Lectures in Group Theory

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More about group homomorphisms

Recall a group-homomorphism $f : G \to G'$ is called an **isomorphism** of a group G onto a group G' if f is one-one and onto G'. In this case, we write $G \simeq G'$ and say that G and G' are **isomorphic**. An isomorphism of a group G onto G is called an **automorphism**. For a group G, $\mathrm{Aut}(G)$, denotes the set of all automorphisms of G. Now, we will give some problems on group-homomorphisms:

Poblem1: Let $f : G \to G'$ be a group-homomorphism. Then

1. f is one-one if and only if $\ker f = \{e\}$.
2. $\ker f \trianglelefteq G$.

Poblem2: Let $f : G \to G'$ be a group-isomorphism. Then

1. $f^{-1} : G' \to G$ is an isomorphism.
2. G is commutative if and only if G' is commutative.
3. For all $a \in G, o(a) = o(f(a))$.

Proof of Problem1:

1. Suppose f is one-one. Let $a \in \ker f$. Then $f(a) = e' = f(e)$. Since f is one-one, we must have $a = e$. Hence, $\ker f = \{e\}$. Conversely, suppose that $\ker f = \{e\}$. Let $a, b \in G$. Suppose $f(a) = f(b)$. Then $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)f(b)^{-1} = e'$. Thus, $ab^{-1} \in \ker f = \{e\}$ and so $ab^{-1} = e$, i.e., $a = b$.
2. Since $e \in \ker f, \ker f \neq \emptyset$. Let $a, b \in \ker f$. Then $f(ab) = f(a)f(b) = e'(e') = e'$, and hence $ab \in \ker f$. Moreover, $f(b^{-1}) = f(b)^{-1} = e'^{-1} = e'$. Thus, $b^{-1} \in \ker f$ and hence $\ker f$ is a subgroup of G. Let $a \in G$ and $h \in \ker f$. Then

$$f(aha^{-1}) = f(a)f(h)f(a^{-1}) = f(a)f(h)f(a)^{-1} = f(a)e'f(a)^{-1} = e'.$$

Therefore, $aha^{-1} \in \ker f$. This proves that $a(\ker f)a^{-1} \subseteq \ker f$.

Proof of Problem2:

1. Since f is one-one and onto G', f^{-1} is one-one and onto G. Now we only need to verify that f^{-1} is a homomorphism. Let $u, v \in G'$. Then there exist $a, b \in G$ such that $f(a) = u$ and $f(b) = v$. This implies that $a = f^{-1}(u), b = f^{-1}(v)$, and $uv = f(a)f(b) = f(ab)$. Thus, $f^{-1}(uv) = ab = f^{-1}(u)f^{-1}(v)$ and so f^{-1} is a homomorphism.
2. Suppose G is commutative. Let $u, v \in G'$. Since f is onto G', there exist $a, b \in G$ such that $f(a) = u$ and $f(b) = v$. Now $uv = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = vu$. Thus, G' is commutative. Conversely, suppose G' is commutative. Let $a, b \in G$. Now $f(ab) = f(a)f(b) = f(b)f(a) = f(ba)$. Since f is one-one, we have $ab = ba$. This proves that G is commutative.
3. **Homework**.