Lectures in Group Theory

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Cosets of a subgroup

Let $H \leq G$ (G a group) and $a \in G$. The sets $aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$ are called the **left and right cosets** of H in G, respectively. If G is commutative, then of course $aH = Ha$. Observe that $eH = H = He$ and that $a = ae \in aH$ and $a = ea \in Ha$. Now, we give some properties of left and right cosets of a subgroup: Let H be a subgroup of a group G and $a, b \in G$. Then

1. $aH = bH$ if and only if $b^{-1}a \in H$.
2. $Ha = Hb$ if and only if $ab^{-1} \in H$.
3. either $aH = bH$ or $aH \cap bH = \emptyset$, i.e., two left cosets are either equal or they are disjoint.
4. $\{aH : a \in G\}$ forms a partition of G.
5. The elements of H are in one-one correspondence with the elements of any left (right) coset of H in G and hence $|H| = |aH| = |Ha|$. Moreover, there is a one-one correspondence of the set of all left cosets of H in G onto the set of all right cosets of H in G.

Cosets of a subgroup

Example: Find all the left distinct cosets of $H = \langle 2 \rangle = \{0, 2, 4\}$ in $(\mathbb{Z}_6, +)$.
Answer:
$0 + H = \{0 + 0, 0 + 2, 0 + 4\} = \{0, 2, 4\} = H,$
$1 + H = \{1 + 0, 1 + 2, 1 + 4\} = \{1, 3, 5\},$
$2 + H = \{2 + 0, 2 + 2, 2 + 4\} = \{2, 4, 0\} = H,$
$3 + H = \{3 + 0, 3 + 2, 3 + 4\} = \{3, 5, 1\} = 1 + H,$
$4 + H = \{4 + 0, 4 + 2, 4 + 4\} = \{4, 0, 2\} = H,$
$5 + H = \{5 + 0, 5 + 2, 5 + 4\} = \{5, 1, 3\} = 1 + H.$
Hence the distinct cosets of H are H, $1 + H$.
Homework:

1. Find all the left distinct cosets of $H = \langle 6 \rangle = \{0, 6\}$ in $(\mathbb{Z}_{12}, +)$.
2. Let $H = \langle 2 \rangle = 2\mathbb{Z}, K = \langle 3 \rangle = 3\mathbb{Z}$ be subgroups of $(\mathbb{Z}, +)$. Find $H + K$. Is $H + K$ subgroup of $\mathbb{Z}$?

Let G be a group. A subgroup H of G is said to be a **normal** subgroup of G, written $H \trianglelefteq G$, if $aH = Ha$ for all $a \in G$.
From the definition of a normal subgroup, it follows that for any group G, $\{e\}$ and G itself are normal subgroups of G. For $a \in G, \emptyset \neq H \subseteq G$, let $aHa^{-1} = \{aha^{-1} : h \in H\}$.
Theorem: A necessary and sufficient condition for a subgroup H of a group G to be a normal subgroup is $aHa^{-1} \subseteq H$ for all $a \in G$.
Proof: First suppose that H is a normal subgroup of G. Let $a \in G$. Let $aha^{-1} \in aHa^{-1}$, where $h \in H$. Since H is a normal subgroup of G, $aH = Ha$. Also, since $ah \in aH$, we have $ah \in Ha$ and so $ah = h'a$ for some $h' \in H$. Thus, $aha^{-1} = h'aa^{-1} = h'e = h' \in H$. Hence, $aHa^{-1} \subseteq H$.
Conversely, suppose $aHa^{-1} \subseteq H$ for all $a \in G$. Let $a \in G$. Let $ah \in aH$, where $h \in H$. Now $aha^{-1} \in aHa^{-1}$ and so $aha^{-1} = h' \in H$. This implies that $ah = h'a \in Ha$. Therefore, $aH \subseteq Ha$. Similarly, we can show that $Ha \subseteq aH$. Hence, $aH = Ha$. Consequently, H is a normal subgroup of G.