Lectures in Group Theory

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Examples on group

Example: Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Consider the subset of complex numbers $C_n = \{\varepsilon^i : i \in \mathbb{Z}\}$. Then C_n is a group under usual multiplication of complex numbers.

Claim: First, note that $C_n = \{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^n = 1\}$.

According to DeMover's Theorem, we have

$$\varepsilon^m = (\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^m = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}.$$

1. Let $\varepsilon^j, \varepsilon^k \in C_n$. Then

$$\varepsilon^j\varepsilon^k = (\cos\frac{2j\pi}{n} + i\sin\frac{2j\pi}{n})(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n})$$
$$\cos\frac{2(j+k)\pi}{n} + i\sin\frac{2(j+k)\pi}{n} = \varepsilon^{j+k} \in C_n.$$

2. It is clear that, for all $\varepsilon^m, \varepsilon^j, \varepsilon^k \in C_n$, we have

$$\varepsilon^m(\varepsilon^j\varepsilon^k) = \varepsilon^m\varepsilon^{j+k} = \varepsilon^{m+j+k},$$
$$(\varepsilon^m\varepsilon^j)\varepsilon^k = \varepsilon^{m+j}\varepsilon^k = \varepsilon^{m+j+k}.$$

More examples on groups

3. Note that

$$\varepsilon^n = \cos\frac{2n\pi}{n} + i\sin\frac{2n\pi}{n} = \cos 2\pi + i\sin 2\pi = 1 = \varepsilon^0$$

acts as the identity element of C_n. More precisely,

$$\varepsilon^m 1 = 1\varepsilon^m = \varepsilon^m.$$

4. If $\varepsilon^m \in C_n$, then $\varepsilon^{-m} \in C_n$ and

$$\varepsilon^m\varepsilon^{-m} = \varepsilon^{-m}\varepsilon^m = \varepsilon^{-m+m} = \varepsilon^0 = 1.$$

Homework: Is $\mathbb{R}^+$ a group under usual multiplication? where $\mathbb{R}^+$ is the set of all positive reals.

Symmetric group

Let $X = \{1,2,3\}$. Consider the set
$\mathfrak{S}_n = \{\text{all bijection maps } X \to X\}$; $n = 3$, namely

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Note that under composition of mappings, the set
$\mathfrak{S}_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ forms a group with identity equal to f_1. For example,

$$f_3 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = f_6 \in \mathfrak{S}_3.$$

Similarly, we have

$$f_2 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_5 \in \mathfrak{S}_3.$$

We need to check that $f_i \circ f_j \in \mathfrak{S}_3$ for all $i, j \in \{1,2,3\}$ (**Homework**).

Symmetric group

In fact, we have the following table

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

From the above table (1), (2) in the definition of a group are satisfied. Moreover, f_1 acts as the identity element of $\mathfrak{S}_3$, hence (3) is true. Finally, from the table above, we get

$$f_1^{-1} = f_1, f_2^{-1} = f_2, f_3^{-1} = f_3, f_4^{-1} = f_4, f_5^{-1} = f_6, f_6^{-1} = f_5.$$

So, (4) is true, and hence $(\mathfrak{S}_3; \circ)$ is a group.