Stiffness

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Definition

Stiffness is the ability of a material to maintain its shape when acted upon by a load.

The concept of stiffness in metals is usually approached through Hooke's Law, which is concerned with the relationship between stress and strain. When a metal is loaded, the stress-strain curve is at first approximately linear and its **slope is a measure of the stiffness** of the metal. If the loading is in tension or compression the value of the slope is known as Young's modulus, or the modulus of elasticity, denoted by, when the loading is in shear it is known as the modulus of rigidity, or shear modulus, denoted by G. These two elastic constants are related through Poisson's ratio, *V*, as follows:

$$G = \frac{E}{2(1 + v)}$$

1-The importance of stiffness

There are three reasons why stiffness is important.
* One is concerned with stable deflections,
* with absorption of energy and
* with failure by instability.

Deflections

Deflections increase as stiffness decreases. For example, the end-deflection, δ, of a cantilever of length L, subjected to an end load P, (Figure1). The deflection δ is given by

$$\delta = \frac{Pl^3}{3EI}$$

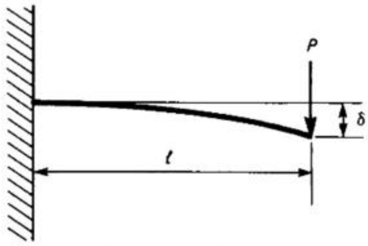

Figure 8.1 A cantilever subjected to an end load.

where **I** is the second moment of area of the cross-section of the cantilever. It follows that if two cantilevers - one of aluminium, the other of steel - are constructed to have identical second moments of area, the deflection in the former will be three times as great as that in the latter since Young's modulus for aluminium is only one-third of that for steel.

** **It** is not possible to produce any significant improvement in the performance of aluminium, or any other metal, by alloying because **Young's modulus is a structure-insensitive property**, and microstructural or composition varation cannot produce more than about 10% variation in either direction. This inability to control Young's modulus within a given material means that **if,** for some reason, the designer is ***compelled to use a material of low stiffness he must compensate for this by increasing the stiffness of his structure, i.e. by increasing its second moment of area***.

Although there is a well-established prejudice against large deflections in massive structures such as **ships, bridges and buildings**, it is not at all clear that movement of the structure as *a whole* is necessarily harmful. A tall **building**, subject to **wind load** at the top, can be regarded as a cantilever and the John Hancock building in Chicago, for example, which is 102 storied high, displays a winds way of 40 cm (15.7in) but there is no suggestion that its overall integrity is thereby threatened.

**Problems also arise in complex assemblies which incorporate materials of differing stiffness because there is then the danger that incompatibilities of deformation can lead to local concentrations of stress and ultimately some form of localized failure. Presumably, recently reported occurrences of cladding blocks, and even whole window frames, falling out of tall buildings are related to this sort of effect.

Attempts to **save weight by using high- strength materials** are also liable to affect stiffness adversely, since although the Young's modulus of the material is not significantly affected by the metallurgical strengthening methods employed, the higher strength allows smaller cross-sections to be employed with a consequent reduction in **I**, the geometric stiffness.

Energy absorption

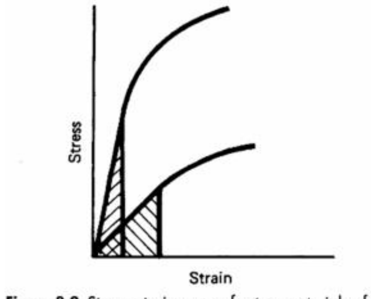

Figure 8.2 Stress–strain curves for two materials of differing stiffness.

When a material is strained it gains elastic strain energy. The energy per unit volume is then equal to the area under the stress-strain curve (Figure2).

Energy per unit volume $= (1/2)\sigma\varepsilon = \sigma^2/2E$
where ε = strain.

Considering a crash barrier required to absorb the kinetic energy of a moving vehicle leaving a roadway, if Young's modulus of the barrier material were reduced by a factor *n*, then the maximum retarding stress would be reduced by $\sqrt{n}$ which would be good for the occupants of the vehicle (except that elastic strain energy is recoverable, so that a highly compliant elastic crash barrier would tend to behave like a catapult).

