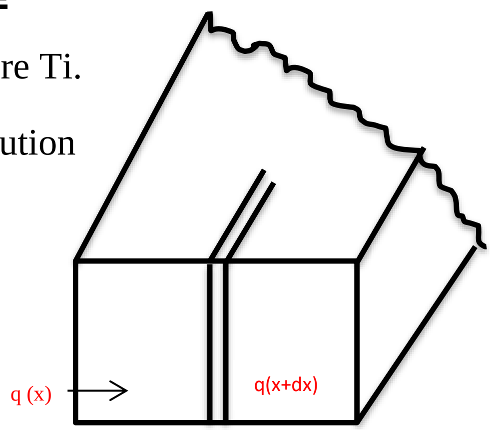
Transient heat flow in a semi-infinite solid



Consider a semi-infinite solid, the initial temperature Ti.

We seek an expression for the temperature distribution in the solid, As a function of the time and x.

$q_{in} - q_{out}$ = accumulation

$$q(x) - q(x+dx) = \rho\, Cp\, v\, \frac{dT}{dt}$$

$$q(x+dx) = q(x) + dx\, \frac{dqx}{dx}$$

x → ∞

y → ∞

z → ∞

Assume length in direction y and z = 1

V= dx * dy * dz = 1* 1* dx= dx

$$\cancel{q(x)} - \cancel{q(x)} - dx\, \frac{dqx}{dx} = \rho\, Cp\, dx\, \frac{dT}{dt}$$

Divide by dx →

$$-\frac{dqx}{dx} = \rho\, Cp\, \frac{dT}{dt}$$

$$q(x) = -k\, \frac{dT}{dx} \quad \text{[Fourier law]}$$

$$-\frac{d}{dx}\left[-k\, \frac{dT}{dx}\right] = \rho\, Cp\, \frac{dT}{dt}$$

$$K\, \frac{d^2T}{dx^2} = \rho\, Cp\, \frac{dT}{dt}$$

$$\frac{d^2T}{dx^2} = \frac{\rho\, Cp}{k}\, \frac{dT}{dt}$$

$$\frac{\rho\, Cp}{k} = \frac{1}{\alpha}$$, where α is thermal diffusivity $= \frac{k}{\rho\, Cp}$

$$\frac{d^2T}{dx^2} = \frac{1}{\alpha} * \frac{dT}{dt} \quad \text{-------------------- 1}$$

The initial condition: t = 0 at T =Ti → T (x, 0) = Ti

It is three cases to find temperature distribution:

Case 1 Constant surface temperature

B.C: T (0, t) = Ts

Equation 1 is solved by lablace transformation method, the solution is:

$$\frac{T-Ts}{Ti-Ts} = \text{erf}\,\frac{x}{2\sqrt{\alpha\, t}} \text{ ------------------ } 2$$

Ts = Surface temperature.

Ti = Initial temperature.

erf = Error function.

To find the heat rate

$$q=\frac{k\,A\,(Ts-Ti)}{\sqrt{\pi\alpha\, t}}\ \exp\left(-\frac{x^2}{4\,\alpha\, t}\right) \text{ --------------------} 3$$

Equation 3 is applied to find q at any part of the body (any value of x).

But at the surface (x =0)

$$q=\frac{k\,A\,(Ts-Ti)}{\sqrt{\pi\alpha\, t}} \text{ ------------------------} 4$$

Case 2 Constant heat flux

B.C: $\frac{q}{A} = -k\,\frac{dT}{dx}\Big|_{x=0}$ The temperature distribution

$$T-T_i=\frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right)-\frac{q_0x}{kA}\left(1-\text{erf}\,\frac{x}{2\sqrt{\alpha\tau}}\right) \text{ --------------------- } 5$$

Case 3 Convection heat transfer.

B.C: heat convection to the surface = heat conduction to the surface.

$h\,A(T-T\infty)\big|_{x=0} = -k\,A\,\frac{dT}{dx}\Big|_{x=0}$ The solution is ⟶

$$\frac{T-T_i}{T_\infty-T_i}=1-\text{erf}\,\overline{X}-\left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha\tau}{k^2}\right)\right]\times\left[1-\text{erf}\left(\overline{X}+\frac{h\sqrt{\alpha\tau}}{k}\right)\right] \text{ --------------------- } 6$$

Where

$$\overline{X}=x/(2\sqrt{\alpha\tau})$$

<u>Example:</u>

A very large slab of copper is initially at temperature of 300 °C, the surface temperature is suddenly lowered at 35 °C .What temperature at depth of 7.5 cm and 4 min. $\alpha = 11.23 * 10^{-5}$ m^2/sec

T (0.075 m , 240 sec) = ?

