Shearing Forces and Bending Moments in Beams

A beam is a structural member that carries loads transversely, that is, perpendicular to its long axis.

When analyzing a beam to determine ***reactions***, ***internal shearing forces***, and ***internal bending moments***, it is helpful to classify the manner of loading, the type of supports, and the type of beam.

Beams are subjected to a variety of loading patterns, including:

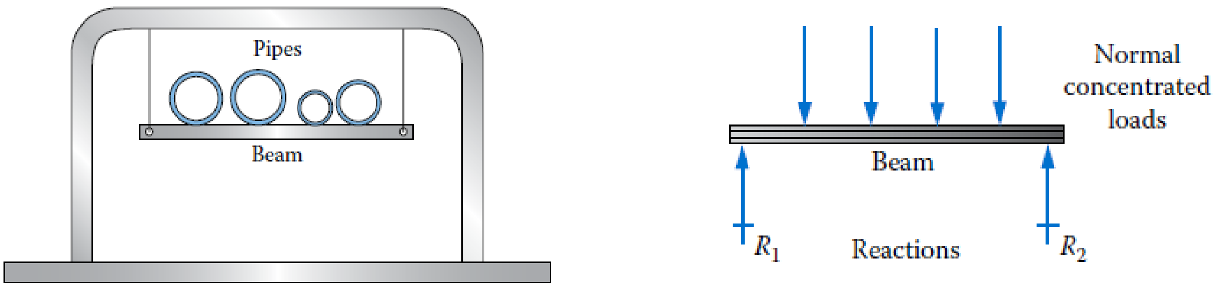
- Normal concentrated loads
- Inclined concentrated loads
- Uniformly distributed loads
- Varying distributed loads
- Concentrated moments

Support types include:

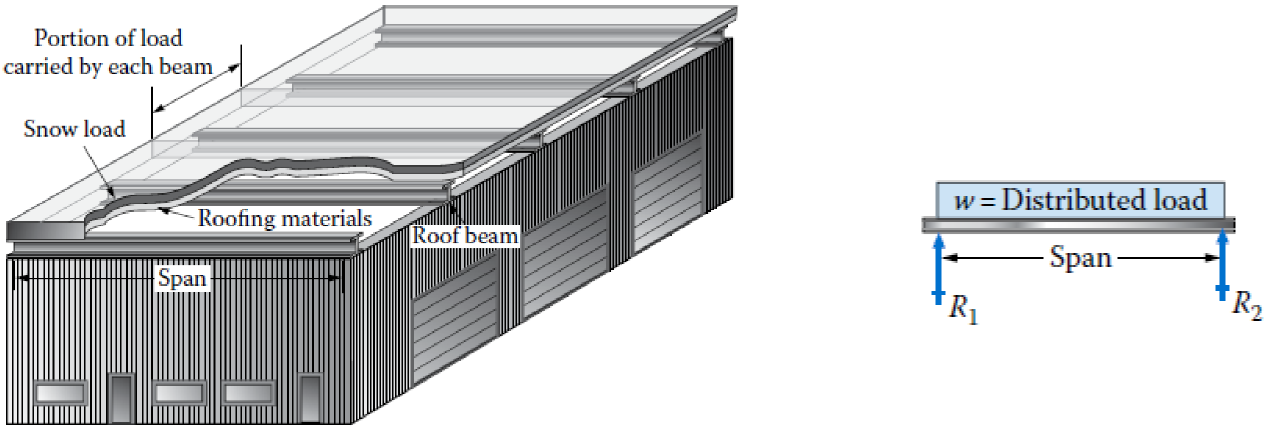
- Simple, roller-type support
- Pinned support
- Fixed support

Beam types include

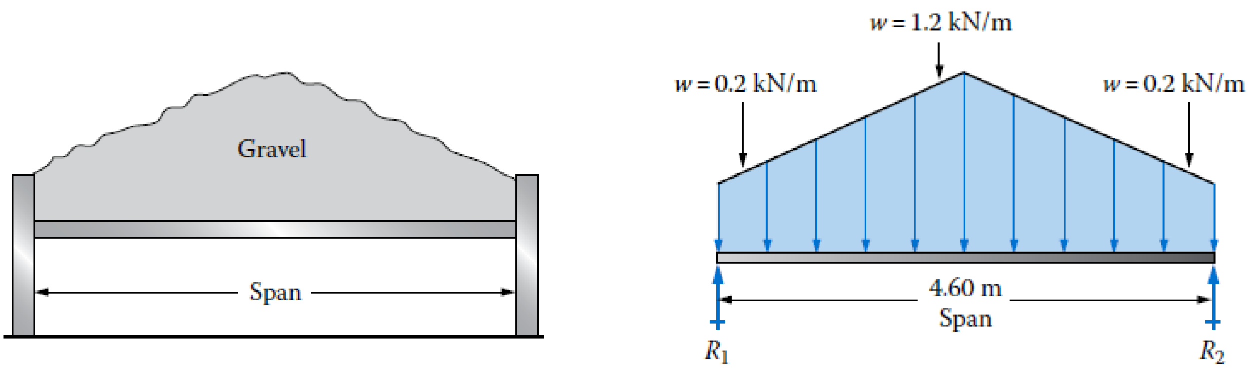
- Simply supported beams or simple beams
- Overhanging beams
- Cantilever beams or cantilevers
- Compound beams
- Continuous beams



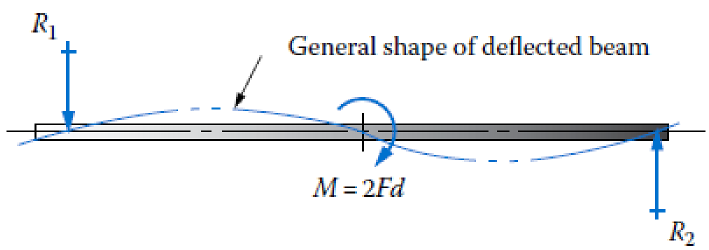
Simple beam with normal concentrated loads



Simple beam with uniformly distributed load



Example of linearly varying distributed load on a simple beam



Concentrated moment on a beam

Type of Supports

- **Simple Support or Roller Support**

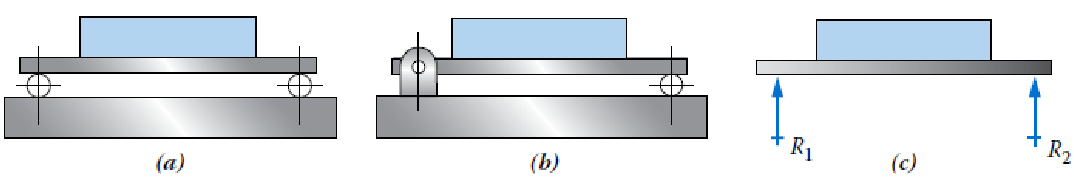
A simple support is one that can resist only forces acting perpendicular to the beam.

- **Pinned Support**

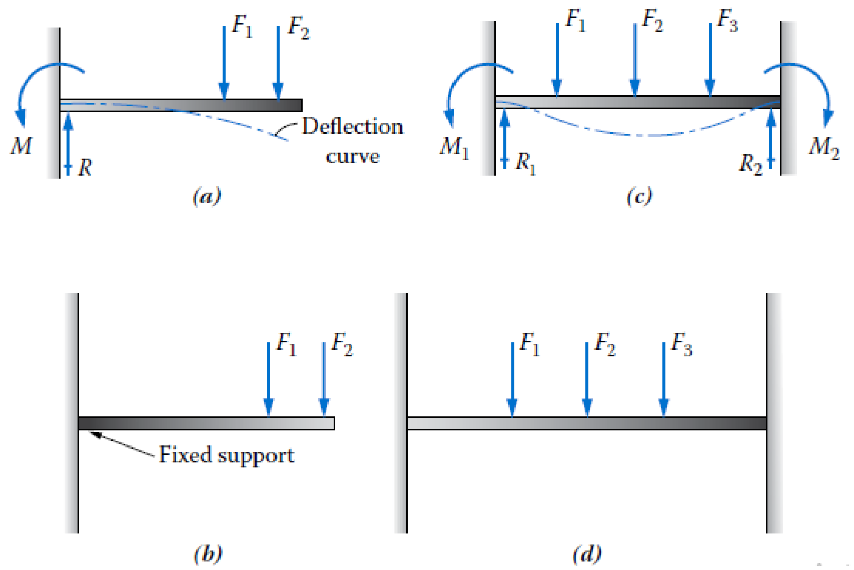
An example of a pinned support is a hinge that can resist forces in any direction but which allows rotation about the axis of the pin in the hinge.

- **Fixed Support**

A fixed support is one that is held solidly such that it resists forces in any direction and also prohibits rotation of the beam at the support.



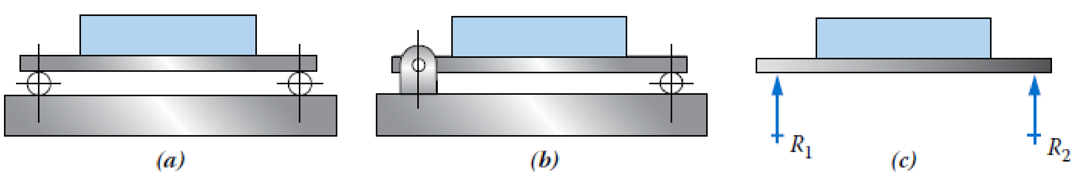
Examples of simple supports: (a) beam on two rollers, (b) beam with pinned support and one roller, and (c) free-body diagram for (a) or (b).



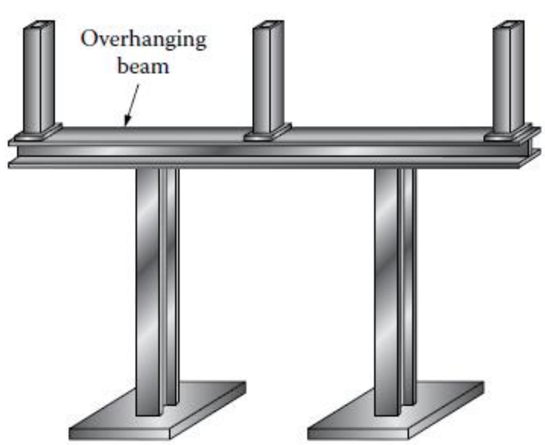
Beams with fixed supports: fixed support for a cantilever and representation of beam with two fixed supports.