Failure by elastic instability

The simpler methods of stress analysis assume that the overall geometry of a body under load does not change sufficiently to invalidate the analysis. For example, simple beam theory makes the assumption that plane sections remain plane. This applies particularly to thin, slender bodies or those incorporating cross-sections of high aspect ratio, that twisting or buckling of the stressed body occurs with the result that failure occurs at loads much lower than those predicted by simple theory. Failure by elastic instability can be general or localized, and some examples are shown in Figure 3.

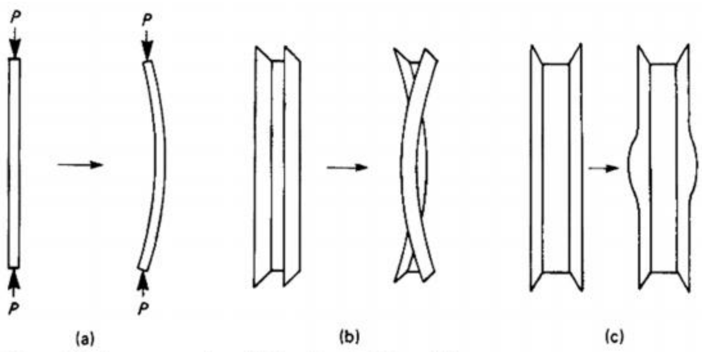

Figure 8.3 Some examples of failure by elastic instability.

2- The stiffness of materials

The elastic moduli of materials cover a wide range from diamond, the stiffest material known with a tensile modulus of 1000GPa to rubbers and plastics at around 0.01GPa. Steel has a tensile modulus of 200GPa which makes it a very useful structural material but the modulus of aluminium, at 70GPa, is low enough to present problems and nylon, with 3GPa, can never find major structural use.
Table 8.1 gives data for the tension moduli of some important materials.

TABLE 8.1

	Stiffness (GPa) E	Density (tonnes/m^3) ρ	Materials selection criteria Minimum weight $\frac{E^{1/2}}{\rho}$	$\frac{E^{1/3}}{\rho}$
Concrete (in compression)	27.0	2.40	2.16[7]	1.25[7]
Oak: parallel to grain	9.5	0.60	5.10[2]	3.53[1]
HM Carbon fibres	400.0	1.95	10.20	3.78
Aluminium N8 alloy	70.0	2.70	3.10[4]	1.53[5]
Steel	207.0	7.80	1.84[8]	0.76[9]
Glass fibre-reinforced concrete	25.0	2.40	2.10[7]	1.22[7]
Glass fibre: 70% resin reinforced plastic mat	10.0	1.50	2.10[7]	1.44[6]
Glass fibre: 50% resin reinforced plastic cloth	14.0	1.70	2.20[7]	1.42[6]
Glass fibre 20% resin reinforced plastic undirectional	48.0	2.00	3.46[3]	1.82[3]
Nylon 33% g.f.	3.5	1.20	1.56[9]	1.27[7]
Titanium	116.0	4.50	2.39[6]	1.08[8]
Unidirectional graphite-epoxy	137.0	1.50	7.80[1]	3.40[2]
45° cross-ply graphite-epoxy	15.0	1.50	2.58[5]	1.64[4]
Polypropylene	0.36	0.90	0.67[10]	0.79[9]

[1]–[10] = order of merit

**Young's modulus in metals at room temperature is not time-dependent and is therefore not influenced by changes in strain rate dependent.

**The behavior of polymeric materials is very different from that of metals. Not only are they much less stiff, but in consequence of their viscoelastic nature their properties are strongly time dependent. Stress-strain curves are strain rate dependent and stiffness moduli increase as strain rat increases. In another sense, this means that plastics under load become less stiff as time goes by the stress-strain relationship indicates that plastics also become less stiff with increasing strain.

**Temperature has a strong effect on the properties of plastics; so much so that only specialized plastics are used at temperatures above 100°C e.g. polyether ether ketone, polyethersulphone, polysulphone and liquid crystal polymers.

