+1)^2} \times \frac{2x}{3u^2}$$

$$\frac{dy}{dx} = \frac{8x}{3u(u+1)^2}$$

Example 13 find the horizontal tangents of $y = x^4 - 2x^2 + 2$




Sol: The horizontal tangents, if any, occur where the slope $\frac{dy}{dx}$ is zero.

$$\frac{dy}{dx} = \frac{d}{dx}x^4 - \frac{d}{dx}2x^2 + \frac{d}{dx}2$$

$$\frac{dy}{dx} = 4x^3 - 4x + 0$$

$$\frac{dy}{dx} = 0 = 4x^3 - 4x$$

$$4x(x^2 - 1) = 0$$

$$x = 0, 1, -1$$

| **x** | $y = x^4 - 2x^2 + 2$ |
|---|---|
| 0 | 2 |
| 1 | 1 |
| -1 | 1 |

Example 14 find the derivative of

1. $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Sol:

$$\frac{dy}{dt} = \frac{d}{dt}x^3 + \frac{d}{dt}\frac{4}{3}x^2 - \frac{d}{dt}5x + \frac{d}{dt}1$$

$$\frac{dy}{dt} = 3x^2 + \frac{4}{3} \times 2x - 5 + 0$$

$$\frac{dy}{dt} = 3x^2 + \frac{8}{3}x - 5$$

2. $y = (x^2 + 1)(x^3 + 3)$

Sol:

$$u = x^2 + 1, \qquad \frac{du}{dx} = 2x$$

$$v = x^3 + 3, \qquad \frac{dv}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x$$

**Or**

$$y = (x^2 + 1)(x^3 + 3) = x^5 + 3x^2 + x^3 + 3$$

$$y = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

- **The Chain Rule**

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at $x$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x.

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Where $dy/du$ is evaluated at $u = g(x)$.

Example 15 find the derivative of

1. $y = 9x^4 + 6x^2 + 1$

Sol:

$9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$

$y = u^2$ and $u = 3x^2 + 1$

$\frac{dy}{du} = 2u, \qquad \frac{du}{dx} = 6x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2u)(6x) = 2(3x^2 + 1) \times 6x$

$= 36x^3 + 12x$

2. $y = \frac{(\sqrt[3]{x^2+2})^2-1}{(\sqrt[3]{x^2+2})^2+1}$

Sol:

$y = \frac{(u)^2-1}{(u)^2+1}$ and $u = \sqrt[3]{x^2+2}$

$$\frac{dy}{du} = \frac{2u(u^2+1) - 2u(u^2-1)}{(u^2+1)^2}$$

$$= \frac{2u^3 + 2u - 2u^3 + 2u}{(u+1)^2}$$

$$\frac{dy}{du} = \frac{4u}{(u+1)^2}$$

$$\frac{du}{dx} = (x^2+2)^{\frac{1}{3}} = \frac{1}{3}(x^2+2)^{\frac{-2}{3}} \times 2x$$

$$\frac{du}{dx} = \frac{2x}{3(x^2+2)^{\frac{2}{3}}} = \frac{2x}{3(\sqrt[3]{x^2+2})^2} = \frac{2x}{3u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{4u}{(u+1)^2} \times \frac{2x}{3u^2}$$

$$\frac{dy}{dx} = \frac{8x}{3u(u+1)^2}$$

- **Derivatives of Trigonometric Functions**

if u is a function $of\ x, then$

1. $\frac{d}{dx}\sin u = \cos u \, . \frac{du}{dx}$

2. $\frac{d}{dx}\cos u = -\sin u \, . \frac{du}{dx}$

3. $\frac{d}{dx}\tan u = sec^2\, u \, . \frac{du}{dx}$

4. $\frac{d}{dx}\sec u = sec\, u \, . \tan u . \frac{du}{dx}$

5. $\frac{d}{dx}\cot u = -csc^2\, u \, . \frac{du}{dx}$

6. $\frac{d}{dx}\csc u = -csc\, u \, . \cot u . \frac{du}{dx}$

Example 16 find the derivative of:

1. $y = x^2 - sinx$

Sol:

$$dy/dx = \frac{d}{dx}x^2 - \frac{d}{dx}sinx \rightarrow dy/dx = 2x - cosx$$

2. $y = x^2 sinx$

$$dy/dx = x^2\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x^2$$

$$dy/dx = x^2\cos x + 2x\sin x$$

3. $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\,.\frac{d}{dx}\sin x - \sin x\,.\frac{d}{dx}x}{x^2}$$

$$\frac{dy}{dx} = \frac{x.\cos x - \sin x}{x^2}$$

4. $y = 5x + \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}5x + \frac{d}{dx}\cos x$$

$$\frac{dy}{dx} = 5 - sin\,x$$

5. $y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\sin x$$

$$\frac{dy}{dx} = \sin x\,(-\sin x) + \cos x\,(\cos x)$$

$$\frac{dy}{dx} = \cos^2 x - \sin^2 x$$

6. $y = \frac{\cos x}{1-\sin x}$

Sol:

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}\cos x - \cos x \,.\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{(1-\sin x)\,(-\sin x) - (\cos x)(-\cos x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-\sin x + sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{1-\sin x}{(1-\sin x)^2} \to \frac{1}{(1-\sin x)}$$

7. $y = sec^2\, 5x$

$$\frac{dy}{dx} = 2\sec 5x \,.\, sec\; 5x \,.\, \tan 5x.\, 5$$

$$\frac{dy}{dx} = 10\sec^2 5x \,.\, \tan 5x$$

8. $y = (5x^3 - x^4)^7$

$$\frac{dy}{dx} = 7(5x^3 - x^4)^6.\,(15x^2 - 4x^3)$$

Example 17 An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of $t$.

Sol:

The velocity is $dx/dt$ .

$$dx/dt = \frac{d}{dt}\cos(t^2 + 1) \, . \, \frac{d}{dt}(t^2 + 1)$$

$$dx/dt = -\, sin(t^2 + 1) \, . \, 2t$$

$$dx/dt = -\, 2t \; sin(t^2 + 1)$$

Example 18 Find the slope of the line tangent to the curve:

1. $y = sin^5 x$

2. $y = \frac{1}{(1-2x)^3}$

Sol:

1. $dy/dx = 5\, sin^4 x \, . \cos x$

2. $y = (1 - 2x)^{-3}$

$$dy/dx = -3(1 - 2x)^{-4} . -2$$

$$dy/dx = 6(1 - 2x)^{-4}$$

$$dy/dx = \frac{6}{(1 - 2x)^4}$$

## • Parametric Equations

Instead of describing a curve by expressing the *y*-coordinate of a point *P(x, y)* on the curve as a function of *x*, it is sometimes more convenient to describe the curve by expressing both coordinates as functions of a third variable *t*.

If $x\ and\ y$ are given as functions $x = f(t), \quad y = g(t)$

over an interval of *t*-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

If all three derivatives exist and $\frac{dx}{dt} \neq 0$

$$\boxed{dy/dx = \frac{dy/dt}{dx/dt}}$$

Example 19 find the value of derivative of $dy/dx$ at $t$=6 if $x = 2t + 3$ $and$ $y = t^2 - 1$

Sol:

$dx/dt = 2, dy/dt = 2t$

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = 6$$

Notice that we are also able to find the derivative $dy/dx$ as a function of x.

From $x = 2t + 3 \rightarrow 2t = x - 3 \rightarrow t = \frac{x-3}{2}$

$$dy/dx = t = \frac{x-3}{2}$$

We can use Chain Rule also

$$t = \frac{x-3}{2}$$

$$y = t^2 - 1 \rightarrow (\frac{x-3}{2})^2 - 1 \rightarrow \frac{x^2-6x+9}{4} - 1$$

$$dy/dx = \frac{1}{4}(2x-6) \rightarrow \frac{x-3}{2}$$

Example 20 find the value of derivative of $d^2y/dx^2$ as a function of $t$, if $x = t - t^2$ $and$ $y = t - t^3$

Sol:

i. Find $y' = dy/dx$

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

ii. Differentiate $y'$ with respect to $t$

$$dy'/dt = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{(1 - 2t).\frac{d}{dt}(1 - 3t^2) - (1 - 3t^2).\frac{d}{dt}(1 - 2t)}{(1 - 2t)^2}$$

$$dy'/dt = \frac{(1 - 2t).(-6t) - (1 - 3t^2).(-2)}{(1 - 2t)^2}$$

$$dy'/dt = \frac{-6t + 12t^2 + 2 - 6t^2}{(1 - 2t)^2} \rightarrow dy'/dt = \frac{6t^2 - 6t + 2}{(1 - 2t)^2}$$

$$dy'/dt = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

iii. Divide $dy'/dt$ by $dx/dt$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{6t^2 - 6t + 2}{(1-2t)^2}}{1-2t} = \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

- **Implicit Differentiation**

When we cannot put an equation $\mathrm{F}(\mathrm{x}, \mathrm{y}) = 0$ in the form $\mathrm{y} = f(x)$, to differentiate it in the usual way, we may still be able $dy/dx$ to find by implicit differentiation.

**1.** Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.

**2.** Collect the terms with $dy/dx$ on one side of the equation.

**3.** Solve for $dy/dx$.

Example 21 find the slope of circle $x^2 + y^2 = 25\ at\ the\ point\ (3, -4)$

Sol:

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope at (3,-4) $= -\frac{3}{-4} = \frac{3}{4}$

Example 22 find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Sol:

$$\frac{d}{dx} y^2 = \frac{d}{dx} x^2 + \frac{d}{dx} \sin xy$$

$$2y\frac{dy}{dx} = 2x + \cos xy\,.\,(y + x\frac{dy}{dx})$$

$$2y\frac{dy}{dx} = 2x + y\cos xy + x\cos xy\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - x\cos xy\frac{dy}{dx} = 2x + y\cos xy$$

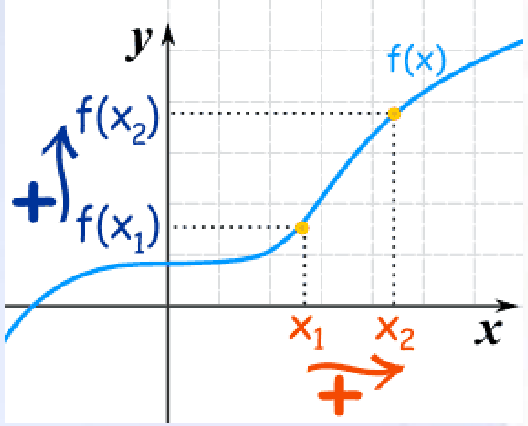
$$\frac{dy}{dx}(2y - x\cos xy) = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

- **Applications of Derivatives**

## Increasing, Decreasing Function

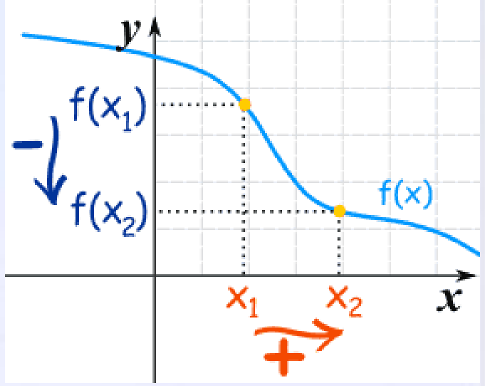
Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be any two points in $I$.




