Lectures in Group Theory

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May 18, 2020

Common Number sets

Natural numbers: The set $\mathbb{N} = \{1, 2, 3, \ldots\}$.
Whole numbers: The set $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$.
Integers: The set $\mathbb{Z} = \{0, \mp 1, \mp 2, \mp 3, \ldots\}$.
Positive integers: The set $\mathbb{Z}^+ = \mathbb{N} = \{1, 2, 3, \ldots\}$.
Negative integers: The set $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$.
Even integers: The set $\mathbb{Z}_e = \{0, \mp 2, \mp 4, \mp 6, \ldots\}$.
Odd integers: The set $\mathbb{Z}_o = \{\mp 1, \mp 3, \mp 5, \ldots\}$.
Prime numbers: The set $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$.
Rational numbers: The set $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.
Real numbers: The set $\mathbb{R} =$ union of rational and irrational numbers.
complex numbers: The set $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$.
Questions:

1. Is the addition of two integers is again an integer?
2. Is the multiplication of two integers is again an integer?
3. Is the addition of two prime numbers is again an prime number?
4. Is the addition of two complex numbers is again an complex number?

Binary operations

Definition: A **binary operation** $*$ on a non-empty set S is a map $* : S \times S \to S$ where

$$(a, b) \in S \times S \mapsto *(a, b) = a * b \in S.$$

Example1: The usual addition is a binary operation on $\mathbb{N}$.

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (a, b) \mapsto +(a, b) = a + b.$$

However, the usual substraction is not binary operation on $\mathbb{N}$, namely

$$- : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (a, b) \mapsto -(a, b) = a - b \text{ (not a map).}$$

Note that $5, 2 \in \mathbb{N}$ but $-(2, 5) = 2 - 5 = -3 \notin \mathbb{N}$.

Example2: The usual addition and multiplication are binary operations on the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

Remark: Henceforth, the pair $(S, *)$ of a non-empty set with its binary operation is called **mathematical system**.

Example3: $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are mathematical systems.

Binary operations

Example4: Define $*$ on the set $H = \{n^2 : n \in \mathbb{Z}\}$ as follows:

$$\forall\, a, b \in H : \; a * b = a + b.$$

Is $*$ a binary operation on H? Is $(H, *)$ a mathematical system?
Answer: It is clear that $H = \{0, 1, 4, 9, 25, \ldots\}$. Note that $1, 4 \in H$, but $1 * 4 = 1 + 4 = 5 \notin H$. Therefore $*$ is not binary operation on H, and hence $(H, *)$ is not mathematical system.
Homework:

1. Is $(\mathbb{P}, +)$ a mathematical system? where $+$ is the usual addition.
2. Is $(\mathbb{Z}^+, \div)$ a mathematical system? where $\div$ is the usual division.
3. Define an operation $*$ on the set $H = \{n^2 : n \in \mathbb{Z}\}$ as follows:

$$\forall\, a, b \in H : \; a * b = a \cdot b.$$

Is $(H, *)$ a mathematical system?