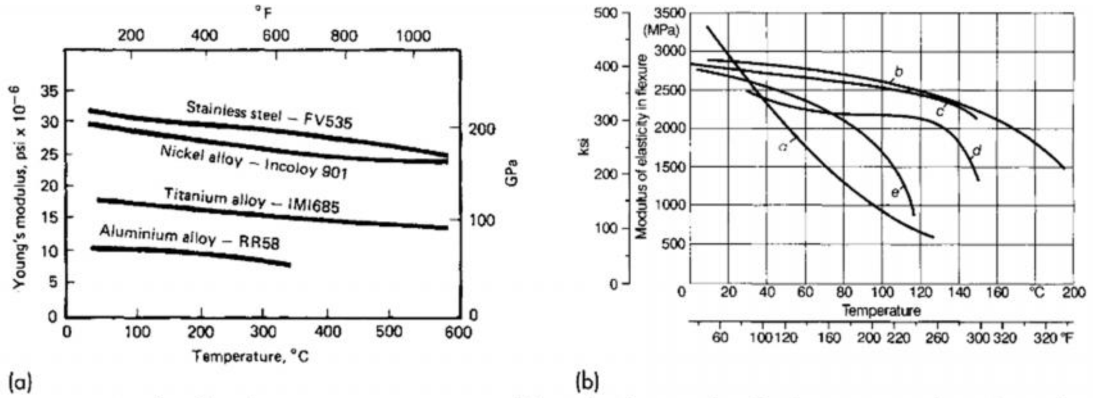

Figure 8.4 (a) The effect of temperature on Young's modulus (taken from Gresham[3]). (b) Temperature dependence of the flexural modulus of elasticity of various engineering thermoplastics[4] *a* polyacetal, *b* polysulphone, *c* polyphenylene ether, *d* polycarbonate, *e* ABS, high temperature grade.

Unlike metals, the stiffness of plastics is not independent of microstructure. The crystalline thermoplastics such as polyethylene and the nylons can vary in their degree of crystallinity depending upon the nature of the processing they have received. Higher stiffness is associated with increased crystallinity.

The stiffness of composites

On a density-compensated basis, the situation is improved as plastic materials are generally of low density. Reinforcing plastics with strong, stiff fibres such as glass or carbon will improve the mechanical properties, the principle behind composites. The axial stiffness of a composite containing continuous, aligned fibres can be calculated from the properties of its constituent parts using the 'Rule of Mixtures'. The derivation of this equation is given in Figure 5. This clearly demonstrates that the composite modulus is dependent on the stiffness of the fibres and the matrix and the