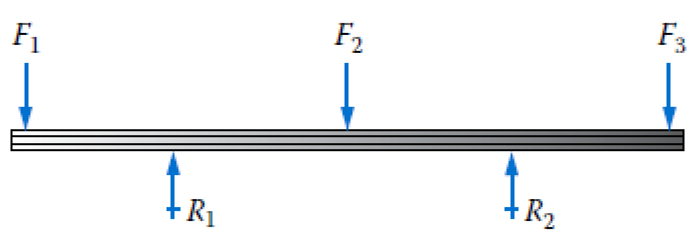
Beams Type

- Simply Supported Beams

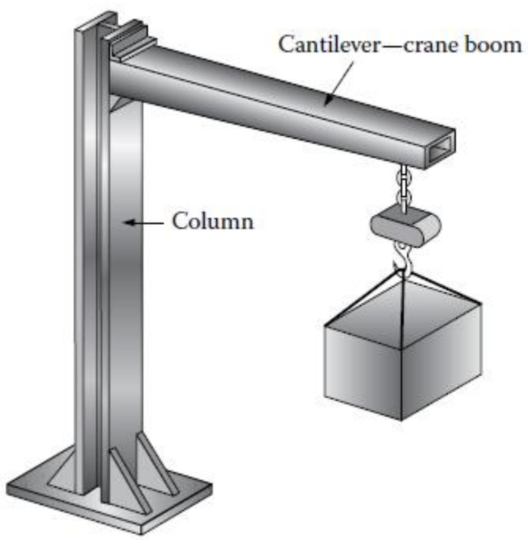


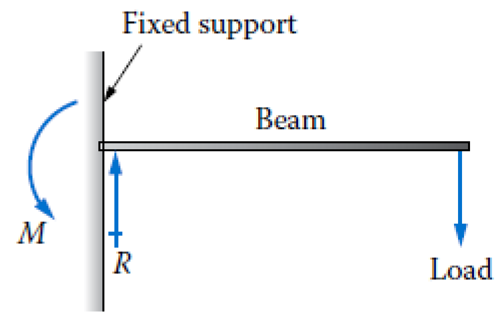
- **Overhanging Beams**



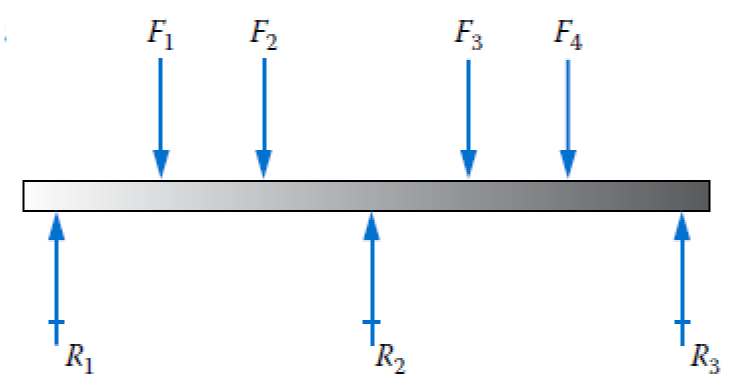


- **Cantilever Beam**

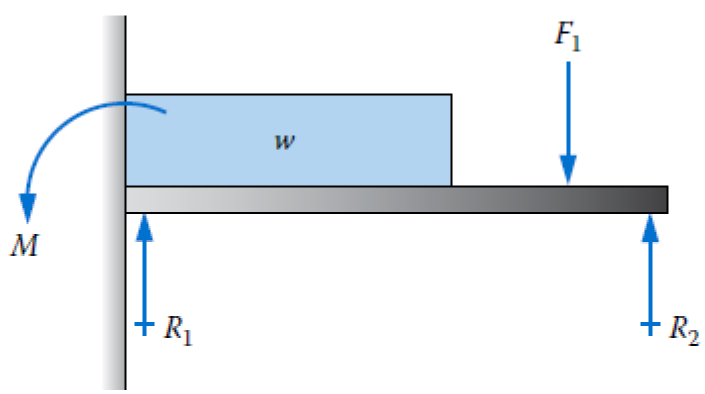




- **Continuous Beam**



- **Supported (Propped) Cantilever Beam**



Reactions at Supports

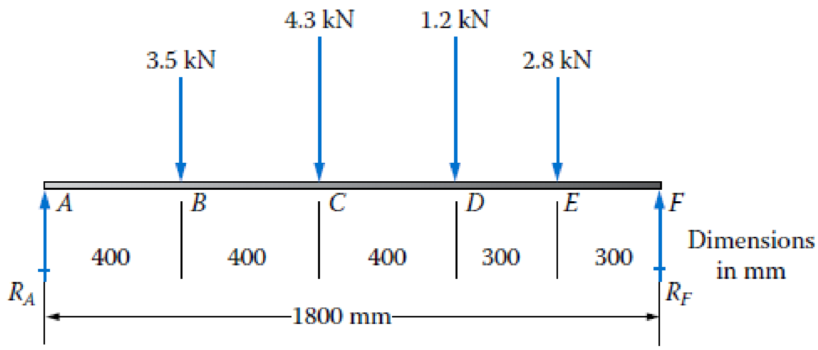
The first step in analyzing a beam to determine its safety under a given loading arrangement is to show completely the loads and support reactions on a free-body diagram.

It is very important to be able to construct free-body diagrams from the physical picture or description of the loaded beam.

Guidelines for Solving for Reactions

1. Draw the free-body diagram.
2. Use the equilibrium equation ΣM = 0 by summing moments about the point of application of one support reaction. The resulting equation can then be solved for the other reaction.
3. Use ΣM = 0 by summing moments about the point of application of the second reaction to find the first reaction.
4. Use ΣF = 0 to check the accuracy of your calculations.

Example (1): Compute the reaction forces in the supports



Solution:

To find the reaction R_F, sum moments about point A:

$$\sum M_A = 0 = 3.5(400) + 4.3(800) + 1.2(1200) + 2.8(1500) - R_F(1800)$$

Note that all forces are in kN and distances in mm. Each moment term has the units of kN · mm. Now solve for R_F:

$$R_F = \frac{[3.5(400) + 4.3(800) + 1.2(1200) + 2.8(1500)]\,\text{kN}\cdot\text{mm}}{1800\ \text{mm}} = 5.82\ \text{kN}$$

Now, to find R_A, sum moments about point F:

$$\sum M_F = 0 = 2.8(300) + 1.2(600) + 4.3(1000) + 3.5(1400) - R_A(1800)$$

$$R_A = \frac{[2.8(300) + 1.2(600) + 4.3(1000) + 3.5(1400)]\,\text{kN}\cdot\text{mm}}{1800\ \text{mm}} = 5.98\ \text{kN}$$

Now use $\Sigma F = 0$ for the vertical direction as a check:

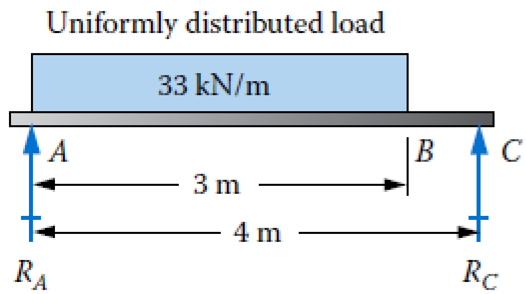
Downward forces: (3.5 + 4.3 + 1.2 + 2.8) kN = 11.8 kN

Upward reactions: (5.82 + 5.98) kN = 11.8 kN (check)

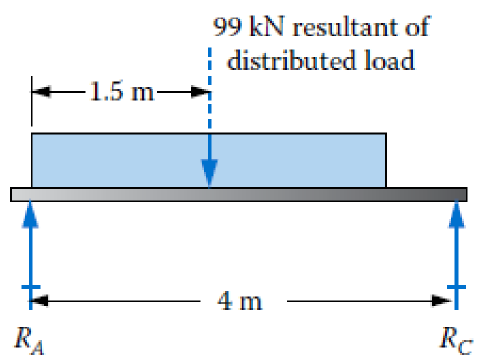
The left reaction is: $R_A = 5.98$ kN

The right reaction is: $R_F = 5.82$ kN

Example (2): Compute the reactions on the beam shown.



Solution:



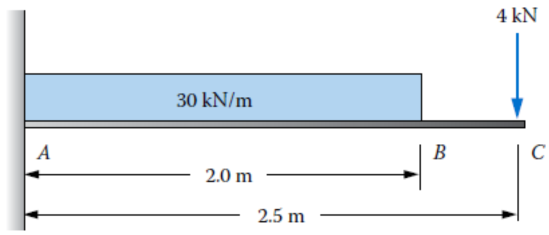
By summing forces in the vertical direction, we obtain

$$R_A = 60 \text{ kN} + 4 \text{ kN} = 64 \text{ kN}$$

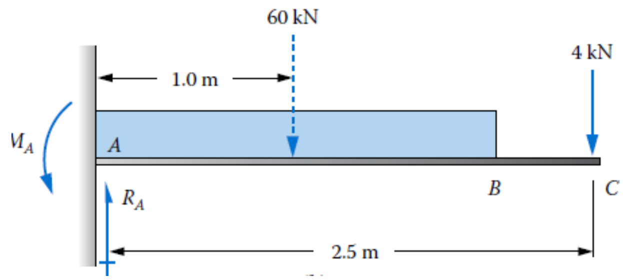
Summing moments about point A yields

$$M_A = 60 \text{ kN}(1.0 \text{ m}) + 4 \text{ kN}(2.5 \text{ m}) = 70 \text{ kN} \cdot \text{m}$$

Example (3): Compute the reactions for the cantilever beam



Solution:



By summing forces in the vertical direction, we obtain

$R_A = 60 \text{ kN} + 4 \text{ kN} = 64 \text{ kN}$

Summing moments about point A yields

$$M_A = 60 \text{ kN}(1.0 \text{ m}) + 4 \text{ kN}(2.5 \text{ m}) = 70 \text{ kN} \times \text{m}$$

Shearing Forces:

Shearing forces are internal forces developed in the material of a beam to balance externally applied forces in order to secure equilibrium of all parts of the beam.

The magnitude of the shearing force in any part of a beam is equal to the algebraic sum of all external forces acting to the left (or right) of the section of interest.

Sign Convention

Positive S.F. Negative S.F.

Notes:

- On any segment of a beam where no loads are applied, the value of the shearing force remains constant.
- A concentrated load (or reaction) on a beam causes an abrupt change in the shearing force in the beam by an amount equal to the magnitude of the load and in the direction of the load.

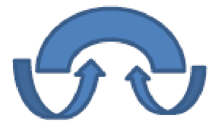
Bending Moments

Bending moments are internal moments developed in the material of a beam to balance the tendency for external forces to cause rotation of any part of the beam.

The magnitude of the bending moment in any part of a beam is equal to the algebraic sum of the moment of forces acting to the left (or right) of the section of interest.

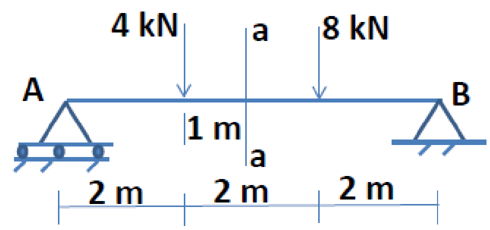
Sign Convention

Positive Bending Moment
(Sagging)



Negative Bending Moment
(Hogging)

Example (4): For the beam shown, find the shearing force and bending moment at section a-a.



Solution

$\Sigma M_B=0$

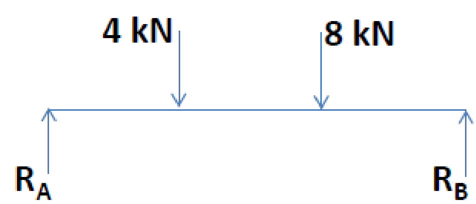
$R_A \times 6 - 4\times4 - 8\times 2 = 0$

RA = 5.33 kN

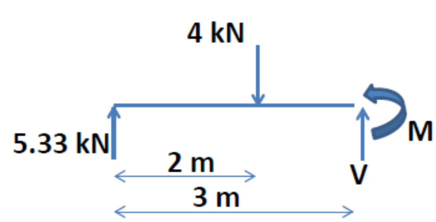
Va-a = RA - 4

= 1.33 kN +ve.

Ma-a = RA × 3 – 4 × 1 = 12 kN.m +ve (sagging)



F.B.D

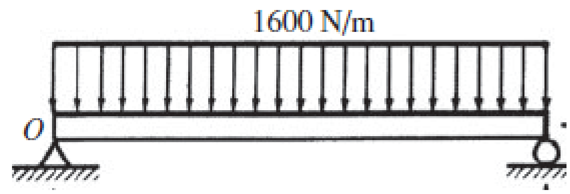


Section a-a

Shear Force and Bending Moment Diagrams

Diagrams, which illustrate the variation in the shear force and bending moment values along the length of the beam under the action of certain loading condition.