Solution

$$\frac{T - Ts}{Ti - Ts} = \text{erf}\,\frac{x}{2\sqrt{\alpha\, t}}$$

$$\frac{x}{2\sqrt{\alpha\, t}} = \frac{0.075}{2\sqrt{11.23 * 10^{-5} * 240}} = 0.2284$$

From table A -1 appendix A (Holman) and using interpolation

erf 0.2284 = 0.2533

$$\frac{T - Ts}{Ti - Ts} = 0.2533$$

T = 0.2533 (300 -35) + 35 = 102.1 °C

<u>Example:</u>

A semi-infinite slab of copper is exposed to a constant heat flux at the surface of $0.5 * 10^6$ W/m^2, so that there is not convection at the surface

a- What is the surface temperature after 5 min if the initial temperature of slab is 20 °C.

b- What is the temperature at the distance of 15 cm from the surface after 5 min and same initial temperature.

Take the $\alpha = 11.23 * 10^{-5}$ m^2/sec , k = 386 W/m .°C

Solution

a- $$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \text{erf}\,\frac{x}{2\sqrt{\alpha\tau}}\right)$$

not value of x ⟶ x = 0 the general equation ⟶

$$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}$$

$$T - 20 = \frac{2 * 0.5 * 10^{6} * (\sqrt{11.23 * 10^{-5} * 300)/\pi)}}{386} \longrightarrow$$

T = 20 + 268.3 = 288.3 °C

b- $$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \operatorname{erf}\frac{x}{2\sqrt{\alpha\tau}}\right)$$

$$\frac{x}{2\sqrt{\alpha t}} = \frac{0.15}{2\sqrt{11.23 * 10^{-5} * 300}} = 0.4086$$

erf 0.4086 from table A-1 and using interpolation = 0.4336

$$\frac{x^2}{4\,\alpha\,t} = \frac{(0.15)^2}{4 * 11.23 * 10^{-5} * 300} = (0.4086)^2 = 0.167$$

$$\sqrt{\alpha t/\pi} = \sqrt{11.23 * 10^{-5} * 300/\pi} = 0.104$$

$$T - 20 = \frac{2 * 0.5 * 10^{6} * (\sqrt{11.23 * 10^{-5} * 300)/\pi)}}{386} * \exp^{-0.167} -$$

$$\frac{0.5 * 10^{6} * 0.15}{386} * [\,1 - 0.4336] \longrightarrow$$

T -20 = 228 – 109.5 =138.5 C°

Example:

A thick concrete stone wall having a uniform temperature of 54 °C is suddenly subjected to an air stream at 10 °C. The heat transfer coefficient is 10 W/m^2.°C. Calculate the temperature in the concrete slab at depth of 7 cm after 30 min.

$\alpha = 7 * 10^{-7}$ m^2/sec , k = 1.37 W/m .°C

T (0.07 m ,1800 sec)= ?

Solution

Case 3

$$\frac{T - T_i}{T_\infty - T_i} = 1 - \operatorname{erf}\bar{X} - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha\tau}{k^2}\right)\right] \times \left[1 - \operatorname{erf}\left(\bar{X} + \frac{h\sqrt{\alpha\tau}}{k}\right)\right]$$

$$\bar{X} = \frac{x}{2\sqrt{\alpha t}} = \frac{0.07}{2\sqrt{7 * 10^{-7} * 1800}} = 0.987$$

$\text{erf}(\overline{X}) = \text{erf } 0.987 = 0.83719$

$$\frac{h\sqrt{\alpha t}}{k} = \frac{10 * \sqrt{7 * 10^{-7} * 1800}}{1.37} = 0.259$$

$$\text{erf}\left(\overline{X} + \frac{h\sqrt{\alpha t}}{k}\right) = \text{erf}(0.987 + 0.259) =$$

$\text{erf}(1.246) = 0.920971$

$$\frac{h\,x}{k} = \frac{10 * 0.07}{1.37} = 0.511$$

$$\frac{h^2 * \alpha t}{k^2} = \frac{10^2 * 7 * 10^{-7} * 1800}{1.37^2} = 0.067$$

$$\frac{T-54}{10-54} = 1 - 0.83719 - \exp^{(0.511+0.067)}\,[1 - 0.920971]$$

$T - 54 = 0.022 * [10 - 54] \longrightarrow$

$T = 53.03\ °C$