1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ then $f$ is said to be increasing on $I$.

2. If $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ then $f$ is said to be decreasing on $I$.

A function that is increasing or decreasing on $I$ is called **monotonic** on $I$.




## First Derivative Test for Monotonic Functions

Suppose that $f$ is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ increasing on $[a, b]$

2. If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ decreasing on $[a, b]$.

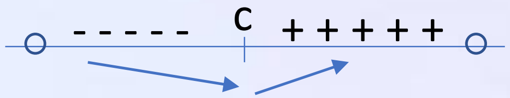
## • Applications of Derivatives

### First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing c except possibly at c itself.

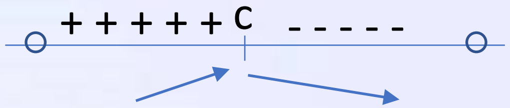
Moving across c from left to right,

1. if $f'$ changes from negative to positive at c, then $f$ has a local minimum at c;



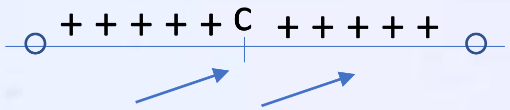

local minimum

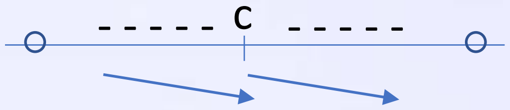
2. if $f'$ changes from positive to negative at c, then $f$ has a local maximum at c;




local maximum

3. if $f'$ does not change sign at c (that is, $f'$ is positive on both sides of c or negative on both sides), then $f$ has no local extremum at c.




An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

- **Applications of Derivatives**

Second Derivative Test for Local Extrema

Suppose $f''$ is continuous on open interval that contains x = c.

1. If $f'(c) = 0 \; and \; f''(c) < 0$, then $f$ has a local maximum at x = c.

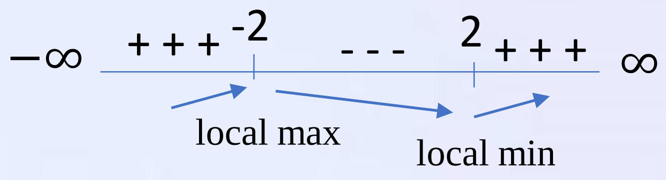
2. If $f'(c) = 0 \; and \; f''(c) > 0$, then $f$ has a local minimum at x = c.

3. If $f'(c) = 0 \; and \; f''(c) = 0$, then the test fails. The function $f$ may has a local maximum, a local minimum, or neither.

Example 1  Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which $f$ is increasing and decreasing. And find if there are any local minimum or maximum at critical points.

Sol:

$f'(x) = 3x^2 - 12 \rightarrow f'(x) = 3(x^2 - 4) \rightarrow f'(x) = 3(x + 2)(x - 2)$




The critical points are $x + 2 = 0 \rightarrow x = -2$ or $x - 2 = 0 \rightarrow x = 2$

For first interval $-\infty < x < -2$

$f'(x) = 3(-3)^2 - 12 \rightarrow 27 - 12 = 15 \rightarrow +$

For second interval $-2 < x < 2$

$f'(x) = 3(0)^2 - 12 \rightarrow 0 - 12 = -12 \rightarrow -$

For third interval $2 < x < \infty$

$f'(x) = 3(3)^2 - 12 \rightarrow 27 - 12 = 15 \rightarrow +$

$f' \therefore f$ *is increasing on* $(-\infty, -2)$ *and* $(2, \infty)$,

*and decresing on* (-2, 2)

To find the local maximum point (x = -2)

$y = (-2)^3 - 12(-2) - 5$

$y = -8 + 24 - 5 \rightarrow y = 11$

the local maximum point is (-2, 11)

To find the local minimum point (x = 2)

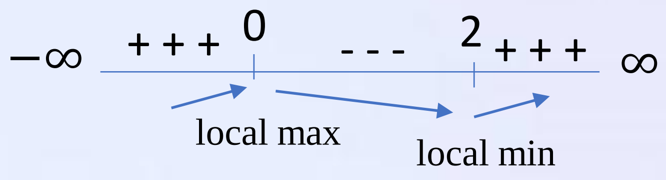
$y = (2)^3 - 12(2) - 5$

$y = 8 - 24 - 5 \rightarrow y = -21$

the local minimum point is (2, -21)

Example 2 Find the critical points of $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$ and identify the intervals on which $f$ is increasing and decreasing. And find if there are any local minimum or maximum at critical points.

Sol:




$$f'(x) = \frac{5}{3} \times 3x^{\frac{5}{3}-1} - \frac{2}{3} \times 15x^{\frac{2}{3}-1} \to 5x^{\frac{2}{3}} - 10x^{\frac{-1}{3}} \to 5x^{\frac{2}{3}} - \frac{10}{x^{\frac{1}{3}}}$$

$$f'(x) = \frac{\left(5x^{\frac{2}{3}} \times x^{\frac{1}{3}}\right) - 10}{x^{\frac{1}{3}}} \to \frac{5x - 10}{x^{\frac{1}{3}}} \to \frac{5(x-2)}{x^{\frac{1}{3}}}$$

The critical points are $x - 2 = 0 \to x = 2$, and $x \neq 0$

For first interval $-\infty < x < 0$

$$f'(x) = \frac{5(-1-2)}{-1^{\frac{1}{3}}} \to \frac{-15}{-1} \to +$$

For second interval $0 < x < 2$

$$f'(x) = \frac{5(1-2)}{1^{\frac{1}{3}}} \to \frac{-15}{1} \to -$$

For third interval $2 < x < \infty$

$$f'(x) = \frac{5(3-2)}{3^{\frac{1}{3}}} \to \frac{5}{1.4} \to +$$

$f' \therefore f$ *is increasing on* $(-\infty, 0)$ *and* $(2, \infty)$,

*and decresing on* $(0, 2)$

To find the local maximum point (x = 0)

$$y = 3(0)^{\frac{5}{3}} - 15(0)^{\frac{2}{3}}$$

$$y = 0$$

the local maximum point is (0, 0)

To find the local minimum point (x = 2)

$$y = 3(2)^{\frac{5}{3}} - 15(2)^{\frac{2}{3}} \rightarrow -14.3$$

the local minimum point is (2, -14.3)

By using Second Derivative Test for Local Extrema

$$f'(x) = \frac{5(x-2)}{x^{\frac{1}{3}}} \rightarrow f''(x) = \frac{5(x-2)}{x^{\frac{1}{3}}} = \frac{10}{3}\left(\frac{x+1}{x^{4/3}}\right)$$

Assignment.

## Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

**(a) concave up** on an open interval $I$ if $f'$ is increasing on $I$.

**(b) concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

## The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval $I$.

**1.** If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.

**2.** If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

## Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

If y is a twice-differentiable function, $y'' = 0$ or undefined at a point of inflection and $y'$ has a local maximum or minimum.

Example 3 Find the intervals on which the following functions are concave up and concave down. Then, if any locate inflection points.

1. $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

Sol:

$-\infty$ + + + -1.22 0.55 $\infty$

$f'(x) = 12x^3 + 12x^2 - 24x$

$f''(x) = 36x^2 + 24x - 24$

$f''(x) = 0 \rightarrow [36x^2 + 24x - 24 = 0] \div 12$

$3x^2 + 2x - 2 = 0$

$x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}, A = 3, B = 2, C = -2$

$x = \frac{-2 \pm \sqrt{4+24}}{6}, x = \frac{-2 \pm \sqrt{28}}{6}, x = \frac{-1 \pm \sqrt{7}}{3}$

Either $x = \frac{-1+\sqrt{7}}{3} \rightarrow x = 0.55$

$y = 3(0.55)^4 + 4(0.55)^3 - 12(0.55)^2 + 2$

$y = -0.69$

or $x = \frac{-1-\sqrt{7}}{3} \rightarrow x = -1.22$

$y = 3(-1.22)^4 + 4(-1.22)^3 - 12(-1.22)^2 + 2$

$y = -16.47$

For first interval $-\infty < x < -1.22$

$f''(x) = 3(-2)^2 + 2(-2) - 2 \rightarrow 12 - 4 - 2 \rightarrow +$

For second interval $-1.22 < x < 0.55$

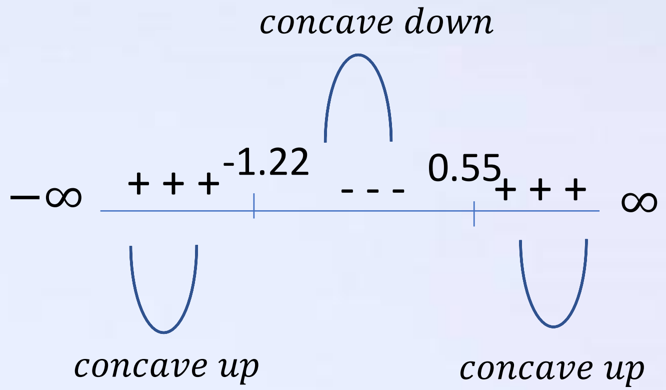
$f''(x) = 3(0)^2 + 2(0) - 2 \rightarrow -$

For third interval $0.55 < x < \infty$

$f''(x) = 3(1)^2 + 2(1) - 2 \rightarrow +$

$\therefore$ *f is concave up on intervals* $(-\infty, -1.22)$ *and* $(0.55, \infty)$,

*and concave down on* (-1.22, 0.55). It has point of inflection at (-1.22, -16.47) and (0.55, -0.69).