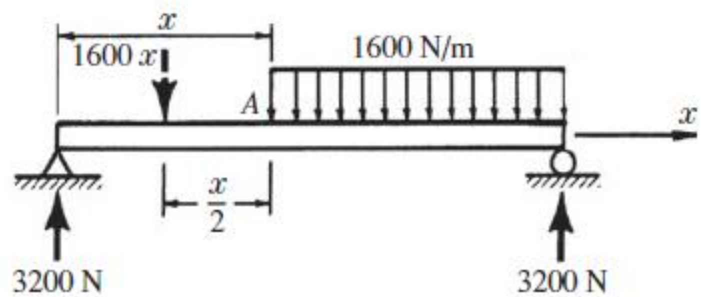
Example (5): Draw the shear and moment diagrams for the beam shown



SOLUTION: The total load on the beam is 6400 N, and from symmetry each of the end reactions is 3200 N.

We shall now consider any cross section of the beam at a distance x from the left end. The shearing force at this section is given by the algebraic sum of the forces to the left of this section and these forces consist of the 3200 N reaction and the distributed load of 1600 N/m extending over a length x. The shearing force at x is then given by:

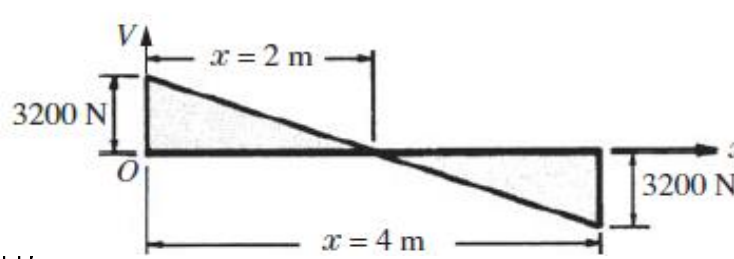
$V(x) = 3200 - 1600x$



Since there are no concentrated loads acting on the beam, this equation is valid at all points along its length. The variation of shearing force along the length of the bar may then be represented by a straight line connecting the two end-point values. The shear is zero at the center of the beam.

$V(x)=0$

$3200-1600x=0$



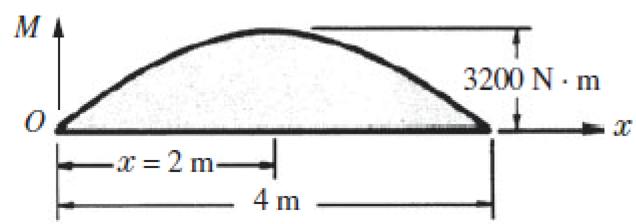
X=2m

The bending moment at the section x is given by the algebraic sum of the moments of the 3200-N reaction and the distributed load of 1600x about an axis through A,

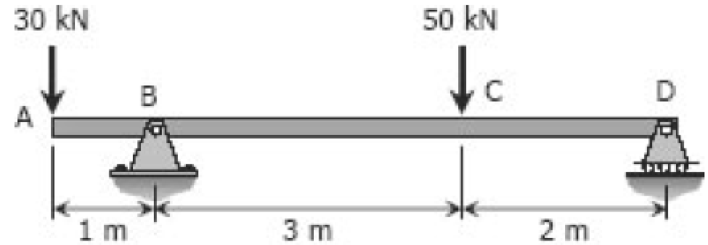
$$M = 3200x - 1600x\left(\frac{x}{2}\right)$$
$$= 3200x - 800x^2$$

From the above equation, it is evident that the bending moment is represented by a parabola along the length of the beam.
Since the bar is simply supported, the moment is zero at either end and, because of the symmetry of loading, the bending moment must be a maximum at the center of the beam where x = 2 m. The maximum bending moment is

$$M_{max} = 3200(2) - 800(2)^2 = 3200 \text{ N} \cdot \text{m}$$



Example (6): Draw the shear and moment diagrams for the beam shown

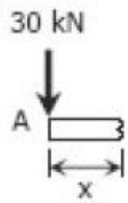


Solution:

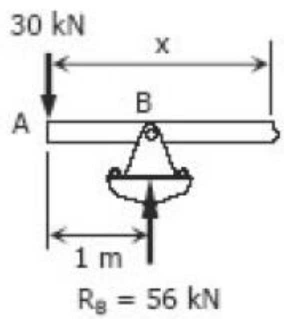
$$\sum M_B = 0$$
$$5R_D + 1(30) = 3(50)$$
$$R_D = 24 \text{ kN}$$

$$\sum M_D = 0$$
$$5R_B = 2(50) + 6(30)$$
$$R_B = 56 \text{ kN}$$

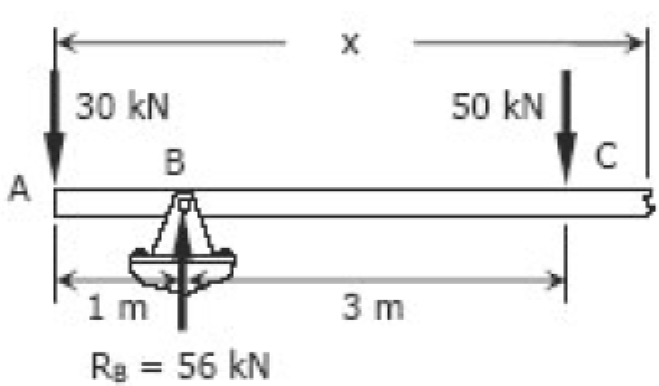
Segment *AB*:
$V_{AB} = -30$ kN
$M_{AB} = -30x$ kN·m



Segment *BC*:
$V_{BC} = -30 + 56$
$= 26$ kN
$M_{BC} = -30x + 56(x - 1)$
$= 26x - 56$ kN·m



Segment *CD*:
$V_{CD} = -30 + 56 - 50$
$= -24$ kN
$M_{CD} = -30x + 56(x - 1) - 50(x - 4)$
$= -30x + 56x - 56 - 50x + 200$
$= -24x + 144$



To draw the Shear Diagram:

(1) In segment AB, the shear is uniformly distributed over the segment at a magnitude of −30 kN.

(2) In segment BC, the shear is uniformly distributed at a magnitude of 26 kN.

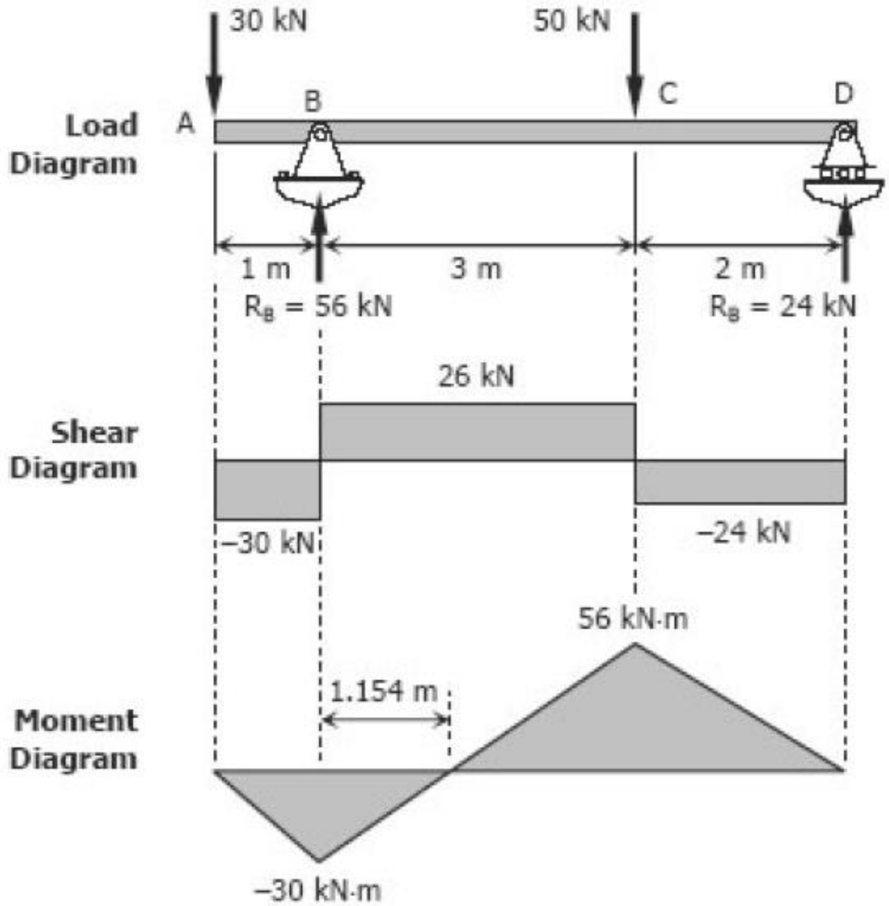
(3) In segment CD, the shear is uniformly distributed at a magnitude of −24 kN.

To draw the Moment Diagram:

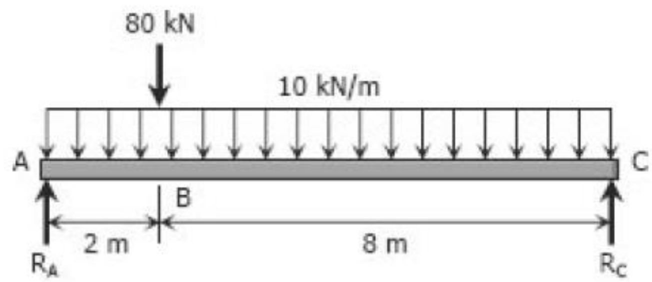
(1) The equation $M_{AB} = -30x$ is linear, at $x = 0$, $M_{AB} = 0$ and at $x = 1$ m, $M_{AB} = -30$ kN·m.

(2) $M_{BC} = 26x - 56$ is also linear. At $x = 1$ m, $M_{BC} = -30$ kN·m; at $x = 4$ m, $M_{BC} = 48$ kN·m. When $M_{BC} = 0$, $x = 2.154$ m, thus the moment is zero at 1.154 m from B.

(3) $M_{CD} = -24x + 144$ is again linear. At $x = 4$ m, $M_{CD} = 48$ kN·m; at $x = 6$ m, $M_{CD} = 0$.



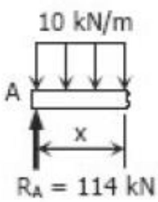
Example (7): Draw the shear and moment diagrams for the beam shown



Solution:

$$\Sigma M_A = 0$$
$$10R_C = 2(80) + 5[10(10)]$$
$$R_C = 66 \text{ kN}$$

$$\Sigma M_C = 0$$
$$10R_A = 8(80) + 5[10(10)]$$
$$R_A = 114 \text{ kN}$$

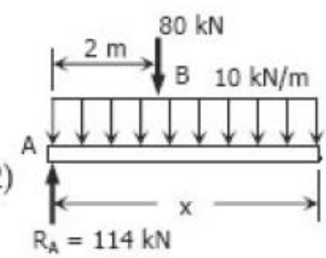


Segment AB:

$$V_{AB} = 114 - 10x \text{ kN}$$
$$M_{AB} = 114x - 10x(x/2)$$
$$= 114x - 5x^2 \text{ kN·m}$$

Segment BC:

$$V_{BC} = 114 - 80 - 10x$$
$$= 34 - 10x \text{ kN}$$
$$M_{BC} = 114x - 80(x - 2) - 10x(x/2)$$
$$= 160 + 34x - 5x^2$$



To draw the Shear Diagram:

(1) For segment AB, $V_{AB} = 114 - 10x$ is linear; at x = 0, V_{AB} = 14 kN; at x = 2 m, V_{AB} = 94 kN.