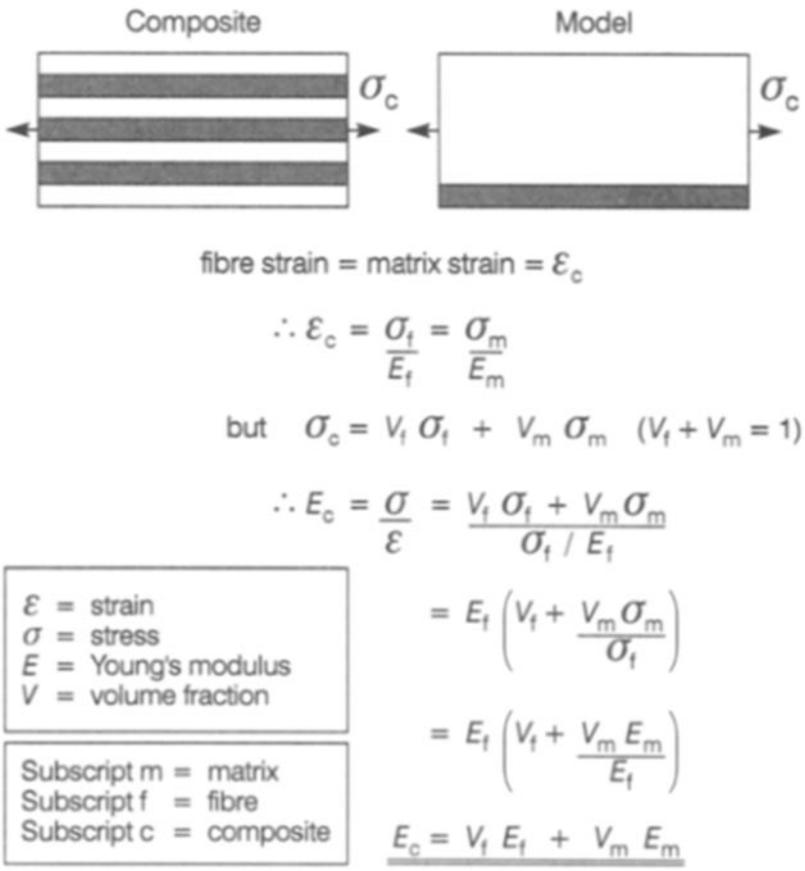

Figure 8.5 Composite 'Rule of Mixtures'.

relative volumes of each. The tensile modulus of polyester resin and glass fibre composites increases from 5% to 25% that of steel as the fibre content increases from 25% to 80% w /w. Despite the fact that the fibres used are generally more dense than the plastic matrices, the specific properties (specific modulus and specific tensile strength) are still very high.

3- The stiffness of sections

The most important structural component subjected to bending is the beam. As a typical example of a beam, consider the cantilever shown in Figure 1. Equation 8.2 shows that, as in most equations of this type, E is accompanied by I, the second moment of area of the cross-section.

Now E is a material property whereas I is a geometric property of the design: it is important to distinguish between material properties and design properties because they may be varied independently. We may define a stiff material as one with a high value of E, whereas a stiff design is one with a high value of I. Thus, if it is desired to use a material with a low value of E because of some other especially favourable property then the designer has the option of overcoming the disadvantage of low material stiffness by increasing the stiffness of the design, i.e. increasing I. Figure6 shows three sections of equal area:

(a) a square cross-section;

(b) a rectangular cross-section of aspect ratio 3 to1; and

(c) a hot-rolled steel section from British Standard 1449 chosen also to have an aspect ratio of 3 to 1.

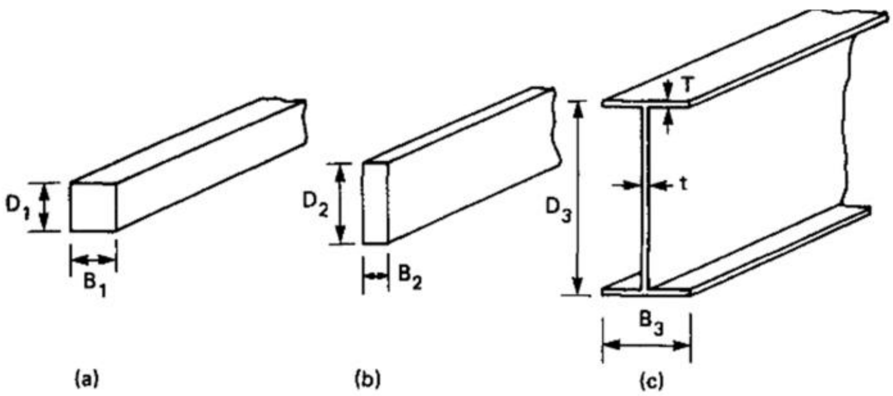

Figure 8.6 Three beams of equal cross-sectional areas (area = 31.40 cm^2): (a) $D_1 = B_1$, I_1 = 82.16cm^4; (b) $D_2 = 3B_2$, I_2 = 246.5cm^4 (c) $D_3 = 3B_2$, I_3 = 4381 cm^4

The efficient disposition of material in the hot- rolled section has increased the second moment of area to more than 50 times that for the square and more than 17 times that for the solid rectangle.

An even more efficient type of section, frequently used for roof decking- is shown in Figure 8.7(d). It is easy to find dimensions such that the second moment of area of the decking section is twice that of the plain rectangular plate shown at (a), whilst simultaneously reducing the cross-sectional area by a factor of 8.