- **Curve Sketching**

Strategy for Graphing y $f(x)$

1. Identify the domain of $f$ and any symmetries the curve may have.

2. Find $y'$ and $y''$.

3. Find the critical points of $f$, and identify the function’s behavior at each one.

4. Find where the curve is increasing and where it is decreasing.

5. Find the points of inflection, if any occur, and determine the concavity of the curve.

6. Identify any asymptotes.

7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

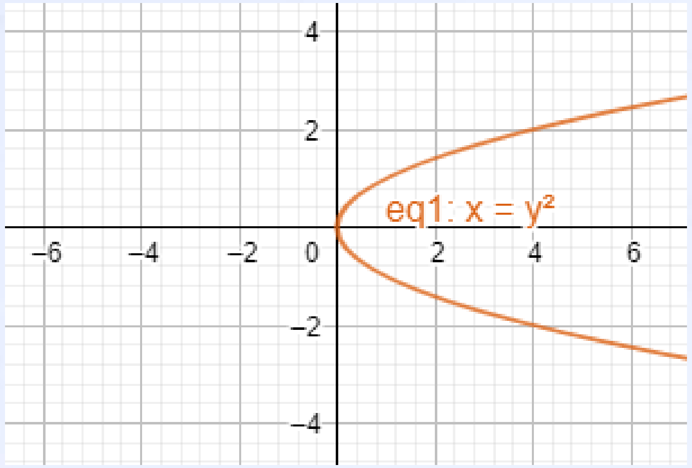
- **Curve Sketching**

The symmetries

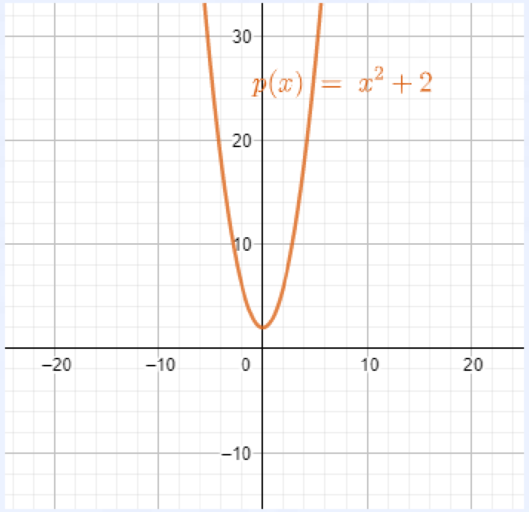
a) The curve has a symmetry about x-axis if $f(x,y) = f(x,-y)$.

b) The curve has a symmetry about y-axis if $f(x,y) = f(-x,y)$. (Even Function)

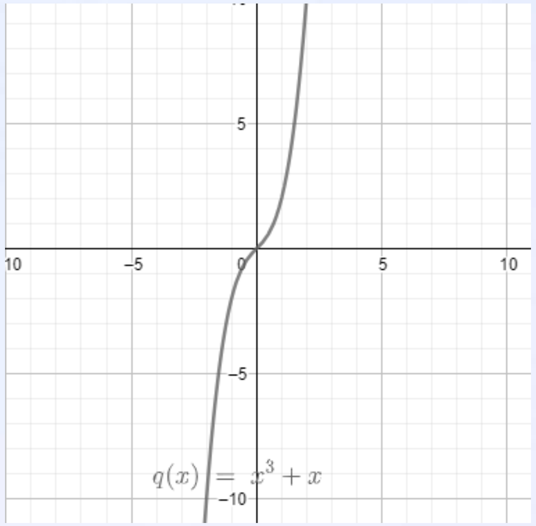
c) The curve has a symmetry about origin if $f(x,y) = f(-x,-y)$. (Odd Function)




$x = y^2$




$y = x^2$




$y = x^3 + x$

Example 9 Sketch the graph of $f(x) = x^4 - 4x^3 + 10$

Sol:

1. Identify the domain of $f$ and any symmetries the curve may have.

*The domain of* $f$ *is* $(-\infty, \infty)$

*symmetries*

$f(1) = 1^4 - 4(1)^3 + 10 \to 7$

$f(-1) = (-1)^4 - 4(-1)^3 + 10 \to 15$

There *are no symmetries about either axis or the origin*.

2. Find $y'$ and $y''$.

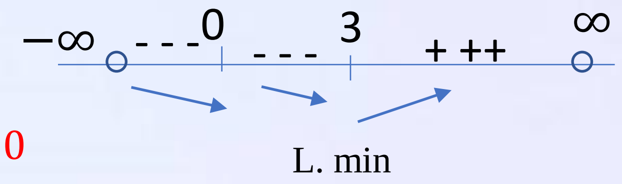
$y' = 4x^3 - 12x^2 \to 4x^2(x-3)$

$y'' = 12x^2 - 24x \to 12x(x-2)$

3. Find the critical points of $f$, and identify the function's behavior at each one.

$y' = 0 = 4x^2(x-3)$

$x = 0 \; or \; x = 3$




For first interval $-\infty < x < 0$

$f'(-1) = 4(-1)^2(-1-3) \to -$

For second interval $0 < x < 3$

$f'(1) = 4(1)^2(1-3) \to -$

For third interval $3 < x < \infty$

$f'(4) = 4(4)^2(4-3) \to +$

4. Find where the curve is increasing and where it is decreasing.

$f' \therefore f\ is\ decreasing\ on\ (-\infty, 0)\ and\ (0,3),$

$and\ incresing\ on\ (3, \infty)$

To find the local minimum point (x = 3)

$y = 3^4 - 4(3)^3 + 10 \rightarrow 81 - 108 + 10 \rightarrow -17$

the local minimum point is (3, -17)

5. Find the points of inflection, if any occur, and determine the concavity of the curve.

$y'' = 0 = 12x(x-2) \rightarrow x = 0\ or\ x = 2$

$-\infty$ + ++ 0 - - - 2 + ++ $\infty$

C. up C. down C. up

For first interval $-\infty < x < 0$

$f''(-1) = 12(-1)(-1-2) \rightarrow +$

For second interval $0 < x < 2$

$f''(1) = 12(1)(1-2) \rightarrow -$

For third interval $2 < x < \infty$

$f''(3) = 12(3)(3-2) \rightarrow +$

To find the inflection at point (x = 0)

$y = 0^4 - 4(0)^3 + 10 \rightarrow 10$

the first point is (0, 10)

To find the inflection at point (x = 2)

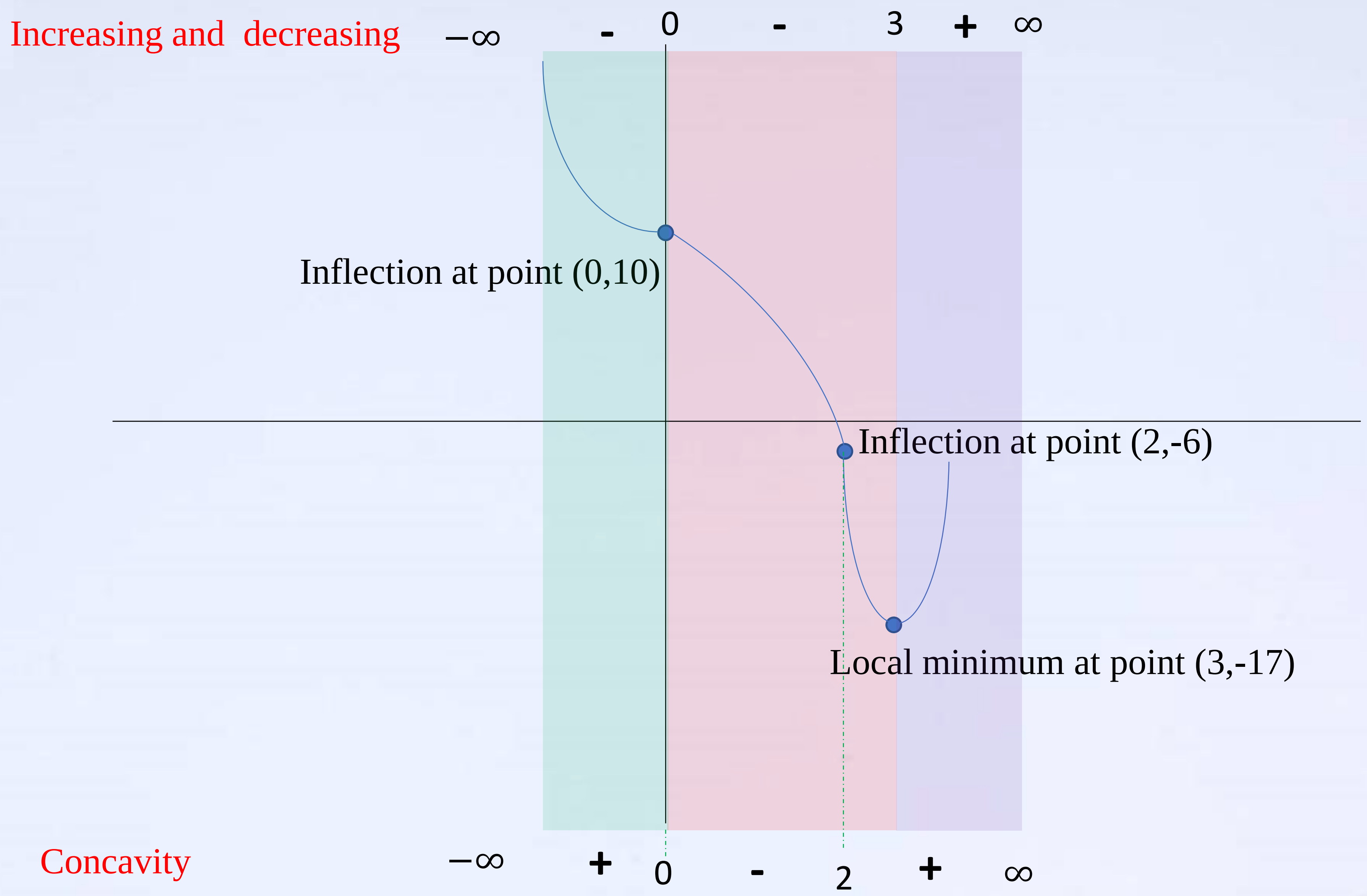
$y = 2^4 - 4(2)^3 + 10 \rightarrow -6$

the second point is (2, -6)

6. Identify any asymptotes.

No asymptotes.