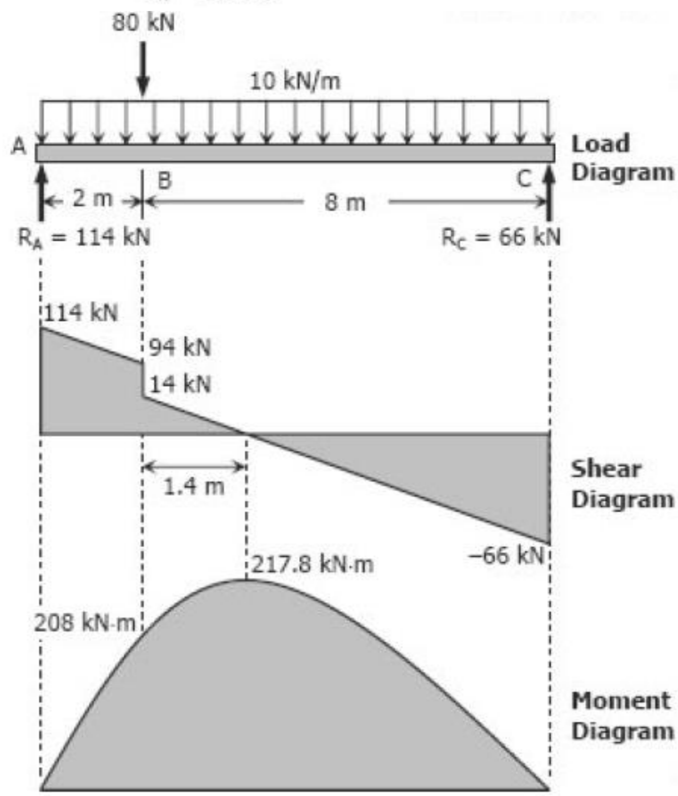
(2) $V_{BC} = 34 - 10x$ for segment BC is linear; at x = 2 m, V_{BC} = 14 kN; at x = 10 m, V_{BC} = −66 kN. When V_{BC} = 0, x = 3.4 m thus V_{BC} = 0 at 1.4 m from B.

To draw the Moment Diagram:

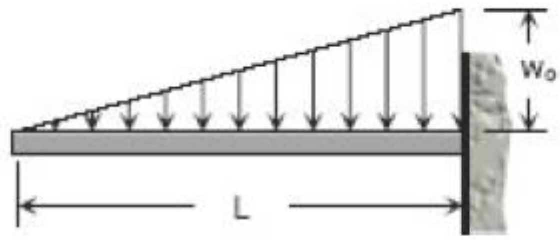
(1) $M_{AB} = 114x - 5x^2$ is a second degree curve for segment AB; at x = 0, M_{AB} = 0; at x = 2 m, M_{AB} = 208 kN·m.

(2) The moment diagram is also a second degree curve for segment BC given by $M_{BC} = 160 + 34x - 5x^2$; at x = 2 m, M_{BC} = 208 kN·m; at x = 10 m, M_{BC} = 0.

(3) Note that the maximum moment occurs at point of zero shear. Thus, at x = 3.4 m, M_{BC} = 217.8 kN·m.



Example (8): Draw the shear and moment diagrams for the beam shown



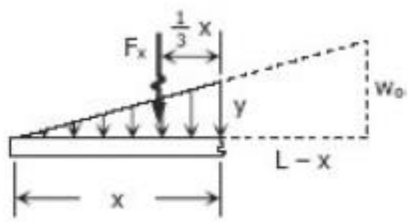
Solution

$$\frac{y}{x} = \frac{w_o}{L}$$

$$y = \frac{w_o}{L}x$$

$$F_x = \tfrac{1}{2}xy$$

$$= \frac{1}{2}x\left(\frac{w_o}{L}x\right)$$



$$= \frac{w_o}{2L}x^2$$

Shear equation:

$$V = -\frac{w_o}{2L}x^2$$

Moment equation:

$$M = -\tfrac{1}{3}xF_x = -\frac{1}{3}x\left(\frac{w_o}{2L}x^2\right)$$

$$= -\frac{w_o}{6L}x^3$$

To draw the Shear Diagram:

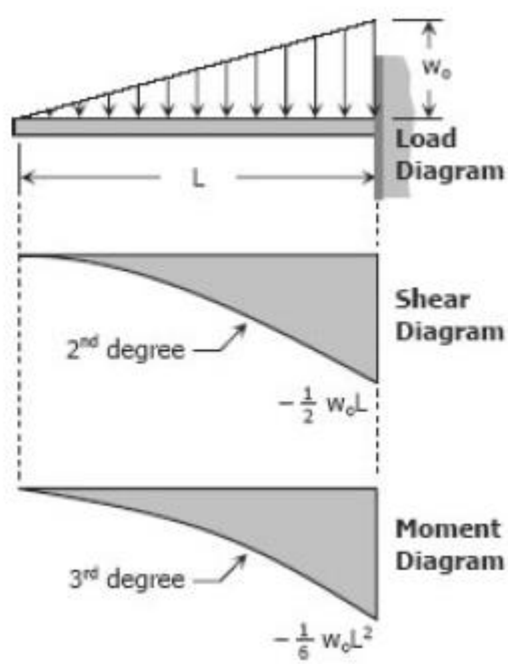
$V = -\frac{w_o}{2L}x^2$ is a second degree curve;

at x = 0, V = 0; at x = L, $V = -\frac{1}{2}w_oL$.

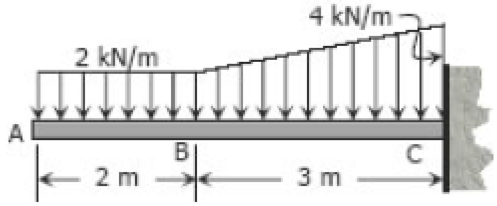
To draw the Moment Diagram:

$M = -\frac{w_o}{6L}x^3$ is a third degree curve; at

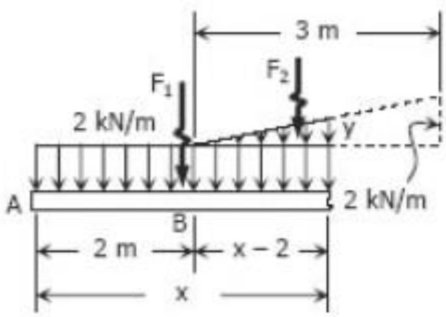
x = 0, M = 0; at x = L, $M = -\frac{1}{6}w_oL^2$.



Example (9): Draw the shear and moment diagrams for the beam shown



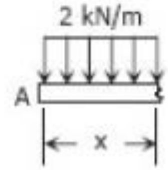
Solution



Segment AB:

$$V_{AB} = -2x \text{ kN}$$

$$M_{AB} = -2x(x/2) = -x^2 \text{ kN·m}$$



Segment BC:

$$\frac{y}{x-2} = \frac{2}{3}$$

$$y = \tfrac{2}{3}(x-2)$$

$$F_1 = 2x$$

$$F_2 = \tfrac{1}{2}(x-2)y = \tfrac{1}{2}(x-2)[\tfrac{2}{3}(x-2)] = \tfrac{1}{3}(x-2)^2$$

$$V_{BC} = -F_1 - F_2 = -2x - \tfrac{1}{3}(x-2)^2$$

$$M_{BC} = -(x/2)F_1 - \tfrac{1}{3}(x-2)F_2 = -(x/2)(2x) - \tfrac{1}{3}(x-2)[\tfrac{1}{3}(x-2)^2] = -x^2 - \tfrac{1}{9}(x-2)^3$$

To draw the Shear Diagram:

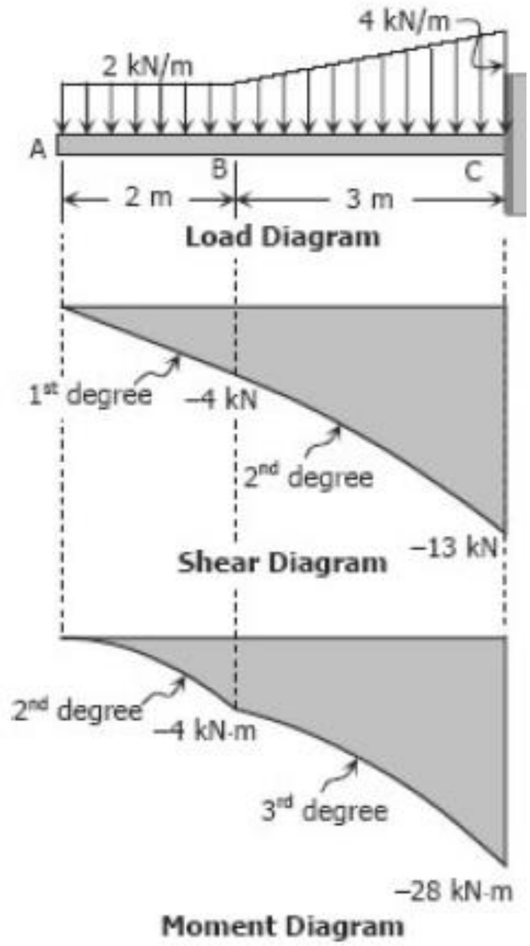
(1) $V_{AB} = -2x$ is linear; at $x = 0$, $V_{AB} = 0$; at $x = 2$ m, $V_{AB} = -4$ kN.

(2) $V_{BC} = -2x - \frac{1}{3}(x-2)^2$ is a second degree curve; at $x = 2$ m, $V_{BC} = -4$ kN; at $x = 5$ m; $V_{BC} = -13$ kN.

To draw the Moment Diagram:

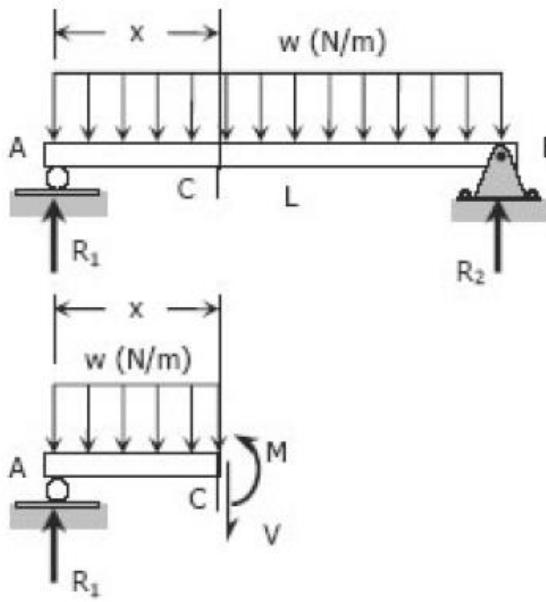
(1) $M_{AB} = -x^2$ is a second degree curve; at $x = 0$, $M_{AB} = 0$; at $x = 2$ m, $M_{AB} = -4$ kN·m.

(2) $M_{BC} = -x^2 - \frac{1}{9}(x-2)^3$ is a third degree curve; at $x = 2$ m, $M_{BC} = -4$ kN·m; at $x = 5$ m, $M_{BC} = -28$ kN·m.



Relationship between Load, Shear, and Moment

The vertical shear at C in the figure shown



$$V_C = (\Sigma F_v)_L = R_1 - wx$$

where $R_1 = R_2 = wL/2$

$$V_C = \frac{wL}{2} - wx$$

$$M_C = (\Sigma M_C) = \frac{wL}{2}x - wx\left(\frac{x}{2}\right)$$

$$M_C = \frac{wLx}{2} - \frac{wx^2}{2}$$

If we differentiate M with respect to x:

$$\frac{dM}{dx} = \frac{wL}{2}\frac{dx}{dx} - \frac{w}{2}2x\frac{dx}{dx}$$

$$\frac{dM}{dx} = \frac{wL}{2} - wx = \text{shear}$$

thus, $$\frac{dM}{dx} = V$$

Thus, the rate of change of the bending moment with respect to x is equal to the shearing force, or **the slope of the moment diagram at the given point is the shear at that point**.

Differentiate V with respect to x gives:

$$\frac{dV}{dx} = 0 - w = \text{load}$$

$$\frac{dV}{dx} = \text{Load}$$

Thus, the rate of change of the shearing force with respect to x is equal to the load or **the slope of the shear diagram at a given point equals the load at that point**.

PROPERTIES OF SHEAR AND MOMENT DIAGRAMS

The following are some important properties of shear and moment diagrams:

1. The area of the shear diagram to the left or to the right of the section is equal to the moment at that section.
2. The slope of the moment diagram at a given point is the shear at that point.
3. The slope of the shear diagram at a given point equals the load at that point.

