Lectures in Group Theory

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Example1: Let us show that $2\mathbb{Z}$ is a normal subgroup of the group $(\mathbb{Z}, +)$.
According to theorem in previous lecture, we must prove that for all $a \in \mathbb{Z} : a + (2\mathbb{Z}) - a \subseteq 2\mathbb{Z}$. But we know that $\mathbb{Z}$ is abelian, so

$$a + (2\mathbb{Z}) - a = a + (-a) + 2\mathbb{Z} = 0 + 2\mathbb{Z} = 2\mathbb{Z} \subseteq 2\mathbb{Z}.$$

Example2: Show that the subgroup

$$H = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

of the group $(\mathfrak{S}_3, \circ)$ is a normal subgroup.
Answer: Here we can use the definition of a normal subgroup. Note that $f \circ H = H \circ f$ for all $f \in \mathfrak{S}_3$ (**Check**).

Quotient group

Let H be a normal subgroup of G. Define an operation on $G/H = \{aH : a \in G\}$ by for all $aH, bH \in G/H$, $(aH)(bH) = abH$. Then G/H is a group (called the **quotient group** of G by H).

Remark: The number of elements in G/H is called index of H in G which is denoted by $[G : H]$. In fact, if G finite, then $[G : H] = \frac{|G|}{|H|}$.

Example: Consider the group $(\mathbb{Z}_8, +)$ and let $H = \{0, 4\}$. Then H is a normal subgroup of $\mathbb{Z}_8$. Now $|H| = 2$ and $|\mathbb{Z}_8| = 8$. Thus, $|\mathbb{Z}_8/H| = \frac{|G|}{|H|} = \frac{8}{2} = 4$ =4. Hence, $\mathbb{Z}_8/H$ has four elements. Now

$0 + H = H = 4 + H$,
$1 + H = \{1, 5\} = 5 + H$,
$2 + H = \{2, 6\} = 6 + H$, and
$3 + H = \{3, 7\} = 7 + H$.

Hence, $\mathbb{Z}_8/H = \{H, 1 + H, 2 + H, 3 + H\}$.

Homework: Consider the subgroup $H = 5\mathbb{Z}$ of the group $(\mathbb{Z}, +)$. Show that H is a normal subgroup of $\mathbb{Z}$, then find the quotient group $\mathbb{Z}/H$. What is the index of H in $\mathbb{Z}$?

Group homomorphism

Let $(G, *)$ and $(G', *')$ be groups and $f : G \to G'$ a function from G into G'. Then f is called a homomorphism of G into G' if for all $a, b \in G, f(a * b) = f(a) *' f(b)$.

Example1: Let e' be the identity element of the group G'. Define $f : G \to G'$ by $f(a) = e'$ for all $a \in G$. Since $f(a * b) = e' = e' *' e' = f(a) *' f(b)$ for all $a, b \in G$, we find that f is a homomorphism from G into G'. This homomorphism is called the **trivial homomorphism**.

Example2: Consider the function $f : (\mathbb{R}, +) \to (\mathbb{R}^+, .)$ between two groups, where $f(x) = e^x$ for all $x \in \mathbb{R}$. Then f is a homomorphism. In fact, if $x, y \in \mathbb{R}$, then

$$f(x + y) = e^{x+y} = e^x.e^y = f(x).f(y).$$

Remark: A group homomorphism $f : G \to G'$ is called

1. an epimorphism if it is onto (surjective),
2. monomorphism if it is one-one (injective),
3. isomorphism if it is 1-1 correspondence (bijective).