Lectures in Group Theory

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Problem3: Let H be a normal subgroup of a group G. Define the function g from G onto the quotient group G/H by $g(a) = aH$ for all $a \in G$. Then g is a homomorphism of G onto G/H and $\ker g = H$. (The homomorphism g is called the **natural homomorphism** of G onto G/H.)
Proof of Problem3: From the definition of g, it follows that g is a function from G onto G/H. To show g is a homomorphism, let $a, b \in G$. Then $g(ab) = (ab)H = (aH)(bH) = g(a)g(b)$. Hence, g is a homomorphism of G onto G/H. Finally, we show that $\ker g = H$. Now $a \in \ker g$ if and only if $g(a) = eH$ if and only if $aH = eH$ if and only if $e^{-1}a \in H$ if and only if $a \in H$. Thus, $\ker g = H$.
Problem4: Every finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +)$ and every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.
Proof of Problem4: Homework.
Problem5: (**Cayley**) Any group G is isomorphic to some subgroup of the group $(S_G, \circ)$ of all permutations of the set G.

Proof of Problem5: Let a be an element of a group G. Define the function $f_a : G \to G$ by for all $b \in G, f_a(b) = ab$. Then $b = c$ if and only if $ab = ac$ if and only if $f_a(b) = f_a(c)$. Thus, f_a is a one-one function of G into G. For any $b \in G, f_a(a^{-1}b) = a(a^{-1}b) = b$. So we find that f a maps G onto G. Hence, f_a is a permutation of G. This implies that $f_a \in S_G$. Let $F(G) = \{f_a : a \in G\}$. Then $F(G)$ is a subset of the set S_G of all permutations on G. Define $g : G \to S_G$ by for all $a \in G, g(a) = f_a$. Then $a = b$ if and only if $ac = bc$ for all $c \in G$ if and only if $f_a(c) = f_b(c)$ for all $c \in G$ if and only if $f_a = f_b$ if and only if $g(a) = g(b)$. This proves that g is a one-one function of G into $F(G)$. Clearly g maps G onto $F(G)$. Now $g(ab) = f_{ab}$ and $g(a) \circ g(b) = f_a \circ f_b$. Also, for all $c \in G$,

$$f_{ab}(c) = (ab)c = a(bc) = f_{a(bc)} = f_a(f_{b(c)}) = (f_a \circ f_b)(c).$$

Thus, $f_{ab} = f_a \circ f_b$. Hence, $g(ab) = g(a) \circ g(b)$ and so g is a homomorphism. This implies that $F(G)$ is a subgroup and G is isomorphic to this subgroup.

Exersises

1. Show that $\mathbb{R}^*$, the group of all nonzero real numbers under multiplication, is not isomorphic to $\mathbb{C}^*$, the group of all nonzero complex numbers under multiplication.
2. Is the group $(\mathbb{Q}, +)$ isomorphic to the group $(\mathbb{Z}, +)$? explain.
3. Prove that any two cyclic groups of the same order are isomorphic.
4. Show that $f : GL(2, \mathbb{R}) \to (\mathbb{R} \backslash \{0\}, .)$, defined by $f(A) = \det A$, is an epimorphism. Find $\ker f$. Is f an isomorphism? explain.
5. Let $\mathbb{R}^*$ be the group of all nonzero real numbers under multiplication. Define $f : \mathbb{R}^* \to \mathbb{R}^*$ by $f(a) = |a|$. Prove that f is a homomorphism which is neither epimorphism nor monomorphism.