7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve



Example 10 Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

Sol:

1. Identify the domain of $f$ and any symmetries the curve may have.

$The\ domain\ of\ f\ is\ (-\infty, \infty)$

$$f(1) = \frac{(1+1)^2}{1+1^2} = 2.$$

$$f(-1) = \frac{((-1)+1)^2}{1+(-1)^2} = 0.$$

*there are no symmetries about either axis or the origin.*

2. Find $y'$ and $y''$.

$$y' = \frac{2(x+1)(1+x^2)-2x(x+1)^2}{(1+x^2)^2} \to \frac{2(x+1)(1+x^2)-2x(x+1)^2}{(1+x^2)^2}$$

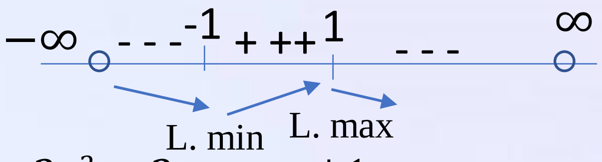
$$y' = \frac{2x+2x^3+2+2x^2-2x^3-4x^2-2x}{(1+x^2)^2} \to \frac{2-2x^2}{(1+x^2)^2}$$

$$y'' = \frac{-4x[(1+x^2)^2]-2(1+x^2)(2x)(2-2x^2)}{(1+x^2)4}$$

$$y'' = \frac{-4x[(1+x^2)]-2\,(2x)(2-2x^2)}{(1+x^2)^3}$$

$$y'' = \frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3} \to \frac{4x^3-12x}{(1+x^2)^3}$$

3. Find the critical points of $f$, and identify the function's behavior at each one.




$$y' = 0 = \frac{2-2x^2}{(1+x^2)^2} \to 2 - 2x^2 = 0 \to 2x^2 = 2 \to x = \pm 1$$

For first interval $-\infty < x < -1$

$f'(-2) = 2 - 2(-2)^2/(1 + (-2)^2)^2 \to -8/25 \to -$

For second interval $-1 < x < 1$

$f'(0) = 2 - 2(0)^2/(1 + (0)^2)^2 \to 2/1 \to +$

For third interval $1 < x < \infty$

$f'(x) = 2 - 2(2)^2/(1 + (2)^2)^2 \to -8/25 \to -$

4. Find where the curve is increasing and where it is decreasing.

$f' \therefore f$ *is decreasing on* $(-\infty, -1)$ *and* $(1, \infty)$,

*and incresing on* (-1, 1)

To find the local maximum point (x = 1)

$$y = \frac{(1+1)^2}{1+1^2} \to \frac{4}{2} = 2$$

the local maximum point is (1, 2)

To find the local minimum point (x = -1)

$$y = \frac{(1+(-1))^2}{1+(-1)^2} \to 0$$

the local minimum point is (-1, 0)

5. Find the points of inflection, if any occur, and determine the concavity of the curve.

$$y'' = 0 = \frac{4x^3 - 12x}{(1+x^2)^3} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

$4x(x^2 - 3) = 0 \to x = 0$ *or* $x = \pm\sqrt{3}$

$-\infty$   - - -   $-\sqrt{3}$   + ++   0   - - -   $\sqrt{3}$   + ++   $\infty$

C. down   C. up   C. down   C. up

For first interval $-\infty < x < -\sqrt{3}$

$f''(-4) = 4(-4)^3 - 12(-4)/(1+(-4)^2)^3 \to (-1024 + 48)$

$/15^2 \to -$

For second interval $-\sqrt{3} < x < 0$

$f''(-1) = 4(-1)^3 - 12(-1)/(1+(-1)^2)^3 \to \frac{8}{8} \to +$

For third interval $0 < x < \sqrt{3}$

$f''(1) = 4(1)^3 - 12(1)/(1+(1)^2)^3 \to \frac{-8}{8} \to -$

For fourth interval $\sqrt{3} < x < \infty$

$f''(4) = 4(4)^3 - 12(4)/(1+(4)^2)^3 \to \frac{1024-48}{15^2} \to +$

To find the inflection at point (x = - $\sqrt{3}$)

$$y = \frac{(-\sqrt{3}+1)^2}{1+(-\sqrt{3})^2} \rightarrow \frac{0.545}{4} = 0.134$$

the first point is (-1.73, 0.134)

To find the inflection at point (x = 0)

$$y = \frac{(0+1)^2}{1+(0)^2} \rightarrow y = 1$$

the second point is (0,1)

To find the inflection at point (x = $\sqrt{3}$)

$$y = \frac{(\sqrt{3}+1)^2}{1+(\sqrt{3})^2} \rightarrow \frac{7.464}{4} = 1.86$$

the third point is (1.73, 1.86)

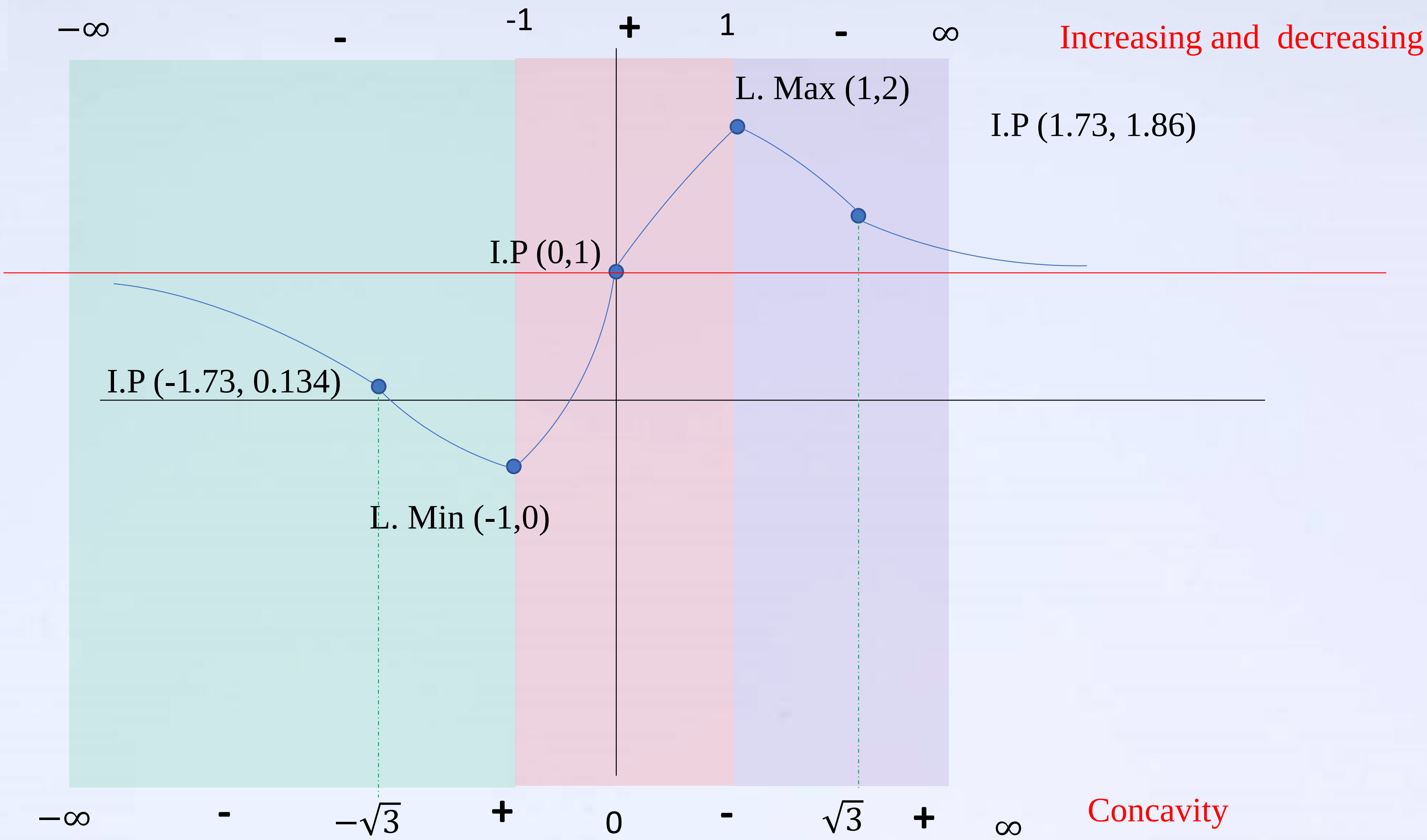
6. Identify any asymptotes.

$$f(x) = \frac{(x+1)^2}{1+x^2} \rightarrow \frac{x^2+2x+1}{1+x^2}$$

No vertical asymptotes, $1 + x^2 \neq 0$

Horizontal asymptotes at y=1

7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve



- Related Rates

## Related Rates Problem Strategy

**1.** *Draw a picture and name the variables and constants*. Use $t$ for time. Assume that all variables are differentiable functions of $t$.

**2.** *Write down the numerical information* (in terms of the symbols you have chosen).

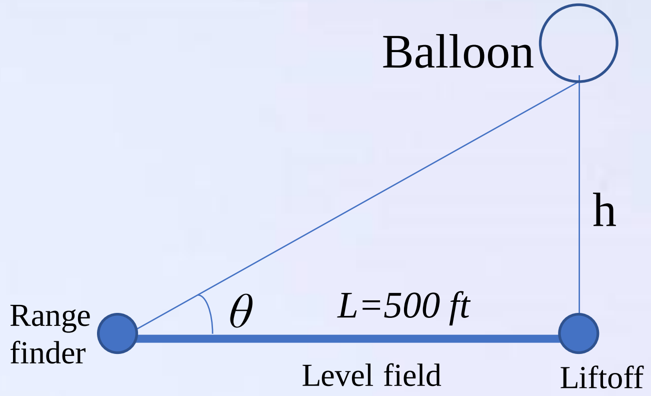
**3.** *Write down what you are asked to find* (usually a rate, expressed as a derivative).

**4.** *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.

**5.** *Differentiate with respect to $t$*. Then express the rate you want in terms of the rate and variables whose values you know.

**6.** *Evaluate*. Use known values to find the unknown rate.

Example 4  A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?




1. Draw a picture and name the variables and constants. The variables in the picture are
θ = the angle in radians the range finder makes with the ground.
h = the height in feet of the balloon.
*The L is constant while* θ *and h are differentiable functions of t.*

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14\,\frac{rad}{min}, \qquad when\ \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want $\frac{dh}{dt}$ $when\ \theta = \frac{\pi}{4}$.

4. *Write an equation that relates the variables y and* $\theta$.