4. The maximum moment occurs at the point of zero shears. This is in reference to property number 2, that when the shear (also the slope of the moment diagram) is zero, the tangent drawn to the moment diagram is horizontal.

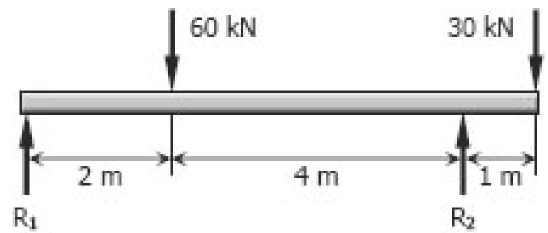
5. When the shear diagram is increasing, the moment diagram is concave upward.

6. When the shear diagram is decreasing, the moment diagram is concave downward.

Examples:

Without writing shear and moment equations, draw the shear and moment diagrams for the beams specified in the following problems.

Example (1):



Solution:

$\Sigma M_A = 0$

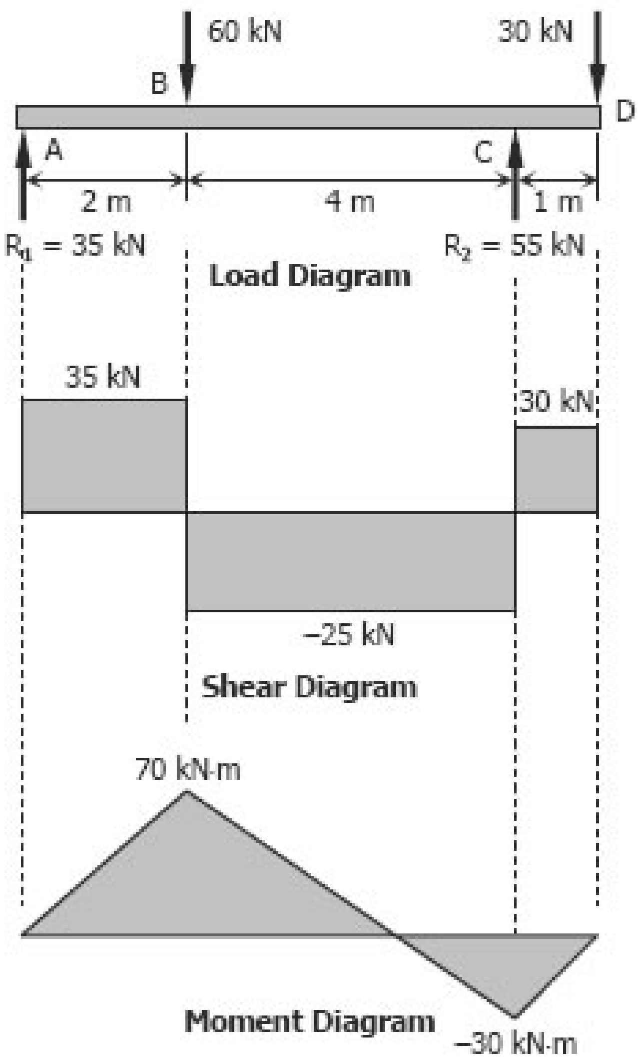
$6R_2 = 2(60) + 7(30)$

$R_2 = 55$ kN

$\Sigma M_C = 0$

$6R_1 + 1(30) = 4(60)$

$R_1 = 35$ kN



To draw the Shear Diagram:

(1) $V_A = R_1 = 35$ kN

(2) $V_B = V_A$ + Area in load diagram − 60 kN
$V_B = 35 + 0 - 60 = -25$ kN

(3) $V_C = V_B$ + area in load diagram + R_2
$V_C = -25 + 0 + 55 = 30$ kN

(4) $V_D = V_C$ + Area in load diagram − 30 kN
$V_D = 30 + 0 - 30 = 0$

To draw the Moment Diagram:

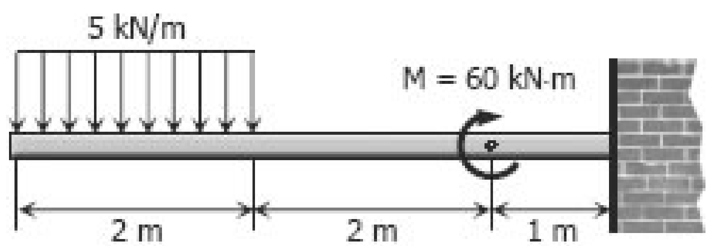
(1) $M_A = 0$

(2) $M_B = M_A$ + Area in shear diagram
$M_B = 0 + 35(2) = 70$ kN·m

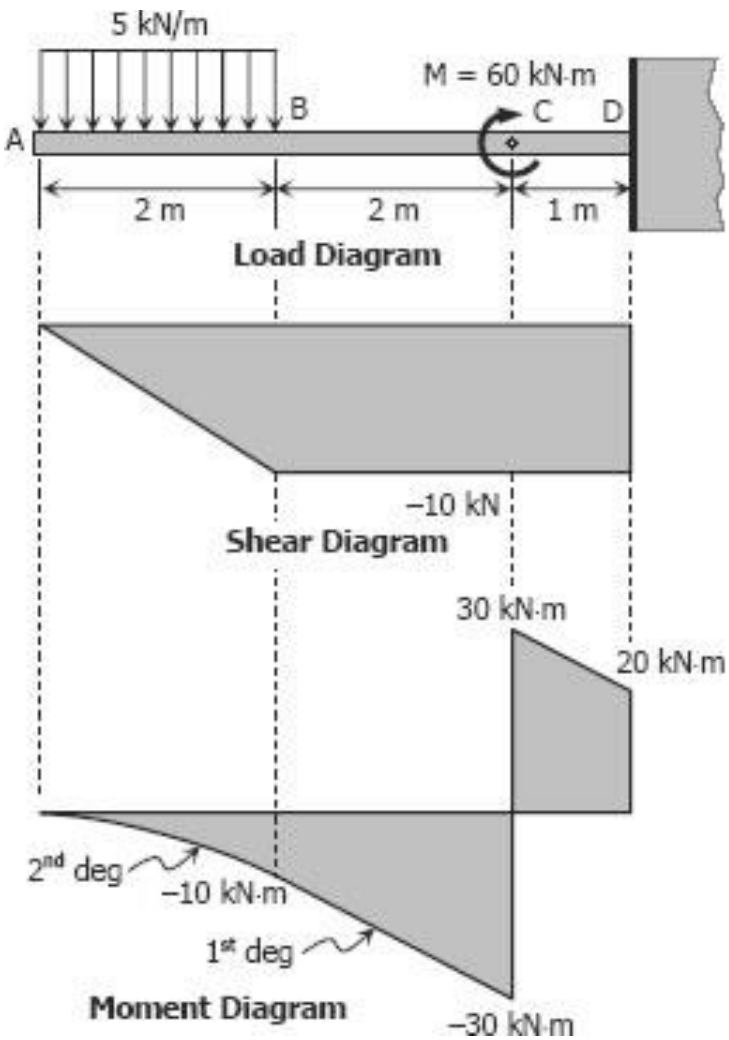
(3) $M_C = M_B$ + Area in shear diagram
$M_C = 70 - 25(4) = -30$ kN·m

(4) $M_D = M_C$ + Area in shear diagram
$M_D = -30 + 30(1) = 0$

Example (2):



Solution:



To draw the Shear Diagram

(1) $V_A = 0$

(2) $V_B = V_A$ + Area in load diagram

$V_B = 0 - 5(2)$

$V_B = -10$ kN

(3) $V_C = V_B$ + Area in load diagram

$V_C = -10 + 0$

$V_C = -10$ kN

(4) $V_D = V_C$ + Area in load diagram

$V_D = -10 + 0$

$V_D = -10$ kN

To draw the Moment Diagram

(1) $M_A = 0$

(2) $M_B = M_A$ + Area in shear diagram

$M_B = 0 - ½ (2)(10)$

$M_B = -10$ kN·m

(3) $M_C = M_B$ + Area in shear diagram

$M_C = -10 - 10(2)$

$M_C = -30$ kN·m

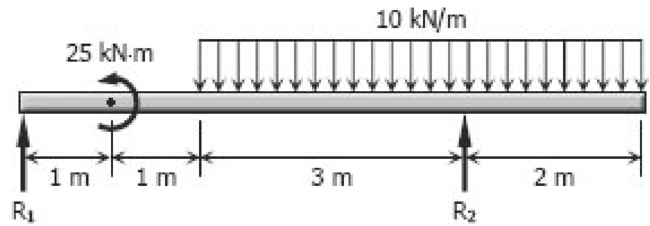
$M_{C2} = -30 + M = -30 + 60 = 30$ kN·m

(4) $M_D = M_{C2}$ + Area in shear diagram

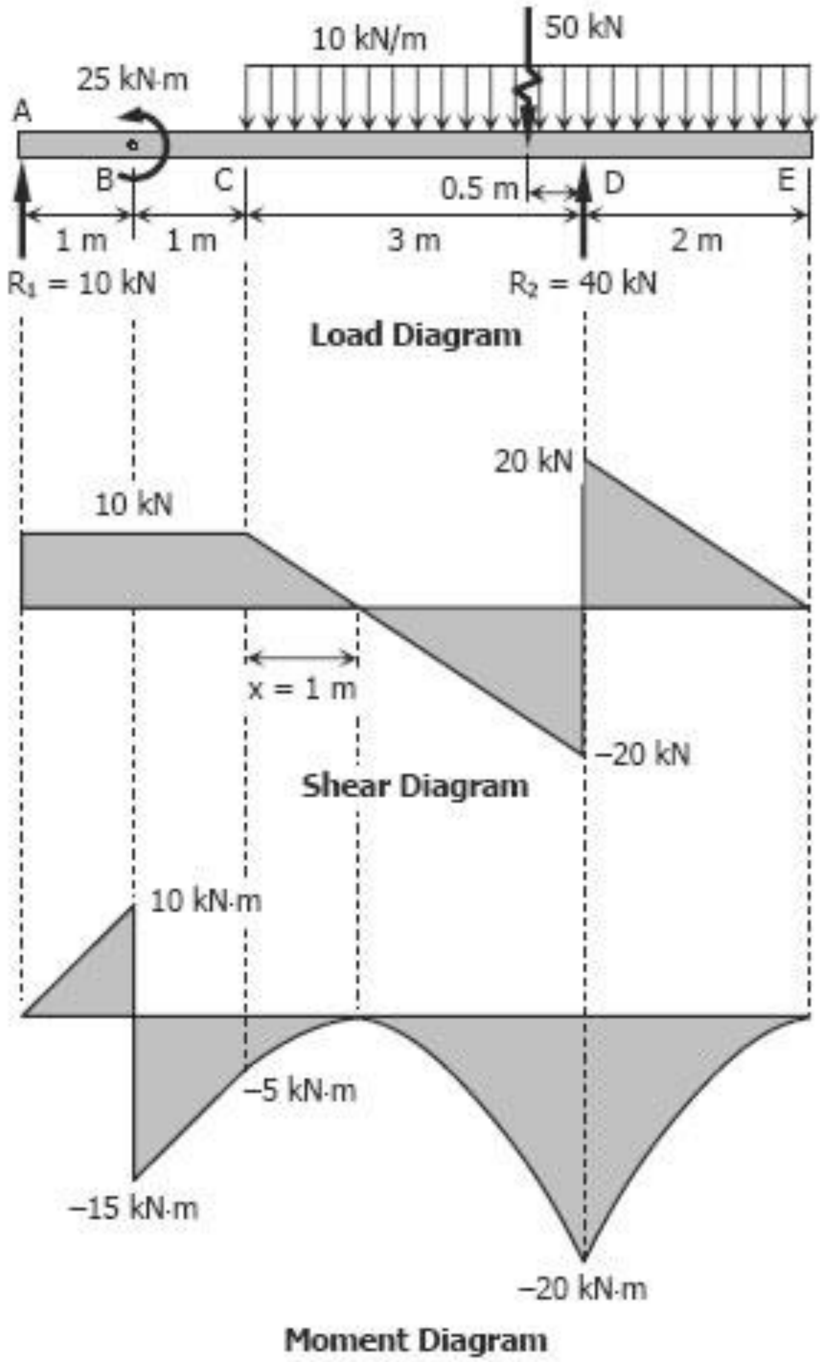
$M_D = 30 - 10(1)$

$M_D = 20$ kN·m

Example (3):