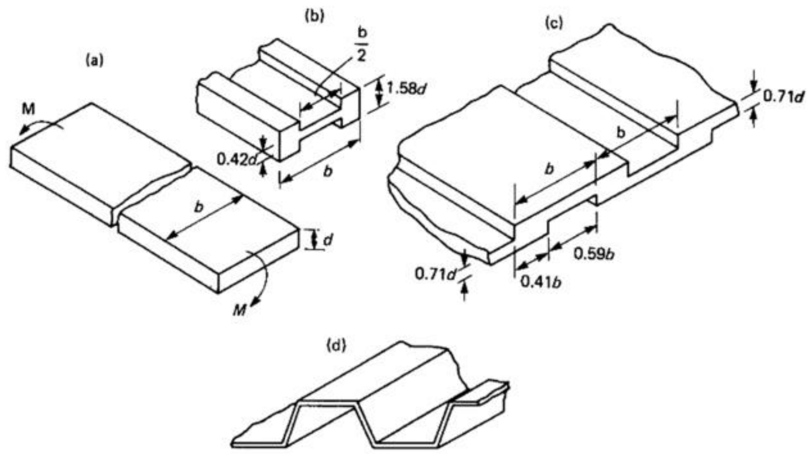

Figure 8.7 (a), (b) and (c): Planks of equal cross-sectional area; (d) decking material.

***The resistance to bending of a section can be increased for a given weight by making it hollow.
Consider, for example, two hollow sections - one square, the other circular - constructed to have areas and depths equal to those of the rectangular section in Figure 8.6(b). The required wall thicknesses are one-eleventh and one-eighth, respectively, of the depth, whilst the second moments of area are increased by factors of 1.66 and 1.17 respectively, as compared with that of the solid rectangular section. The square shows the higher increase because more of the cross-section is further from the plane of bending.
It is generally true that ***the stiffness of a section can be increased by placing as much as possible of the material as far as possible from the axis of bending. The extent to which this has been achieved can be measured by the **radius of gyration**. This is defined by putting the second moment of area of the section, I, equal to Ak^2 where A is the area of the cross-section and k is the radius of gyration. Thus, although stiffness is increased by both A and k, the square term means that the latter is more effective.

Failure of a strut

When long slender structural members are subjected to uniaxial compressive loads they are known as struts and failure occurs by overall flexural buckling (Figure 3. (a)). The longer and more slender the struts are, the smaller is the failure load. The standard formula for the failure load of a strut was developed by Euler and can be expressed as follows:

$$\text{Euler buckling load, } P_E = \frac{\pi^2 EI}{l^2} \qquad (8.3)$$

where E = Young's modulus; I = second moment of area; L = length of strut. If I, the second moment area, is written as Ak^2 where A is the area of the cross-section and k is the radius of gyration, Euler's equation can be put into terms of stress:

$$\text{Euler buckling stress, } \sigma_E = \frac{\pi^2 E}{(l/k)^2}$$

The ratio l/k is known as the slenderness ratio of the strut and Euler's equation only agrees with the measured failure load (or stress) of a strut when the slenderness ratio is rather high. When it is very low, i.e. when the strut is short and stubby, Euler's buckling stress becomes greater than the yield stress in compression of the material of which the strut is made, and it is obvious that failure will then occur by crushing in simple compression rather than by buckling. The relationship between the Euler buckling stress, σ_E the slenderness ratio is shown in Figure 8 σ_{YS} the yield stress.

For steel, Young's modulus E is taken as 200GN/m/, and yield stress σ_{YS} 250MN/m^2. Then E/ σ_{YS} = 800 and

$$\left(\frac{l}{k}\right)^2 = \frac{800\pi^2}{(\sigma_E/\sigma_{YS})}$$

For aluminium, E is taken as 68 GN/m^2 and σ_{YS} as 230 MN/m^2 so that

$$\frac{E}{\sigma_{YS}} = 300 \quad \text{and} \quad \left(\frac{l}{k}\right)^2 = \frac{300\pi^2}{\sigma_E/\sigma_{YS}}$$

These results indicate (Figure 8) that whereas steel starts to buckle rather than yield at a slenderness ratio of 89, aluminium buckles at a corresponding figure of 54, but this idealized relationship overestimates the buckling stresses actually measured in practice. Experimental results lie within the shaded areas

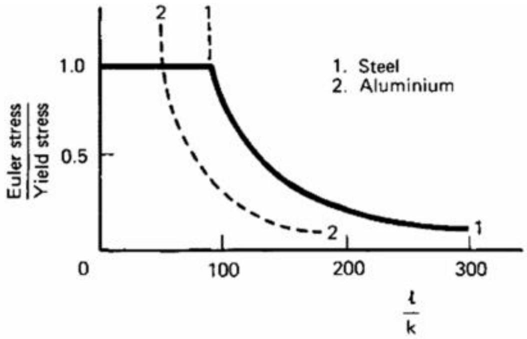

Figure 8.8 Idealized buckling behaviour of steel and aluminium struts. Solid line = steel; dotted line = aluminium.