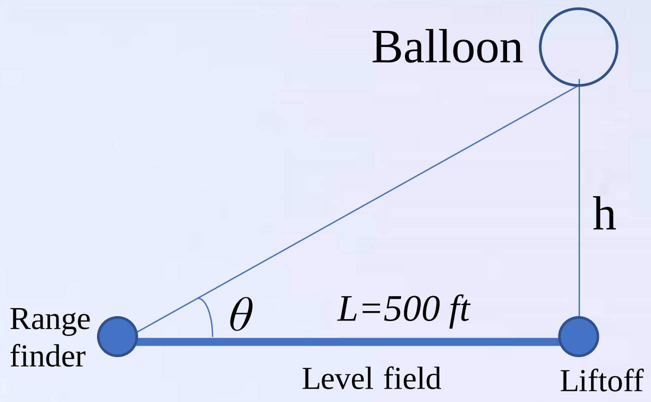
$$\tan\theta = \frac{h}{500} \rightarrow h = 500\ tan\theta$$

5. Differentiate with respect to t using the Chain Rule. The result tells how $\frac{dh}{dt}$ (which we want) is related to $\frac{d\theta}{dt}$(which we know).

$$\frac{d}{dt}h = 500\frac{d}{dt}tan\theta \rightarrow \frac{dh}{dt} = 500(sec^2\theta)\frac{d\theta}{dt}$$

6. *Evaluate with* $\theta = \frac{\pi}{4}$ *and* $\frac{d\theta}{dt} = 0.14$ *to find* $\frac{dh}{dt}$.

$$\frac{dh}{dt} = 500(1.414^2) \times 0.14 \rightarrow \frac{dh}{dt} = 140\ ft/min$$



Example 5 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

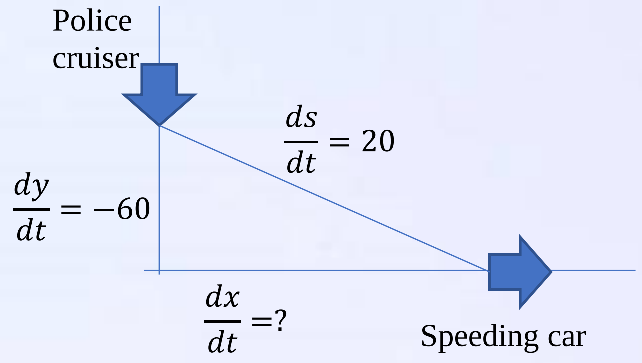
1. Draw a picture and name the variables and constants. The variables in the picture are

x = position of car at time t

*y = position of cruiser at time t*

*s = distance between car and cruiser at time t.*

*The x, y and s are are differentiable functions of t.*




2. *Write down the additional numerical information.*

$$x = 0.8\,mi, y = 0.6\,mi, \quad \frac{dy}{dt} = -60\,mph, \quad \frac{ds}{dt} = 20\,mph$$

3. *Write down what we are to find.* We want $\frac{dx}{dt}$.

4. *Write an equation that relates the variables* $\theta x, y$ *and* $s$.

$$s^2 = x^2 + y^2$$

Need s value $s^2 = x^2 + y^2 \to s^2 = 0.8^2 + 0.6^2 \to s^2 = 1 \to s = 1$

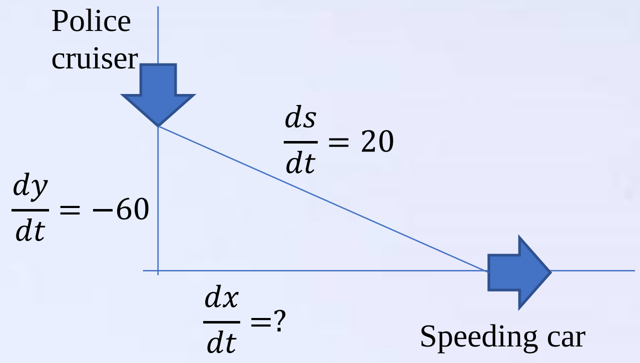
5. Differentiate with respect to t .

$$\frac{d}{dt}s^2 = \frac{d}{dt}x^2 + \frac{d}{dt}y^2 \to 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

6. $Evaluate\ to\ find\ \frac{dh}{dt}$.

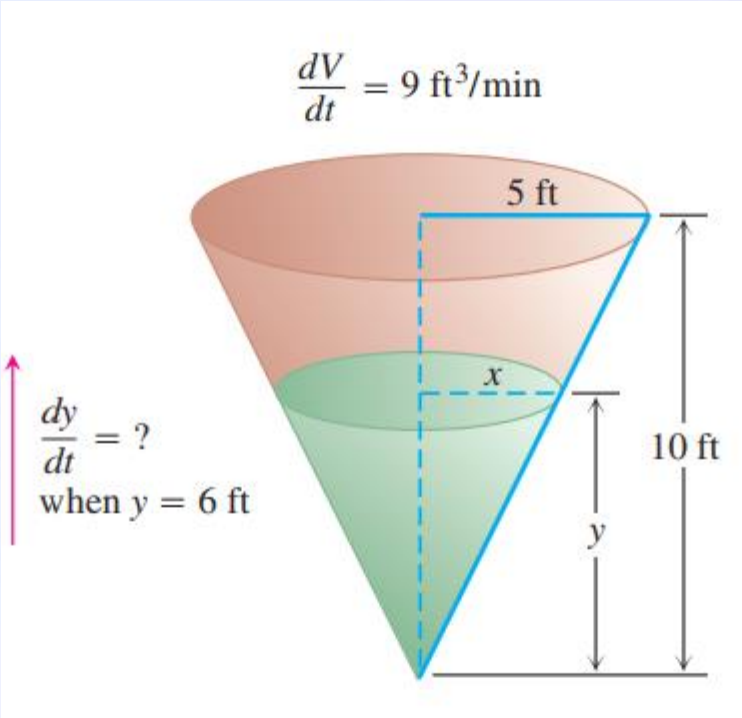
$$2(1)(20) = 2(0.8)\frac{dx}{dt} + 2(0.6)(-60) \to 40 = 1.6\frac{dx}{dt} - 72$$

$$112 = 1.6\frac{dx}{dt} \to \frac{dx}{dt} = 70\ mph$$



Example 6  Water runs into a conical tank at the rate of *9 $ft^3$/min*. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

1. Draw a picture and name the variables and constants. The variables in the picture are
V = volume *$ft^3$* of the water in the tank at time t (min).
x = radius (*ft*) of the surface of the water at time *t*.
y = depth (*ft*) of water in tank at time *t*.
*The V, x and y are are differentiable functions of t.*




2. *Write down the additional numerical information.*

$x = ?, y = 6\ ft, \quad \frac{dV}{dt} = 9 ft^3/min, r = 5\ ft, h = 6\ ft.$

3. *Write down what we are to find*. We want $\frac{dy}{dt}$.

4. *Write an equation that relates the variables $x, y$ and $V$.*

$V = \frac{1}{3}\pi x^2 y \quad \ldots 1$

Because no information is given about x and $\frac{dx}{dt}$ at the time in question, we need to eliminate x by using the similar triangles to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \rightarrow x = \frac{y}{2} \quad \ldots 2$$

By substitute equ.2 in 1

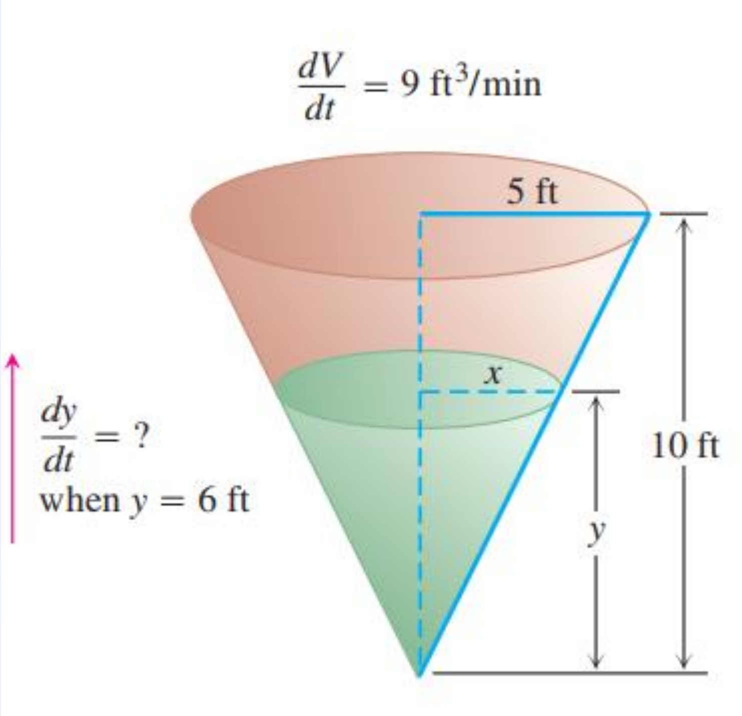
$$V = \frac{1}{3}\pi(\frac{y}{2})^2 y \rightarrow V = \frac{\pi}{12}\, y^3$$

5. Differentiate with respect to t .

$$\frac{\mathrm{d}}{\mathrm{dt}}V = \frac{\pi}{12}\frac{\mathrm{d}}{\mathrm{dt}}y^3 \rightarrow \frac{\mathrm{d}V}{\mathrm{dt}} = \frac{\pi}{4} \times y^2 \times \frac{\mathrm{dy}}{\mathrm{dt}}$$

6. F$ind\,\frac{dy}{dt}$.

$$9 = \frac{\pi}{4} \times 6^2 \times \frac{\mathrm{dy}}{\mathrm{dt}} \rightarrow \frac{\mathrm{dy}}{\mathrm{dt}} = \frac{9 \times 4}{\pi \times 36} \rightarrow \frac{\mathrm{dy}}{\mathrm{dt}} = \frac{1}{\pi} ft/min$$



Example 7  A 17 ft ladder is leaning against building. The foot of the ladder is 8 ft from the base of the building and it's sliding away from the building at 3 ft/s.

a. How fast is the top of the ladder sliding down the wall of the building?

b. How fast is the area rate of the triangle formed by the ladder, wall, and ground changing then?

c. Find the rate at which the angle between the ladder and the ground changing at this instance?

