Lectures in Group Theory

MOHAMMED ALABBOOD

University of Basrah-College of Science
Department of Mathematics

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Group of matrices

Let $M_{2\times 3}(\mathbb{R})$ be the set of all 2×3 matrices with real entries. Then $M_{2\times 3}(\mathbb{R})$ forms a group under addition of matrices.
Recall that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

1. From above, it is obvious that $A+B\in M_{2\times 3}(\mathbb{R})$ whenever $A, B\in M_{2\times 3}(\mathbb{R})$.
2. For all $A, B, C\in M_{2\times 3}(\mathbb{R})$, we have $A+(B+C)=(A+B)+C$ (**Check!**).
3. $O=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ acts as the identity element of $M_{2\times 3}(\mathbb{R})$. More precisely,
$$A+O=O+A=A, \forall A\in M_{2\times 3}(\mathbb{R}).$$
4. For all $A\in M_{2\times 3}(\mathbb{R})$, A has an inverse, namely $-A\in M_{2\times 3}(\mathbb{R})$ with
$$A+(-A)=(-A)+A=O$$

General linear group

Consider the set $GL_2(\mathbb{R}) = \{A \in M_{2\times 2} : \det A \neq 0\}$. Then $GL_2(\mathbb{R})$ forms a group under multiplication of matrices.

1. Let $A, B \in GL_2(\mathbb{R})$. Then $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $\det A \neq 0$, and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ with $\det B \neq 0$. Furthermore,
$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$
and $\det AB = \det A \det B \neq 0$. Thus, $AB \in GL_2(\mathbb{R})$.
2. For all $A, B, C \in GL_2(\mathbb{R})$, we have $A(BC) = (AB)C$ (**Check!**).
3. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as the identity element of $GL_2(\mathbb{R})$. More precisely,
$$AI = IA = A, \forall A \in GL_2(\mathbb{R}).$$
4. For all $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL_2(\mathbb{R})$, A has an inverse, namely

General linear group

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Note that $\det A^{-1} = 1/\det A \neq 0$ (why?). Moreover, we have

$$AA^{-1} = A^{-1}A = I.$$

Homework:

1. Is $GL_2(\mathbb{R})$ a group under usual addition of matrices?
2. Is $M_{2\times 2}$ a group under usual multiplication of matrices?