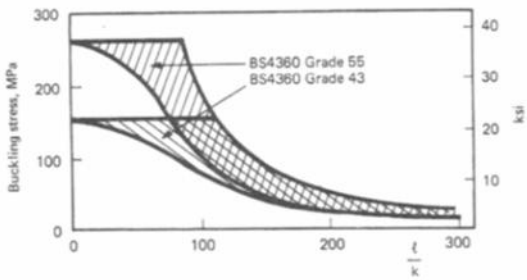

Figure 8.9 Buckling behaviour of two grades of structural steels.

Buckling of a panel

When a plate is subjected to an end load *P* which lies in the plane of the plate it is described as a panel. If **t** is the thickness of a panel of width **b,** then the vertical stress sustained by the panel is given by **P/bt**. If the panel is thick enough, failure will occur by **plastic crashing** when the applied end attain the yield stress of the material. Thinner panels, however, fail by buckling at a lower value of stress given by

$$\sigma_B = \frac{\pi^2 E}{3\,(1 - v^2)\,(b/t)^2} \qquad (8.4)$$

This equation is similar to the Euler equation for buckling of a strut with the thinness ratio of the panel taking the place of the slenderness ratio of the strut. The two types of behaviour are, however, rather different because whereas Eulerian buckling is the result of an overall instability, panel buckling is a form of local instability. In practical terms this means that whereas the strength of a strut disappears virtually to zero immediately buckling is initiated, a buckled panel will continue to support a Significant, although much lower, load and it would be wasteful not to allow for this residual strength. Although the stress analysis of a buckled panel is rather complex it is accepted that the distribution of stress across the width of an end-loaded buckled plate is not uniform, varying

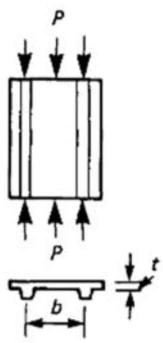

Figure 8.11 A panel subjected to an in-plane end load.

from a minimum at the center of width to maxima at the two edges. (Figure 8.11). In equation *(8.4)* ,*b* is the distance between stiffeners.

4 Materials selection criteria for stiffness

Deflection of beam

As shown in figure 8.1, the deflection of a cantilever beam, δ, is given by $\delta = PL^3/3EI$. If the cross-section of the beam is square, of breadth b, then the stiffness

$$\frac{P}{\delta} = \frac{Eb^4}{4l^3} \quad \text{and} \quad b = \left[\frac{4l^3}{E} \cdot \frac{P}{\delta}\right]^{1/4}$$

The weight of the beam is

$$lb^2\rho = l\rho\left[\frac{4l^3}{E} \cdot \frac{P}{\delta}\right]^{1/2} = 2l^{5/2}\left(\frac{P}{\delta}\right)^{1/2}\frac{\rho}{E^{1/2}}$$

$$(4l^6b^2)^{1/3}.\left(\frac{P}{\delta}\right)^{1/3}\frac{\rho}{E^{1/3}}$$

where ρ is the density.

Therefore, for a given stiffness P/δ, the weight of the beam is minimized when $E^{1/2}/\rho$ is maximized. $E^{1/2}/\rho$ is therefore the materials selection criterion.

