Lectures in Group Theory

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Let $(G,*)$ be a group with identity e, and let H be a nonempty set of G. Then $(H,*)$ is said to be subgroup of $(G,*)$, written $H \leq G$ if

1. $\forall\, a, b \in H : a * b \in H$,
2. $\forall\, a \in H : a^{-1} \in H$.

Example1: The set of even integers, namely $2\mathbb{Z}$ is a subgroup of $(\mathbb{Z}, +)$.
Claim:

1. Let $2n, 2m \in 2\mathbb{Z}$. Then : $2n + 2m = 2(n+m) \in 2\mathbb{Z}$,
2. Let $a = 2k \in 2\mathbb{Z}$. Then $a^{-1} = -2k \in 2\mathbb{Z}$ $\square$

Example2: It is clear that, under the usual addition, $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$.
Example3: Let $(G,*)$ be a group with identity e, and let $a \in G$. Define $H = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$. Prove that $H \leq G$.
Proof:

1. Let $a^n, a^m \in H$. Then $a^n * a^m = a^{n+m} \in H$
2. Let $a^n \in H$. Then $(a^n)^{-1} = a^{-n} \in H$ $\square$

Subgroups

Example4: Show that, under the usual addition, each subgroup of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for some non-negative integer n.
Proof: Let H be a subgroup of $\mathbb{Z}$. Want to prove that $H = n\mathbb{Z}$ for some non-negative integer n. There is a positive integer k in H (Why?). Let n be the smallest positive integer in $H \subseteq \mathbb{Z}$. It is clear that $nj \in H$ for all $j \in \mathbb{Z}$, and hence $n\mathbb{Z} \subseteq H$.
Conversely, suppose that $h \in H$. Then by Division algorithm there are integers $q, r \in \mathbb{Z}$ such that $h = nq + r$, where $0 \leq r < n$. Note that $r = h - nq \in H$ and hence r must be equal to zero (i.e, $h = nq$).
Otherwise, we have contradiction with minimality of n.
Thus, $H \subseteq n\mathbb{Z}$ and hence $H = n\mathbb{Z}$.
Remark: In similar way to that in Example 4, one can show that a subgroup of $(\mathbb{Z}_n, +)$ has the $k\mathbb{Z}_n$, where k divides n.
Example5: All subgroups of $(\mathbb{Z}_6, +)$ are: $1\mathbb{Z}_6 = \mathbb{Z}_6$, $2\mathbb{Z}_6$, $3\mathbb{Z}_6$ and $6\mathbb{Z}_6 = \{0\}$.

Subgroups

Theorem: Let H, K be subgroups of a group $(G, *)$. Then $H \cap K$ is a subgroup of G.

Proof:

1. $\forall\, a, b \in H \cap K$, we have $a, b \in H, a, b \in K$.
   So, $a * b \in H$ and $a * b \in K$ and hence $a * b \in H \cap K$.
2. $\forall\, a \in H \cap K$, we have $a \in H$ and $a \in K$.
   So, $a^{-1} \in H$ and $a^{-1} \in K$ and hence $a^{-1} \in H \cap K$.

Homework:

1. Find all subgroups of $(\mathbb{Z}_{12}, +)$.
2. Let $H_1, H_2, \ldots, H_k$ be subgroups of a group $(G, *)$. Prove that $\bigcap_{i=1}^{k} H_i$ is a subgroup of G.
3. Is the union of two subgroups of a group $(G, *)$ again subgroup of G? explain.