Solution:



$$\Sigma M_D = 0$$
$$5R_1 = 50(0.5) + 25$$
$$R_1 = 10 \text{ kN}$$

$$\Sigma M_A = 0$$
$$5R_2 + 25 = 50(4.5)$$
$$R_2 = 40 \text{ kN}$$

To draw the Shear Diagram

(1) $V_A = R_1 = 10$ kN

(2) $V_B = V_A$ + Area in load diagram
$V_B = 10 + 0 = 10$ kN

(3) $V_C = V_B$ + Area in load diagram
$V_C = 10 + 0 = 10$ kN

(4) $V_D = V_C$ + Area in load diagram
$V_D = 10 - 10(3) = -20$ kN
$V_{D2} = -20 + R_2 = 20$ kN

(5) $V_E = V_{D2}$ + Area in load diagram
$V_E = 20 - 10(2) = 0$

(6) Solving for x:
$x / 10 = (3 - x) / 20$
$20x = 30 - 10x$
$x = 1$ m

To draw the Moment Diagram

(1) $M_A = 0$

(2) $M_B = M_A$ + Area in shear diagram
$M_B = 0 + 1(10) = 10$ kN·m
$M_{B2} = 10 - 25 = -15$ kN·m

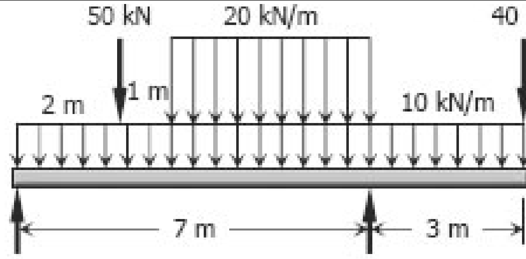
(3) $M_C = M_{B2}$ + Area in shear diagram
$M_C = -15 + 1(10) = -5$ kN·m

(4) $M_x = M_C$ + Area in shear diagram
$M_x = -5 + ½ (1)(10) = 0$

(5) $M_D = M_x$ + Area in shear diagram
$M_D = 0 - ½ (2)(20) = -20$ kN·m

(6) $M_E = M_D$ + Area in shear diagram
$M_E = -20 + ½ (2)(20) = 0$

Example (4):



Solution:

$\Sigma M_D = 0$

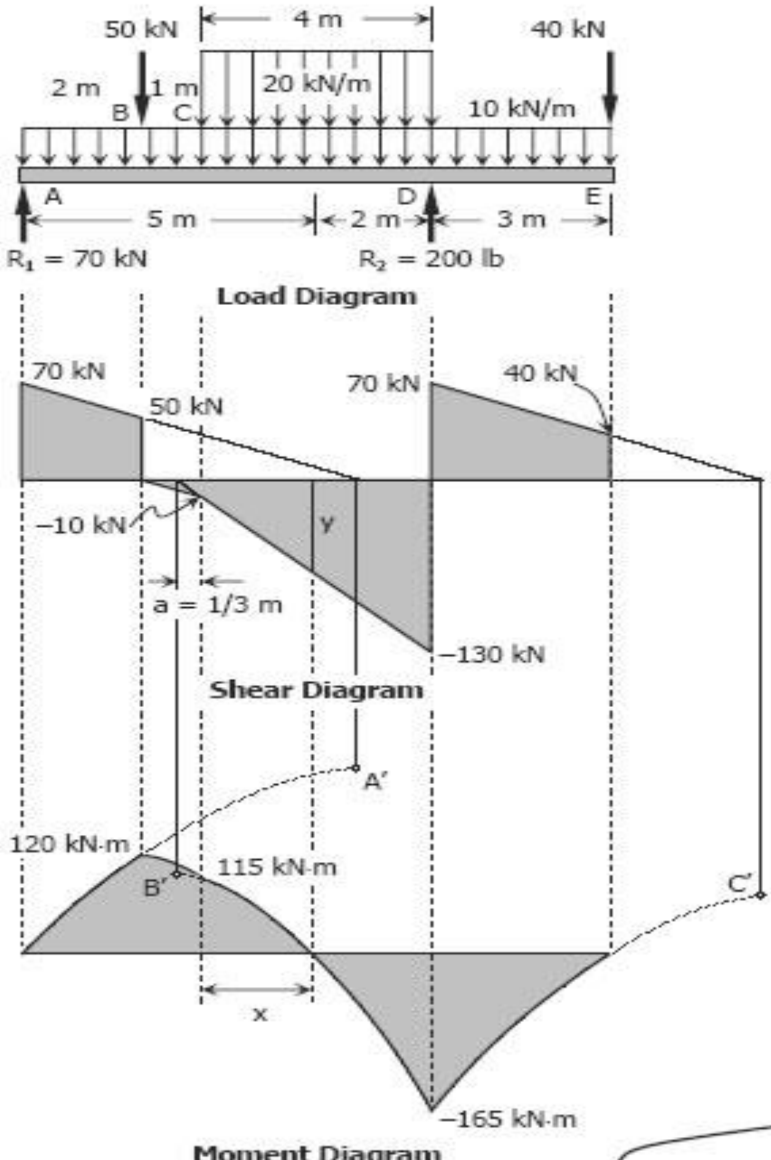
$7R_1 + 40(3) = 5(50) + 10(10)(2) + 20(4)(2)$

$R_1 = 70$ kN

$\Sigma M_A = 0$

$7R_2 = 50(2) + 10(10)(5) + 20(4)(5) + 40(10)$

$R_2 = 200$ lb



To draw the Shear Diagram

(1) $V_A = R_1 = 70$ kN

(2) $V_B = V_A +$ Area in load diagram
$V_B = 70 - 10(2) = 50$ kN
$V_{B2} = 50 - 50 = 0$

(3) $V_C = V_{B2} +$ Area in load diagram
$V_C = 0 - 10(1) = -10$ kN

(4) $V_D = V_C +$ Area in load diagram
$V_D = -10 - 30(4) = -130$ kN
$V_{D2} = -130 + R_2$
$V_{D2} = -130 + 200 = 70$ kN

(5) $V_E = V_{D2} +$ Area in load diagram
$V_E = 70 - 10(3) = 40$ kN
$V_{E2} = 40 - 40 = 0$

To draw the Moment Diagram

(1) $M_A = 0$

(2) $M_B = M_A +$ Area in shear diagram
$M_B = 0 + ½ (70 + 50)(2) = 120$ kN·m

(3) $M_C = M_B +$ Area in shear diagram
$M_C = 120 - ½ (1)(10) = 115$ kN·m

(4) $M_D = M_C +$ Area in shear diagram
$M_D = 115 - ½ (10 + 130)(4)$
$M_D = -165$ kN·m

(5) $M_E = M_D +$ Area in shear diagram
$M_E = -165 + ½ (70 + 40)(3) = 0$

(6) Moment curves AB, CD and DE are downward parabolas with vertices at A′, B′ and C′, respectively. A′, B′ and C′ are corresponding zero shear points of segments AB, CD and DE.

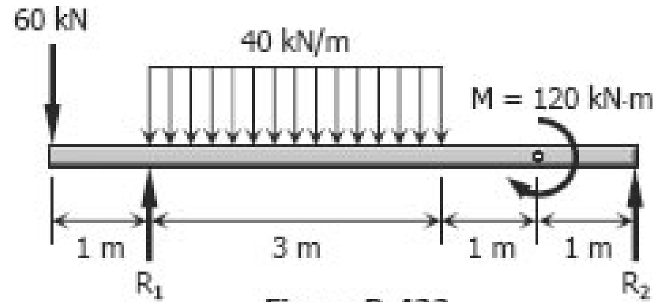
(7) Solving for point of zero moment:
$a / 10 = (a + 4) / 130$
$130a = 10a + 40$
$a = 1/3$ m

$y / (x + a) = 130 / (4 + a)$
$y = 130(x + 1/3) / (4 + 1/3)$
$y = 30x + 10$

$M_C = 115$ kN·m
$M_{zero} = M_C +$ Area in shear
$0 = 115 - ½ (10 + y)x$
$(10 + y)x = 230$
$(10 + 30x + 10)x = 230$
$30x^2 + 20x - 230 = 0$
$3x^2 + 2x - 23 = 0$
$x = 2.46$ m
zero moment is at 2.46 m from C

Another way to solve the location of zero moment is by the squared property of parabola (see Problem 434). This point is the appropriate location for construction joint of concrete structures.

Example (5):



$\Sigma M_E = 0$

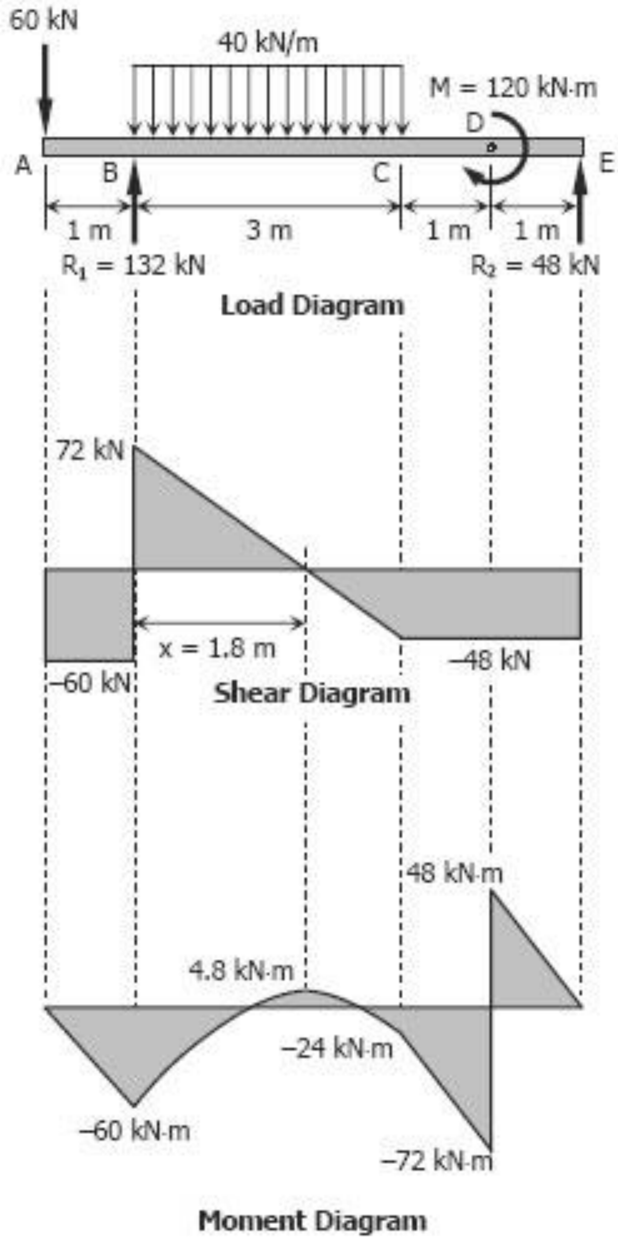
$5R_1 + 120 = 6(60) + 40(3)(3.5)$

$R_1 = 132$ kN

$\Sigma M_B = 0$

$5R_2 + 60(1) = 40(3)(1.5) + 120$

$R_2 = 48$ kN



To draw the Shear Diagram

(1) $V_A = -60$ kN

(2) $V_B = V_A$ + Area in load diagram
$V_B = -60 + 0 = -60$ kN
$V_{B2} = V_B + R_1 = -60 + 132 = 72$ kN

