Lectures in Group Theory

Mohammed Alabbood

University of Basrah-College of Science
Department of Mathematics

May 18, 2020

Semigroups

Definition: A **semigroup** is a mathematical system $(S, *)$ such that $*$ is an associative binary operation on S, i.e.

$$\forall\, a, b, c \in S:\ a * (b * c) = (a * b) * c.$$

Example1: $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are semigroups.
Note that for all a, b, c in the above systems, we have

$$a + (b + c) = (a + b) + c.$$

Example2: $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are semigroups.
Note that for all a, b, c in the above systems, we have

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Example3: It is clear that $(\mathbb{Z}, -)$ is a mathematical system, but it is not semigroup.
Note that $1, 2, 3 \in \mathbb{Z}$ and

$$1 - (2 - 3) = 2 \neq -4 = (1 - 2) - 3.$$

Example4: Define $*$ on the set $\mathbb{Z}$ as follows: $\forall\, a, b \in \mathbb{Z}: \; a * b = ab^2$. Is $*$ a binary operation on $\mathbb{Z}$? Is $(\mathbb{Z}, *)$ a semigroup?
Answer: Let $a, b, c \in \mathbb{Z}$. Then $a * b = ab^2 \in \mathbb{Z}$ and hence $*$ is a binary operation on $\mathbb{Z}$.
On the other hand $(\mathbb{Z}, *)$ is not semigroup.

$$a * (b * c) = a * bc^2 = a(bc^2)^2 = ab^2c^4 \text{ and}$$
$$(a * b) * c = ab^2 * c = ab^2c^2 = ab^2c^2.$$

Thus, $a * (b * c) \neq (a * b) * c$, and hence $(\mathbb{Z}, *)$ is not semigroup.
Homework:

1. Define $*$ on the set $\mathbb{R}$ as follows: $\forall\, a, b \in \mathbb{R}: \; a * b = a^b$. Is $*$ a binary operation on $\mathbb{R}$? Is $(\mathbb{R}, *)$ a semigroup?
2. Define $*$ on the set $\mathbb{R}$ as follows: $\forall\, a, b \in \mathbb{R}: \; a * b = ab + 1$. Is $*$ a binary operation on $\mathbb{R}$? Is $(\mathbb{R}, *)$ a semigroup?

Monoids

Definition: A **monoid** is a semigroup $(S, *)$ such that there exist $e \in S$ (called the **identity** of S) with

$$a * e = e * a = a \, \forall a \in S.$$

Example1: $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are monoids with $e = 0$. Note that for all a in the above systems, we have

$$a + 0 = 0 + a = a.$$

Example2: $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are monoids with $e = 1$. Note that for all a in the above systems, we have

$$a \cdot 1 = 1 \cdot a = a.$$

Example3: It is clear that $(\mathbb{N}, +)$ is a semigroup, but it is not monoid. In fact, there is no element in $e \in \mathbb{N}$ with $e + a = a + e = a$.