Lectures in Group Theory

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Order of element in a group

Definition: Let a be an element of a group $(G, *)$ with identity e. Then

$$a^n := \begin{cases} a * a * \ldots * a(n - \text{times}) & \text{if } n \in \mathbb{Z}^+, \\ e & \text{if } n = 0, \\ a^{-1} * a^{-1} * \ldots * a^{-1}(|n| - \text{times}) & \text{if } n \in \mathbb{Z}^-. \end{cases}$$

Definition: The order of an element g in group $(G, *)$ with identity e is defined by $|g| = o(g) = \min\{n \in \mathbb{Z}^+ : g^n = e\}$ (if such n exists), otherwise $|g| = \infty$.

Example1: Let us find the order of each element in the group $(\mathbb{Z}_4, +)$.
We know that $\mathbb{Z}_4 = \{0, 1, 2, 3\}$.
$0^1 = 0$ implies $|0| = 1$. By the above definition, we have
$1^1 = 1$, $1^2 = 1 + 1 = 2$, $1^3 = 1 + 1 + 1 = 3$, $1^4 = 1 + 1 + 1 + 1 = 4 = 0$.
Thus, $|1| = 4$.
$2^1 = 2$, $2^2 = 2 + 2 = 4 = 0$. Thus, $|2| = 2$.
$3^1 = 3$, $3^2 = 3 + 3 = 6 = 2$, $3^3 = 3 + 3 + 3 = 1$, $3^4 = 3 + 3 + 3 + 3 = 0$.
Thus, $|3| = 4$.

Example2: The only element of finite order in $(\mathbb{Z}, +)$ is 0.
Theorem: Let $(G, *)$ be a group with identity e, and let $a \in G$ with $|a| = n$, $a^m = e$, $n, m \in \mathbb{Z}^+$. Then $n \leq m$.
Proof: By definition n is the smallest positive integer such that $a^n = e$. Thus, $n \leq m$.
Homework:

1. Find the order of each element in $(\mathbb{R} \backslash \{0\}, \cdot)$.
2. Find the order of each element in $(\mathbb{Z}_6, +)$ and $(\mathfrak{S}_3, \circ)$.
3. How many element of order 8 in $(\mathbb{Z}_8, +)$?
4. How many element of order 2 in $(\mathfrak{S}_4, \circ)$?

Elementary properties of groups (**Homework**)

Let $(G, *)$ be a group with identity e, and let $a, b, a_i \in G$. Then

1. If $n, m \in \mathbb{Z}$. Then
   1. $(a * b)^n = a^n * b^n$ if $a * b = b * a$,
   2. $a^n * a^m = a^{n+m}$,
   3. $(a^n)^m = a^{nm}$.
2. $(a^{-1})^{-1} = a$.
   Hint: It is enough to prove that $a * a^{-1} = a^{-1} * a = e$.
3. $(a_1 * a_2 * \ldots * a_k)^{-1} = a_k^{-1} * \ldots * a_2^{-1} * a_1^{-1}$.
   Hint: It is enough to prove that
   $$(a_1 * a_2 * \ldots * a_k) * (a_k^{-1} * \ldots * a_2^{-1} * a_1^{-1}) = e,$$
   and
   $$(a_k^{-1} * \ldots * a_2^{-1} * a_1^{-1}) * (a_1 * a_2 * \ldots * a_k) = e.$$
4. Cancellation laws hold in G (**Done**).
5. Let a, b are elements of finite order in G, then
   1. $|a| = |a^{-1}| = |b * a * b^{-1}|$,
   2. $|a * b| = |b * a|$.

Elementary properties of groups (**Homework**)

6. If $|a| = n$ and $b = a^m$, then $|b| = \frac{n}{\gcd(m,n)}$.
7. The equations $a * x = b$ and $y * a = b$ have unique solutions in G.
8. If G is abelian, then $(a * b)^{-1} = a^{-1} * b^{-1}$.