(3) $V_C = V_{B2}$ + Area in load diagram
$V_C = 72 - 3(40) = -48$ kN

(4) $V_D = V_C$ + Area in load diagram
$V_D = -48 + 0 = -48$ kN

(5) $V_E = V_D$ + Area in load diagram
$V_E = -48 + 0 = -48$ kN
$V_{E2} = V_E + R_2 = -48 + 48 = 0$

(6) Solving for x:
$x / 72 = (3 - x) / 48$
$48x = 216 - 72x$
$x = 1.8$ m

To draw the Moment Diagram

(1) $M_A = 0$

(2) $M_B = M_A$ + Area in shear diagram
$M_B = 0 - 60(1) = -60$ kN·m

(3) $M_x = M_B$ + Area in shear diagram
$M_x = -60 + ½\,(1.8)(72) = 4.8$ kN·m

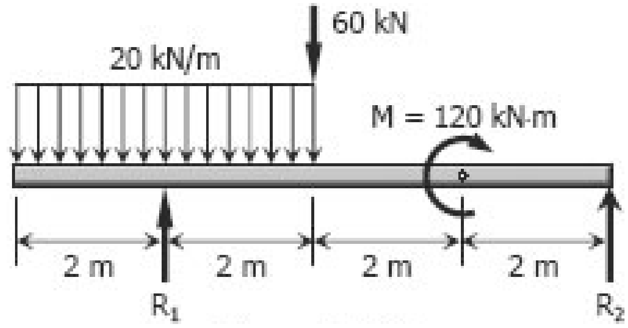
(4) $M_C = M_x$ + Area in shear diagram
$M_C = 4.8 - ½\,(3 - 1.8)(48) = -24$ kN·m

(5) $M_D = M_C$ + Area in shear diagram
$M_D = -24 - ½\,(24 + 72)(1) = -72$ kN·m
$M_{D2} = -72 + 120 = 48$ kN·m

(6) $M_E = M_{D2}$ + Area in shear diagram
$M_E = 48 - 48(1) = 0$

(7) The location of zero moment on segment BC can be determined using the squared property of parabola. See the solution of Problem 434.

Example (6):



$\Sigma M_E = 0$

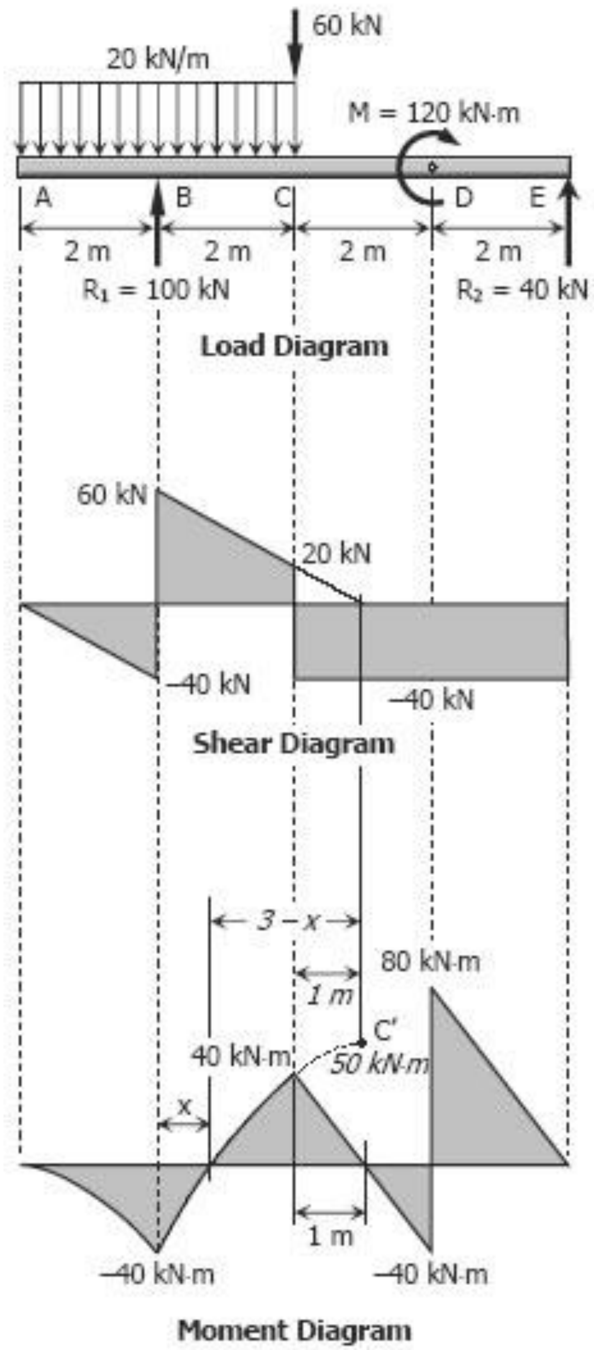
$6R_1 + 120 = 20(4)(6) + 60(4)$

$R_1 = 100$ kN

$\Sigma M_B = 0$

$6R_2 = 20(4)(0) + 60(2) + 120$

$R_2 = 40$ kN



To draw the Shear Diagram

(1) $V_A = 0$

(2) $V_B = V_A$ + Area in load diagram
$V_B = 0 - 20(2) = -40$ kN
$V_{B2} = V_B + R_1 = -40 + 100 = 60$ kN

(3) $V_C = V_{B2}$ + Area in load diagram
$V_C = 60 - 20(2) = 20$ kN
$V_{C2} = V_C - 60 = 20 - 60 = -40$ kN

(4) $V_D = V_{C2}$ + Area in load diagram
$V_D = -40 + 0 = -40$ kN

(5) $V_E = V_D$ + Area in load diagram
$V_E = -40 + 0 = -40$ kN
$V_{E2} = V_E + R_2 = -40 + 40 = 0$

To draw the Moment Diagram

(1) $M_A = 0$

(2) $M_B = M_A$ + Area in shear diagram
$M_B = 0 - ½ (40)(2) = -40$ kN-m

(3) $M_C = M_B$ + Area in shear diagram
$M_C = -40 + ½ (60 + 20)(2) = 40$ kN-m

(4) $M_D = M_C$ + Area in shear diagram
$M_D = 40 - 40(2) = -40$ kN-m
$M_{D2} = M_D + M = -40 + 120 = 80$ kN-m

(5) $M_E = M_{D2}$ + Area in shear diagram
$M_E = 80 - 40(2) = 0$

(6) Moment curve BC is a downward parabola with vertex at C′. C′ is the location of zero shear for segment BC.

(7) Location of zero moment at segment BC:
By squared property of parabola:
$(3 - x)^2 / 50 = 3^2 / (50 + 40)$
$3 - x = 2.236$
$x = 0.764$ m from B

Deflection of Beams

The deformation of a beam is expressed in terms of the deflection of the beam from its original unloaded position. The deflection is measured from the original neutral surface to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

Figure 1 represents the beam in its original undeformed state and Fig.2 represents the beam in the deformed configuration it has assumed under the action of the load.

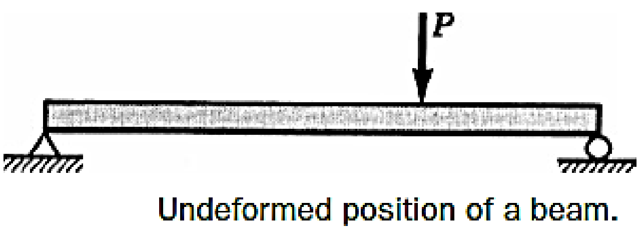


Figure 1

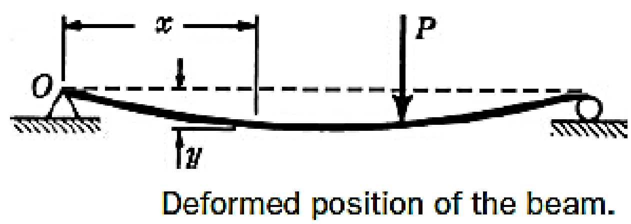


Figure 2

The displacement y is defined as the deflection of the beam. Often it will be necessary to determine the deflection y for every value of x along the beam. This relation is the elastic curve or deflection curve of the beam.

Differential Equation of the Elastic Curve

An expression for the curvature at any point along the curve representing the deformed beam is readily available from differential calculus. It is:

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$$

In this expression, dy/dx represents the slope of the curve at any point; and for small beam deflections this quantity and in particular its square are small in comparison to unity and may reasonably be neglected. This assumption of small deflections simplifies the expression for curvature into

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2}$$

Hence for small deflections

$$EI\frac{d^2y}{dx^2} = M$$

This is the differential equation of the deflection curve of a beam loaded by lateral forces. It is called the Euler-Bernoulli equation of bending of a beam. In any problem it is necessary to integrate this equation to obtain an algebraic relationship between the deflection y and the coordinate x along the length of the beam.

Deflection by Integration

The double-integration method for calculating deflections of beams merely consists of integrating the above Equation.
The first integration yields the slope dy/dx at any point in the beam and the second integration gives the deflection y for any value of x.
The bending moment M must, of course, be expressed as a function of the coordinate x before the equation can be integrated.

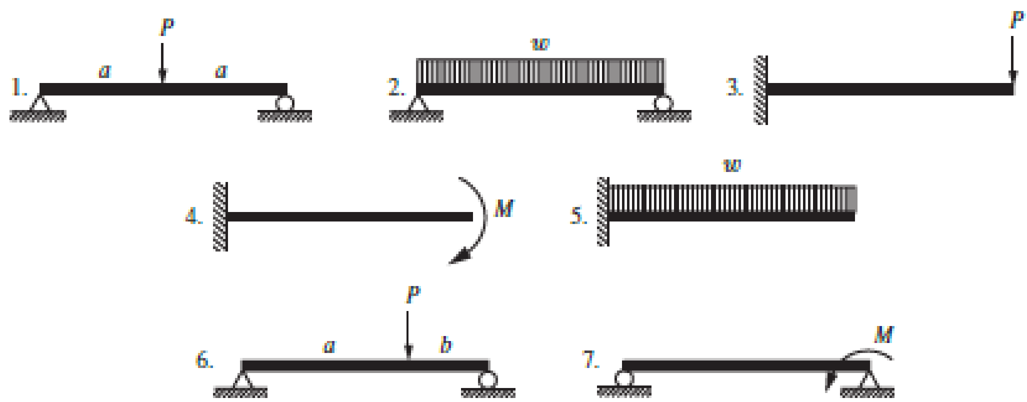
Deflections Using Singularity Functions

The quantity $\langle x - a \rangle$ vanishes if $x < a$ but is equal to $(x - a)$ if $x > a$.
There are several possible approaches for using singularity functions for the determination of beam deflections.
Perhaps the simplest is to employ the approach which the bending moment is written in terms of singularity functions in the form of one equation valid along the entire length of the beam. Two integrations lead to the equation for the deflected beam in terms of two constants of integration which must be determined from boundary conditions. Integration of the singularity functions proceeds directly and in the same manner as simple power functions. Thus, the approach is direct and avoids the problem of the determination of a pair of constants corresponding to each region of the beam (between loads), as in the case of double integration.

Deflections Using Superposition

The equations that describe the deflection of a beam and the stresses due to the applied loads are all linear (i.e., if a load is doubled, the stresses and deflections are also doubled). Thus, we are able to superpose the contributions from each separate load to obtain the resultant effect of several loads. The contributions of each separate load are typically available from previous work for deflections and stresses. If only the maximum deflection is of interest, as is often the case, the results of several simple beams are presented in Table (1) for quick reference.