$z = 17\ ft, x = 8\ ft, y = ?, \frac{dx}{dt} = 3\frac{ft}{s}, \frac{dy}{dt} = ?$

*a.* The ladder sliding down rate

- Find relation between $x, y\ and\ z$

$z^2 = x^2 + y^2$

Also need y value,

$17^2 = 8^2 + y^2$

$y^2 = 289 - 64$

$y^2 = 255$

$y = 15$

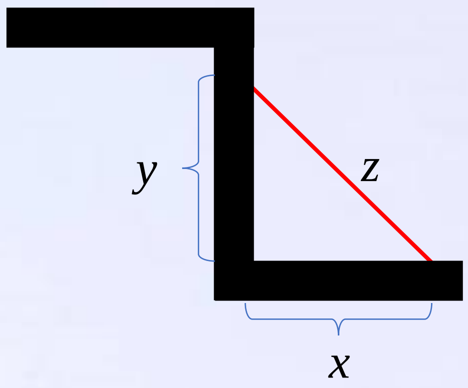
- Find d/dt of $z^2 = x^2 + y^2$

$\frac{d}{dt}z^2 = \frac{d}{dt}x^2 + \frac{d}{dt}y^2$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$2y\frac{dy}{dt} = -2x(3) \rightarrow \frac{dy}{dt} = -\frac{2(8)(3)}{2(15)}$

$\frac{dy}{dt} = -\frac{2(8)(3)}{2(15)} \rightarrow \frac{dy}{dt} = -\frac{8}{5} \rightarrow -1.6\frac{ft}{s}$



*b.* The area changing rate

- Find area relation

$$a = \frac{1}{2}xy$$

- Find da/dt of

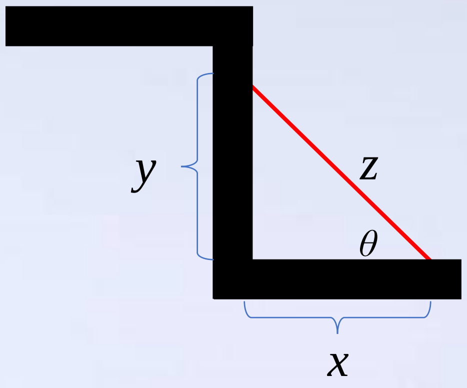
$$\frac{d}{dt}[a = \frac{1}{2}xy]$$

$$\frac{da}{dt} = \frac{1}{2}[\frac{dx}{dt}y + \frac{dy}{dt}x]$$

$$\frac{da}{dt} = \frac{1}{2}[(3)(15) + (-1.6)(8)]$$

$$\frac{da}{dt} = 16.1\,\frac{ft^2}{s}$$

*c.* The angle changing rate




- Find area relation

$$sin\theta = \frac{y}{z}$$

- Find da/dt of

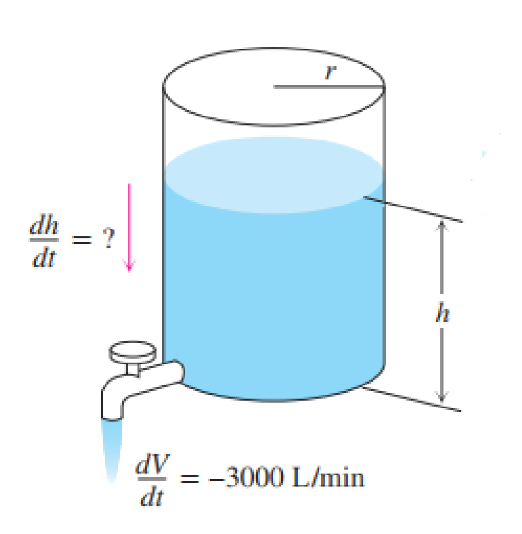
$$\frac{d}{dt}[sin\theta = \frac{1}{z}y]$$

$$cos\theta\frac{d\theta}{dt} = \frac{1}{z}\frac{dy}{dt}$$

$$\frac{8}{17}\frac{d\theta}{dt} = \frac{1}{17}(-1.6)$$

$$\frac{d\theta}{dt} = \frac{1}{8}(-1.6) \rightarrow \frac{d\theta}{dt} = -0.2\frac{rad}{s}$$

Example 8 How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?




Calling its radius *r*, the height of the fluid *h*, he volume of the fluid *V*.
*r* is constant, but *V* and *h* change.

$$h = ?, r = ?, \frac{dV}{dt} = -3000\ L/min$$

The fluid level drop rate (*dh/dt*)

- first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h.
- $V = 1000\pi r^2 h$ *(cubic meter contains 1000 L)*
- Find d/dt of $V = 1000\pi r^2 h$

$$\frac{d}{dt}V = 1000\pi r^2 (\frac{d}{dt}h)$$

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}$$

$$-3000 = 1000\pi r^2 \frac{dh}{dt}$$

$$-\frac{3}{\pi r^2} = \frac{dh}{dt}$$

- **Optimization**

Solving Applied Optimization Problems

1. **Read the problem**. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?

2. **Draw a picture**. Label any part that may be important to the problem.

3. **Introduce variables**. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

4. **Write an equation for the unknown quantity**. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.

5. **Test the critical points and endpoints in the domain of the unknown**. Use what you know about the shape of the function’s graph. Use the first and second derivatives to identify and classify the function’s critical points

Example 1  Find two numbers whose sum is 16 and whose product is a maximum

Sol:

$$Sum = S = x + y, \qquad Product = P = x \times y$$

$$16 = x + y$$

$$16 - x = y$$

$$P = x \times y$$

$$P = x \times (16 - x)$$

$$P = 16x - x^2$$

Maximum at $P' = 0$

$$0 = 16 - 2x$$

$$8 - x = 0$$

$$x = 8$$

$$y = 16 - 8 = 8$$

The maximum product is $P = x \times y \rightarrow 8 \times 8 = 64$

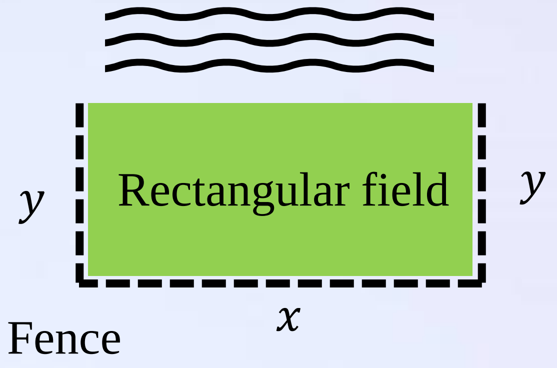
| **x** | $x + y = 16$ | $x \times y$ |
|---|---|---|
| 1 | 15 | 15 |
| 2 | 14 | 28 |
| 3 | 13 | 36 |
| 4 | 12 | 48 |
| 5 | 11 | 55 |
| 6 | 10 | 60 |
| 7 | 9 | 63 |
| 8 | 8 | 64 |

Example 2 A farmer has 3000 ft of fencing and wants to create a rectangular field along a river. He needs no fence along the river. What is the largest area of the rectangular field?

Sol:

$Fence = x + 2y, \qquad 3000 = x + 2y$

$3000 - 2y = x \ldots 1$




$Area\ of$ rectangular field $(A) = x \times y$

$A = (3000 - 2y) \times y$

$A = 3000y - 2y^2$

Largest area at $A' = 0$

$0 = 3000 - 4y \rightarrow y = 750\ ft$

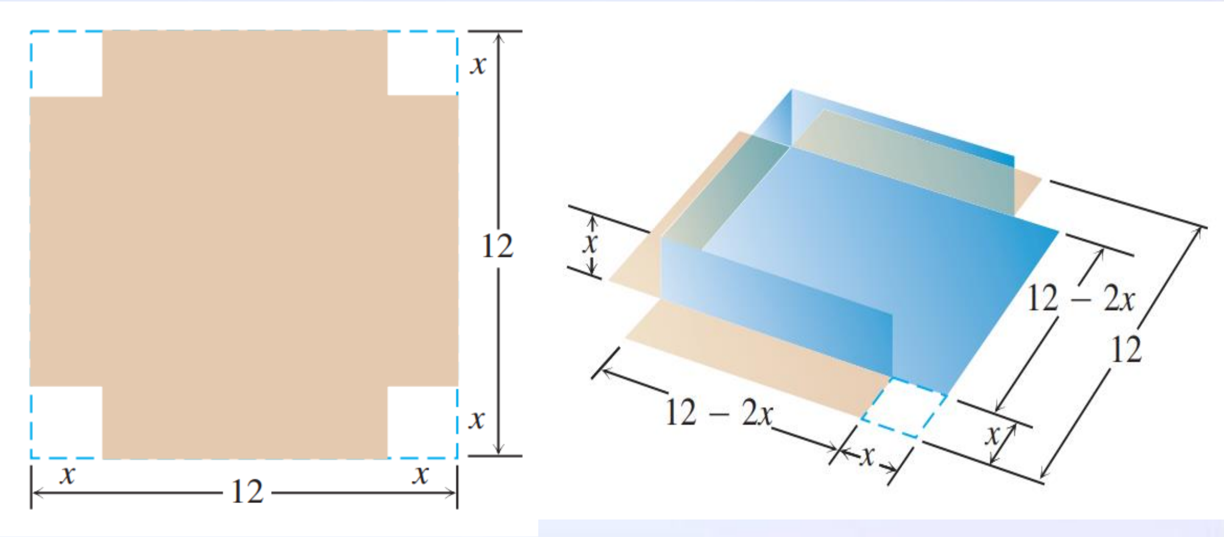
Find x from Eq.1

$3000 - 2(750) = x$

$x = 1500$

$A_{max} = x \times y \rightarrow 1500 \times 750 = 1125000\ \ ft^2$

Example 3 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?