However, the designer can increase the geometric stiffness of the beam by control of the aspect ratio of the cross-section. If he replaces the square cross-section of the beam with a rectangular section of depth *d* and breadth *b*, it is sensible to hold *b* constant and allow *d* to vary. In this case it turns out that the weight of the beam is given by

And the materials selection criterion becomes $E^{1/3}/\rho$

Buckling of a strut

The Euler buckling load, $P_E = (\pi^2 EI)/l^2$. Since a strut is free to buckle in any lateral direction there is no point in considering other than axisymmetric sections. Assume, therefore, that the strut is a round rod of diameter *d*, for which the second moment of area is

$$I = \frac{\pi d^4}{64} \quad \therefore d^4 = \frac{64.P_E l^2}{\pi^3 E}$$

Structural efficiency

$$\frac{\text{Load}}{\text{Weight}} = \frac{\pi^2 E}{l^2} \cdot \frac{\pi d^4}{64} \cdot \frac{4}{\pi d^2} \cdot \frac{1}{p} = \frac{\pi^2 E d^2}{16 l^3 p}$$

$$= \frac{\pi^2 E}{16 l^3 p} \cdot \frac{8 P_E^{1/2}\, l}{\pi^{3/2} E^{1/2}}$$

$$= \frac{\pi^{1/2}}{2} \left(\frac{P_E}{l^4}\right)^{1/2} \frac{E^{1/2}}{p}$$

$\therefore$ The materials selection criterion is $E^{1/2}/p$.

Buckling of a panel

The buckling stress of a solid panel in compression is

$$\sigma_B = \frac{\pi^2}{3(1 - v^2)} \cdot E \cdot \left[\frac{t}{b}\right]^2$$

which, taking $v = 0.3$, becomes $3.62\, E\, (t/b)^2$.
Buckling load $P = 3.62\, E\, (t^2/b^2).\, tb$

$$\therefore t^3 = \frac{1}{3.62} \frac{Pb}{E}$$

Structural efficiency =

$$\frac{\text{Load}}{\text{Weight}} = 3.62 \frac{E t^3}{b} \cdot \frac{1}{tbp}$$

$$= 3.62\frac{E}{p}\cdot\frac{t^2}{b^2} = \frac{3.62}{b^2}\cdot\frac{E}{p}\left(\frac{Pb}{3.62E}\right)^{2/3}$$

$$= 1.54\frac{E^{1/3}}{p}\left(\frac{P}{b^2}\right)^{2/3}$$

The materials selection criterion is thus $E^{1/3}/p$.

5 -Comparison of materials selection criteria

We are now in a position to examine the performance of several constructional materials in terms of the criteria that have been developed (Table 8.1). The orders of merit revealed in the table demonstrate above all the importance of density in weight-sensitive applications. Steel, which in absolute terms has the highest Young's modulus of all the materials considered, ranks bottom, equal with polypropylene, in terms of $(E^{1/3}/\rho)$. The two best materials, wood and carbon-fibre-reinforced-plastic (CFRP), are both materials of low density and, further, when the GRP reinforced with 30% glass fibre in the form of chopped strand mat is compared with the version containing 50% woven roving it is seen that although the increased glass content has raised the absolute value of stiffness significantly, this has been offset by the concurrent increase in density.

The effect of anisotropy is worth noting. The best values are produced by the most anisotropic materials - oak, unidirectional GRP and unidirectional CFRP. Wood in the form of plywood panels, and CFRP and GRP as cross-ply or random laminations, are much less competitive.

Although polymers in general are highly compliant materials, their low densities enable them to find wide use in small-scale applications, because whether injection-moulded or laminated, it is a simple matter to provide additional stiffness where **it** is most needed by local thickening of cross-sections without undue increase in weight. **It** is common practice to stiffen plastic sections with the use of ribs, which can be incorporated into the injection moulding process relatively easily. As already mentioned, additional stiffening is frequently applied to metallic structures by welding on (sometimes riveting), stiffeners, but this procedure is less convenient for several reasons. Being an extra operation, it introduces additional cost;

since metals are dense, the increase in weight is not insignificant; where welding is involved, careful consideration at the design stage is necessary because of the propensity for welds to introduce defects and reduce the fatigue resistance of the structure.

The theory of beam stiffening is put into good practice with the sandwich construction of composite panels, where stiff composite laminates are bonded each side of a light, rigid, polymeric foam such as PVC, or similar materials e.g. balsa. This construction is used for ocean-going racing yachts, for example. The sandwich is essentially acting as an I-beam in terms of the stiffness increase.