Sol:

The volume of the box is:

$V = hlw \rightarrow V(x) = x(12-2x)^2 = 0, x = 0 \; or \; x = 6$

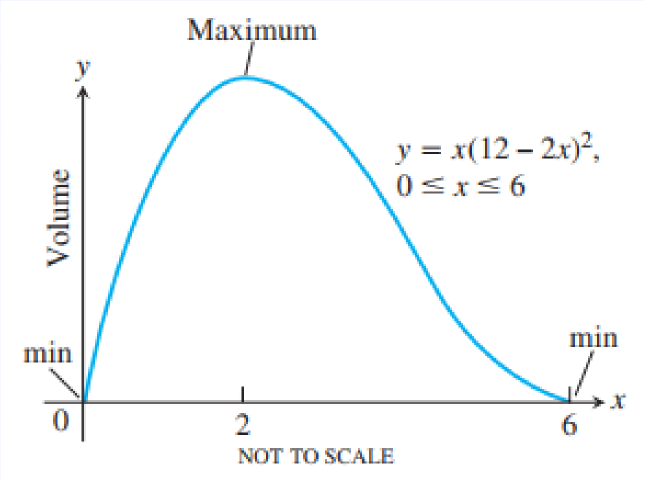
The domain of $V$ is the interval $0 \le x \le 6$

To find Max. value, find the first derivative of V with respect to x:

$V(x) = x(12-2x)^2 = 4x^3 - 48x^2 + 144x$

$$\frac{dV}{dt} = 12x^2 - 96x + 144 \rightarrow \frac{dV}{dt} = 12(x^2 - 8x + 12)$$

$$\frac{dV}{dt} = 12(x-2)(x-6)$$



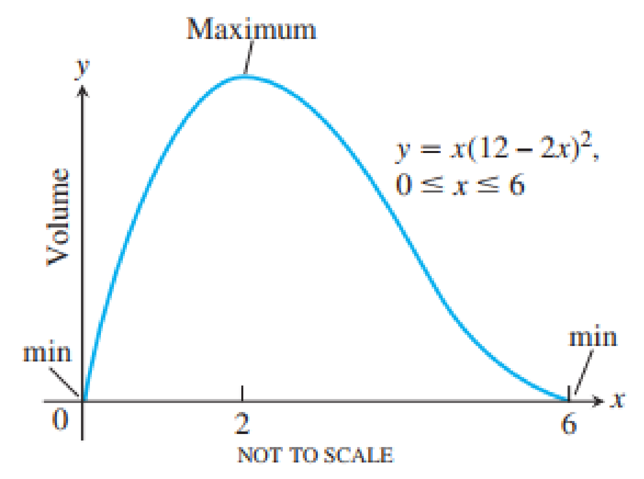
$$\frac{dV}{dt} = 12(x-2)(x-6) \rightarrow x = 2 \; or \; x = 6$$

Of the two zeros, $x = 2 \; and \; x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list.

Critical-point value: V(2) = $2(12-2(2))^2$ = 128

Endpoint values: V(0) = 0, V(6) = 0.

The maximum volume is $128 \; in^3$. The cutout squares should be 2 in. on a side



## Economics

Suppose that:

*r(x)* = the **revenue** from selling x items

*c(x)* = the **cost of producing** the x items

*p(x)* = *r(x)* – *c(x)* = the **profit** from producing and selling x items.

The marginal revenue, marginal cost, and marginal profit when producing and selling x items are:

$\frac{dr}{dx}$ = marginal revenue

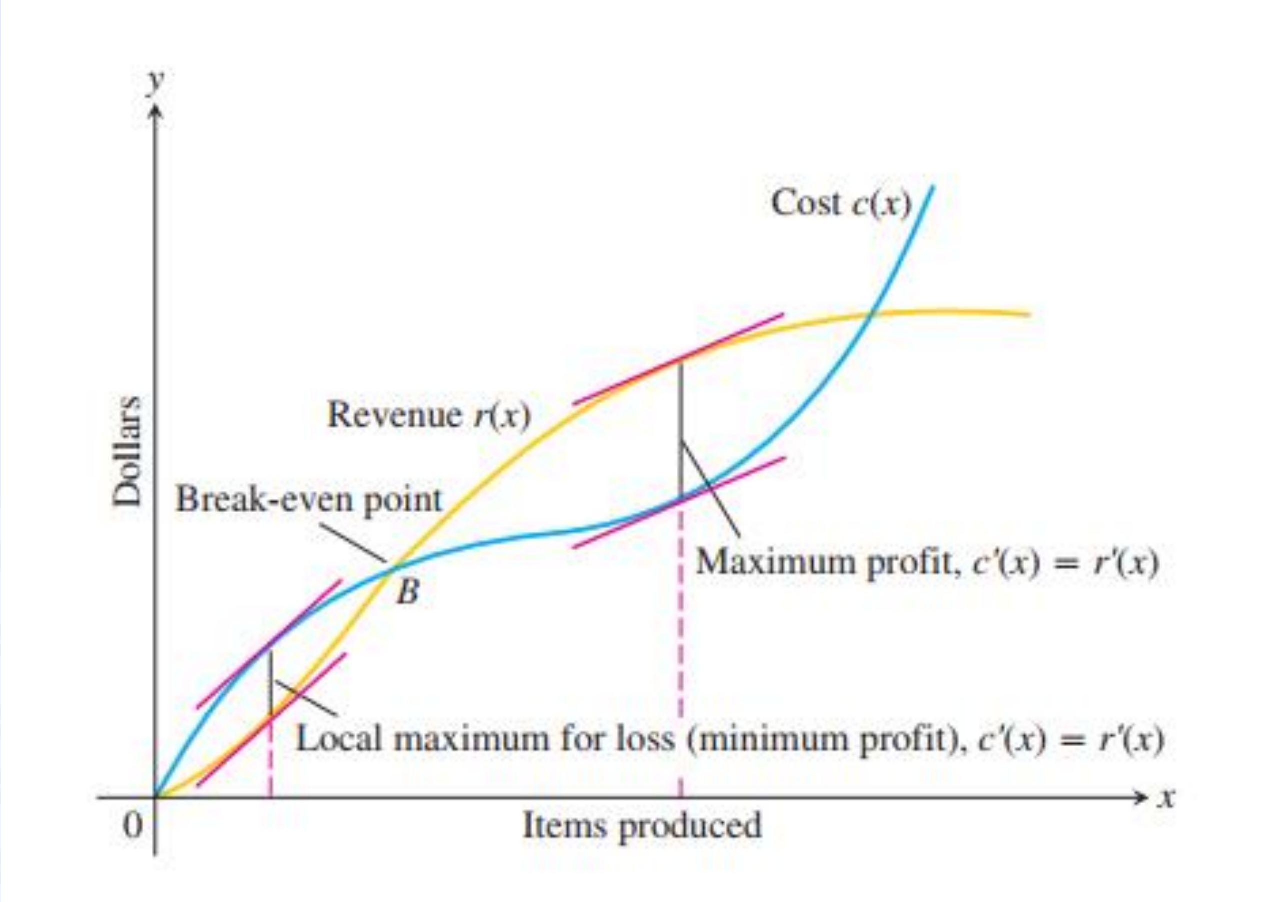
$\frac{dc}{dx}$ = marginal cost,

$\frac{dp}{dx}$ = marginal profit.

The maximum value of profit occurs at a production level $\frac{dp}{dx} = 0$

$p'(x) = r'(x) - c'(x) \rightarrow 0 = c'(x) - r'(x) \rightarrow \boldsymbol{c'(x) = r'(x)}$

The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.



Example 4 Suppose that $\mathrm{r}(x) = 9x$ and $\mathrm{c}(x) = x^3 - 6x^2 + 15x$ where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

Sol:

The maximum at $\boldsymbol{c'(x) = r'(x)}$

$c'(x) = 3x^2 - 12x + 15, \qquad r'(x) = 9$

$3x^2 - 12x + 15 = 9 \rightarrow 3x^2 - 12x + 6 = 0$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, A = 3, B = -12, C = 6$$

$$x = \frac{12 \pm \sqrt{72}}{6} \rightarrow x = \frac{12 + \sqrt{72}}{6} = 3.414 \text{ or } x = \frac{12 - \sqrt{72}}{6} = 0.586$$

The possible production levels for maximum profit are x = 0.586 thousand units or x = 3.414 thousand units

The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = r''(x) - c''(x)$

$c''(x) = 6x - 12, \qquad r'(x) = 0$

$p''(x) = -c''(x) = 12 - 6x$

$p''(x) = -c''(x) = 12 - 6x$

By the Second Derivative Test:

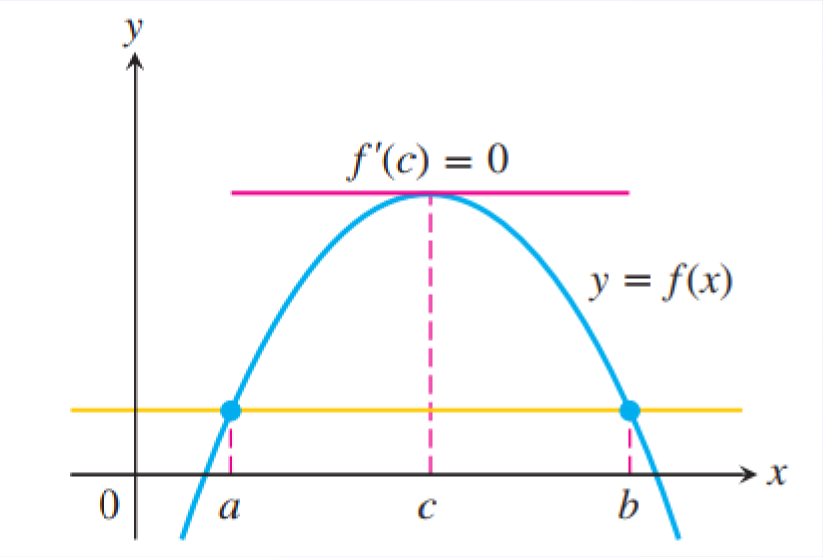
at x = 0.586 , $p''(x) = 12 - 6(0.586) = +$ concave up.

at x = 3.414 , $p''(x) = 12 - 6(3.414) = -$ concave down.

A maximum profit occurs at x = 3.414

- **Rolle's Theorem**

Two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal.




Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If

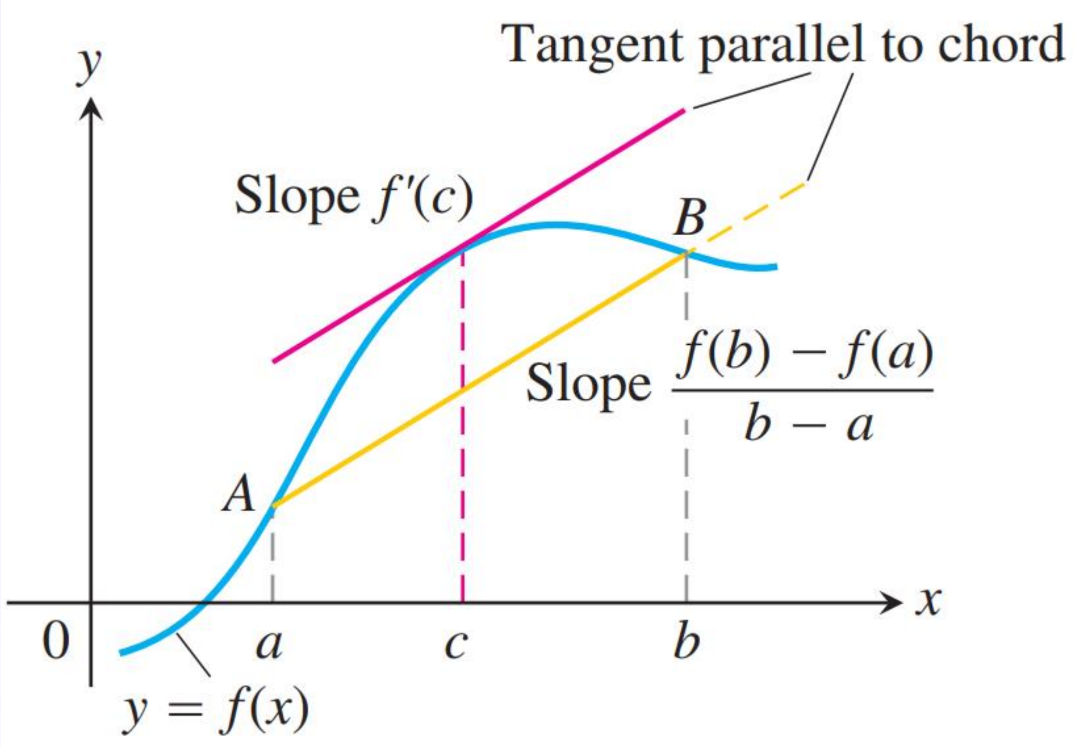
$$f(a) = f(b),$$

then there is at least one number $c$ in $(a, b)$ at which

$$f'(c) = 0.$$

- **The Mean Value Theorem**

is a *slanted* version of Rolle's Theorem. There is a point where the tangent is parallel to chord AB.




Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Example 1 Is the M.V.T is applicable on the following function ? If so find the value or values of c.

1. $f(x) = x - 2\sin x \qquad 0 \le x \le 2\pi,$ 2. $f(x) = x^{2/3} \qquad -8 \le x \le 8$ , 3. $f(x) = x^{2/3} \qquad 0 \le x \le 8$

Sol:

$f(x) = x - 2\sin x$ is continuous on [0, 2π]

$f'(x) = 1 - 2\cos x$ is differentiable on (0, 2π)

∴ the M.V.T. is applicable on [0, 2π]

To find c

$$\frac{f(b) - f(a)}{b - a} = f'(c) \rightarrow$$

$f(a) = 0 - 2\sin 0 \rightarrow f(a) = 0$

$f(b) = 2\pi - 2\sin 2\pi \rightarrow f(b) = 2\pi - 0 = 2\pi$

$f'(x) = 1 - 2\cos x \rightarrow f'(c) = 1 - 2\cos c$

$$\frac{2\pi - 0}{2\pi - 0} = 1 - 2\cos c \rightarrow 1 - 2\cos c = 1$$

$-2\cos c = 0 \rightarrow \cos c = 0$

$\therefore c = \pm\frac{n\pi}{2}$ , $n = 1, 3, 5$ ... ...

$c_1 = \frac{\pi}{2}, \quad c_2 = \frac{3\pi}{2}$ on the interval [0, 2π]

Example 1 Is the M.V.T is applicable on the following function ? If so find the value or values of c.

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Sol:

2. $f(x) = x^{2/3}$ is continuous on [-8, 8]

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\times\sqrt[3]{x}}$ is not differentiable on (-8, 8)

$\therefore$ the M.V.T. is not applicable on [-8, 8]

3. $f(x) = x^{2/3}$ is continuous on [0, 8]

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\times\sqrt[3]{x}}$ is differentiable on (0, 8)

$\therefore$ the M.V.T. is applicable on [0, 8]

To find c

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$f(a) = 0^{2/3} \to f(a) = 0$

$f(b) = 8^{2/3} \to f(b) = 4$

$$f'(c) = \frac{2}{3 \times \sqrt[3]{c}}$$

$$\frac{4 - 0}{8 - 0} = \frac{2}{3\times\sqrt[3]{c}} \to 4 = 3 \times \sqrt[3]{c}$$

$$\sqrt[3]{c} = \frac{4}{3}$$

$\therefore$ c $= \frac{64}{27}$ on the interval [0, 8]

- **L-Hopital's Rule**

Indeterminate Forms $\frac{0}{0}, \ \frac{\infty}{\infty}, \infty \times 0, \infty - \infty$

$0, \ \infty$ are acceptable values

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ *and* $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 2 Using L-Hopital's Rule of :

$$1.\ \lim_{x\to 0}\frac{3x-sinx}{x},\quad 2.\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x},\quad 3.\ \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2},\quad 4.\ \lim_{x\to 0}\frac{x-\sin(x)}{x^3}$$

Sol:

$$1\text{-}\ \lim_{x\to 0}\frac{3x-sinx}{x}\to\frac{\left(\frac{d}{dx}\right)3x-sinx}{\left(\frac{d}{dx}\right)x}\Bigg|x=0\to\frac{3-cosx}{1}\Bigg|x=0\to\frac{3-\cos(0)}{1}\to\frac{3-1}{1}=2$$

$$2\text{-}\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}\to\frac{\left(\frac{d}{dx}\right)(1+x)^{\frac{1}{2}}-1}{\left(\frac{d}{dx}\right)x}\Bigg|x=0\to\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1}\Bigg|x=0\to\frac{1}{2\sqrt{1+x}}\Bigg|x=0\to\frac{1}{2\sqrt{1+0}}=\frac{1}{2}$$

Example 2 Using L-Hopital's Rule of :

$$1.\ \lim_{x\to 0}\frac{3x-sinx}{x},\quad 2.\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x},\quad 3.\ \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2},\quad 4.\ \lim_{x\to 0}\frac{x-\sin(x)}{x^3}$$

Sol:

$$3.\ \lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}\to\frac{\left(\frac{d}{dx}\right)\sqrt{1+x}-1-\frac{x}{2}}{\left(\frac{d}{dx}\right)x^2}\Bigg|x=0\to\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-0-\frac{2}{4}}{2x}\Bigg|x=0\to\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}\Bigg|x=0\to\frac{\frac{1}{2\sqrt{1+0}}-\frac{1}{2}}{2(0)}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x}\to\frac{\left(\frac{d}{dx}\right)\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{\left(\frac{d}{dx}\right)2x}\Bigg|x=0\to\frac{\frac{1}{2}(-\frac{1}{2})(1+x)^{-\frac{3}{2}}}{2}\Bigg|x=0\to\frac{-\frac{1}{4\times\sqrt{(1+x)^3}}}{2}\Bigg|x=0$$

$$\left.\frac{-\frac{1}{4\times\sqrt{(1+x)^3}}}{2}\right|x=0 \rightarrow \frac{-\frac{1}{4\times\sqrt{(1+0)^3}}}{2}=\frac{-\frac{1}{4}}{2}=-\frac{1}{8}$$

$$4.\ \lim_{x\to0}\frac{x-\sin(x)}{x^3}\rightarrow\frac{\left(\frac{d}{dx}\right)x-sinx}{\left(\frac{d}{dx}\right)x^3}\rightarrow\left.\frac{1-cosx}{3x^2}\right|x=0\rightarrow\frac{1-\cos(0)}{3(0)^2}\rightarrow\frac{1-1}{0}=\frac{0}{0}$$

$$\lim_{x\to0}\frac{1-cosx}{3x^2}\rightarrow\frac{\left(\frac{d}{dx}\right)1-cosx}{\left(\frac{d}{dx}\right)3x^2}\rightarrow\left.\frac{sinx}{6x}\right|x=0\rightarrow\frac{sin0}{6(0)}=\frac{0}{0}$$

$$\lim_{x\to0}\frac{sinx}{6x}\rightarrow\frac{\left(\frac{d}{dx}\right)sinx}{\left(\frac{d}{dx}\right)6x}\rightarrow\left.\frac{cosx}{6}\right|x=0\rightarrow\frac{\cos(0)}{6}=\frac{1}{6}$$

Example 3 Using L-Hopital's Rule of :

*1.* $\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$ , 2. $\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x}$ , 3. $\lim_{x\to 0} (\frac{1}{sinx} - \frac{1}{x})$

Sol:

1- $\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x} \to \frac{\sec \pi/2}{1+\tan \pi/2} \to \frac{\infty}{1+\infty} = \frac{\infty}{\infty}$

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x} \to \frac{\left(\frac{d}{dx}\right)\sec x}{\left(\frac{d}{dx}\right)1+\tan x}\Bigg|x=\frac{\pi}{2} \to \frac{\sec x \tan x}{sec^2 x}\Bigg|x=\frac{\pi}{2} \to \frac{\sin x}{\cos x} \times \cos x\Bigg|x=\frac{\pi}{2} \to \quad \sin\frac{\pi}{2} = 1$$

2- $\lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x} \to \frac{\left(\frac{d}{dx}\right)x-2x^2}{\left(\frac{d}{dx}\right)3x^2+5x}\Bigg|x=\infty \to \frac{1-4x}{6x+5}\Bigg|x=\infty \to \frac{1-4(\infty)}{6(\infty)+5} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{1-4x}{6x+5} \to \frac{\left(\frac{d}{dx}\right)1-4x}{\left(\frac{d}{dx}\right)6x+5}\Bigg|x=\infty \to \frac{-4}{6} = -\frac{2}{3}$$

Example 3 Using L-Hopital's Rule of :

*1.* $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x}$ , 2. $\lim\limits_{x\to\infty} \frac{x-2x^2}{3x^2+5x}$ , 3. $\lim\limits_{x\to 0} (\frac{1}{sinx} - \frac{1}{x})$

Sol:

$$3.\ \lim_{x\to 0} (\frac{1}{sinx} - \frac{1}{x}) \to \lim_{x\to 0} (\frac{x-\sin x}{x \sin x}) = \frac{0-\sin 0}{0 \sin 0} = \frac{0}{0}$$

$$\lim_{x\to 0} (\frac{x-\sin x}{x \sin x}) \to \frac{\left(\frac{d}{dx}\right) x-\sin x}{\left(\frac{d}{dx}\right) x \sin x}\Bigg| x = 0 \to \frac{1-\cos x}{sin\ x+x \cos x}\Bigg| x = 0 \to \frac{1-\cos 0}{sin\ 0+0 \cos 0} = \frac{1-1}{0+0} = \frac{0}{0}$$

$$\frac{\left(\frac{d}{dx}\right) 1-\cos x}{\left(\frac{d}{dx}\right) sin\ x+x \cos x}\Bigg| x = 0 \to \frac{\sin x}{cos\ x+\cos x-x \sin x}\Bigg| x = 0 \to \frac{\sin x}{2cos\ x-x \sin x}\Bigg| x = 0 \to \frac{\sin 0}{2cos\ 0-0 \sin 0} \to \frac{0}{2-0} = 0$$