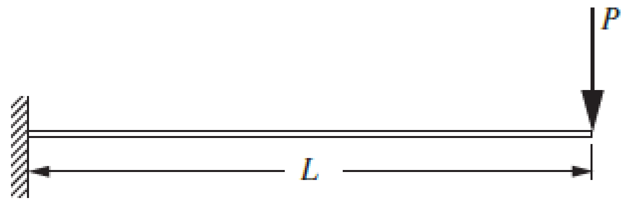
Table 1. Beam Deflection Formulas

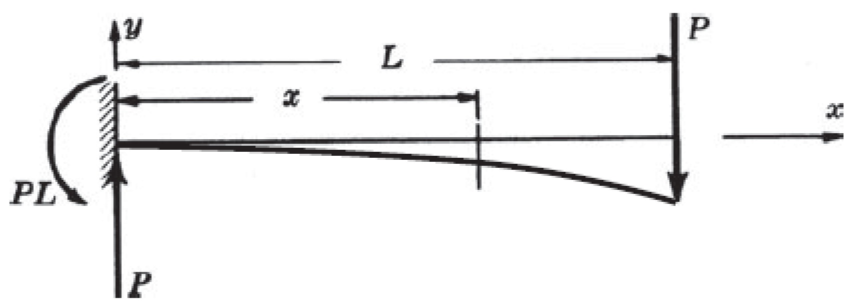


	Max Shear	Max Moment	Max Deflection	Max Beam Slope
1	$P/2$	$PL/4$	$PL^3/48EI$	$PL^2/16EI$
2	$wL/2$	$wL^2/8$	$5wL^4/384EI$	$wL^3/24EI$
3	P	PL	$PL^3/3EI$	$PL^2/2EI$
4	0	M	$ML^2/2EI$	ML/EI
5	wL	$wL^2/2$	$wL^4/8EI$	$wL^3/6EI$
6	$\frac{Pa}{L}$	$\frac{Pab}{L}$	$y_{max} = \frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}LEI}$ at $x = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_{left} = -\frac{Pb(L^2-b^2)}{6LEI}$
			$y_{center} = \frac{Pb(3L^2-4b^2)}{48EI}$	$\theta_{right} = \frac{Pa(L^2-a^2)}{6LEI}$
7	$\frac{M}{L}$	M	$y_{max} = \frac{\sqrt{3}ML^2}{27EI}$ at $x = \frac{L}{\sqrt{3}}$	$\theta_{left} = -\frac{ML}{6EI}$
			$y_{center} = \frac{ML^2}{16EI}$	$\theta_{right} = \frac{ML}{3EI}$

EXAMPLE (1):

The cantilever beam shown is 3 m long and loaded by an end force of 20 kN. The cross section has I =60.7×10^{-6} m^4. Find the maximum deflection and slope of the beam. Take E = 200 GPa. Neglect the weight of the beam.





$$M = -PL + Px$$

The differential equation of the bent beam is then

$$EI\frac{d^2y}{dx^2} = -PL + Px$$

This equation is readily integrated once to yield

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2} + C_1$$

which represents the equation of the slope, where C1 denotes a constant of integration. This constant may be evaluated by use of the condition that the slope dy/dx of the beam at the wall is zero since the beam is rigidly clamped there. Equation above is true for all values of x and y, and if the condition x = 0 is substituted we obtain 0 = 0 + 0 + C1 or C1 = 0.

Next, integration of the equation yields:

$$EIy = -PL\frac{x^2}{2} + \frac{Px^3}{6} + C_2$$

where C2 is a second constant of integration. Again, the condition at the supporting wall will determine this constant. At x = 0, the deflection y is zero since the bar is rigidly clamped. Then

0 = 0 + 0 + C2 or C2 = 0.

The deflection is maximum at the right end of the beam (x = L), under the load P,

$$EIy_{\max} = \frac{-PL^3}{3}$$

or

$$\Delta_{\max} = \frac{PL^3}{3EI}$$

$$\Delta_{\max} = \frac{PL^3}{3EI} = \frac{(20000)(3)^3}{3(200\times10^9)(60.7\times10^{-6})} = 0.0148\,\text{m} \qquad \text{or} \qquad 14.8\,\text{mm}$$

$$EI\frac{dy}{dx} = -PLx + \frac{Px^2}{2}$$

At the free end, $x = L$, and

$$EI\left(\frac{dy}{dx}\right)_{x=L} = -PL^2 + \frac{PL^2}{2}$$

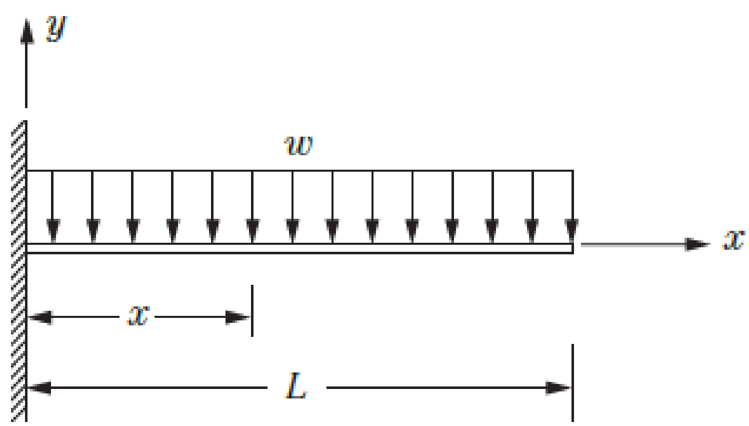
The slope at the end is thus

$$\left(\frac{dy}{dx}\right)_{x=L} = \frac{-PL^2}{2EI} = \frac{-20\,000\times 3^2}{2(200\times 10^9)(60.7\times 10^{-6})} = -0.00741 \qquad \text{or} \qquad -0.425°$$

Note: $dy/dx = \tan\theta \cong \theta$ since the slope of beams is very small.

Example (2):

Determine the deflection curve of a cantilever beam subject to the uniformly distributed load w, shown



Solution:

The force due to the distributed load w is $w(L-x)$. This force acts at the midpoint of this length of beam to the right of x and thus its moment arm from x is $\frac{1}{2}(L-x)$. The bending moment at the section x is thus given by

$$M = -\frac{w}{2}(L-x)^2$$

the negative sign being necessary since downward loads produce negative bending.
The differential equation describing the bent beam is thus

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2 \tag{1}$$

The first integration yields

$$EI\frac{dy}{dx} = \frac{w}{2}\frac{(L-x)^3}{3} + C_1 \tag{2}$$

The constant may be evaluated by using $(dy/dx)_{x=0} = 0$. We find $C_1 = -wL^3/6$. We thus have

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \tag{3}$$

The next integration yields

$$EIy = -\frac{w}{6}\frac{(L-x)^4}{4} - \frac{wL^3}{6}x + C_2 \tag{4}$$

At the clamped end, the deflection is zero so that

$$0 = \frac{-wL^4}{24} + C_2 \qquad \text{or} \qquad C_2 = \frac{wL^4}{24}$$

The final form of the deflection curve of the beam is thus

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \tag{5}$$

The deflection is maximum at the right end of the bar ($x = L$) and there we have from Eq. (5)

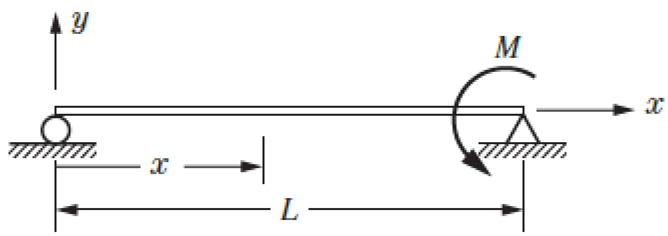
$$EIy_{\max} = -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{wL^4}{8}$$

where the negative value denotes that this point on the deflection curve lies below the x-axis. The magnitude of the maximum deflection is

$$\Delta_{\max} = \frac{wL^4}{8EI} \tag{6}$$

Example (3):

A simply supported beam is loaded by a couple M as shown. The beam is 2 m long and of square cross section 50 mm on a side. If the maximum permissible deflection in the beam is 5 mm, and the allowable bending stress is 150 MPa, find the maximum allowable moment M. Use E = 200 GPa.



Solution:

$$R_L = R_R = \frac{M}{L} \qquad \text{and thus} \qquad M_x = \frac{M}{L}x$$

The differential equation describing the bent beam is thus

$$EI\frac{d^2y}{dx^2} = \frac{M}{L}x \tag{1}$$

Integrating twice

We may now determine the two constants of integration through use of the fact that the beam deflection is zero at each end. When $x = 0$, $y = 0$, so from Eq. (3) we have

$$0 = 0 + 0 + C_2 \qquad C_2 = 0$$

Next, when $x = L$, $y = 0$, so we have from Eq. (3)

$$0 = \frac{ML^2}{6} + C_1L \qquad \therefore C_1 = -\frac{ML}{6}$$

The desired equation of the deflection curve and the equation for the slope is

$$EIy = \frac{Mx^3}{6L} - \frac{MLx}{6} \tag{4}$$

and

$$EI\frac{dy}{dx} = \frac{Mx^2}{2L} - \frac{ML}{6} \tag{5}$$

The point of peak deflection occurs when the slope given by Eq. (5) is zero. This provides

$$x = \frac{L}{\sqrt{3}} \tag{6}$$

Returning to Eq. (4), the maximum deflection is

$$y_{\max} = \frac{M}{6LEI}\left(\frac{L}{\sqrt{3}}\right)^3 - \frac{ML}{6EI}\left(\frac{L}{\sqrt{3}}\right) = -\frac{ML^2\sqrt{3}}{27EI} \tag{7}$$

Let us now consider that the maximum deflection in the beam is 5 mm. According to Eq. (7), we have

$$0.005 = \frac{M(2)^2\sqrt{3}}{27(200\times10^9)[(0.05)(0.05)^3/12]} \qquad \text{or} \qquad M = 2030\ \text{N}\cdot\text{m} \tag{8}$$

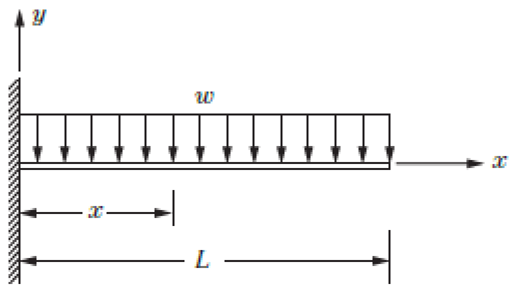
We shall now assume that the allowable bending stress of 150 MPa is set up in the outer fibers of the beam at the section of maximum bending moment. The maximum bending moment occurs at the right end with magnitude *M*. There

$$\sigma_{\max} = \frac{Mc}{I} \qquad 150\times10^6 = \frac{M(0.025)}{(0.05)(0.05)^3/12} \qquad \text{or} \qquad M = 3125\ \text{N}\cdot\text{m} \tag{9}$$

Thus the maximum allowable moment is given by Eq. (8) and is 2030 N · m.

Example (4):

Determine the deflection curve of a cantilever beam subject to the uniformly distributed load w, shown



$$M = -\frac{w}{2}(L-x)^2$$

the negative sign being necessary since downward loads produce negative bending.

The differential equation describing the bent beam is thus

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2 \qquad (1)$$

The first integration yields

$$EI\frac{dy}{dx} = \frac{w}{2}\frac{(L-x)^3}{3} + C_1 \qquad (2)$$

The constant may be evaluated by using $(dy/dx)_{x=0} = 0$. We find $C_1 = -wL^3/6$. We thus have

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \qquad (3)$$

The next integration yields

$$EIy = -\frac{w}{6}\frac{(L-x)^4}{4} - \frac{wL^3}{6}x + C_2 \qquad (4)$$

At the clamped end, the deflection is zero so that

$$0 = \frac{-wL^4}{24} + C_2 \qquad \text{or} \qquad C_2 = \frac{wL^4}{24}$$

The final form of the deflection curve of the beam is thus

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \tag{5}$$

The deflection is maximum at the right end of the bar ($x = L$) and there we have from Eq. (5)

$$EIy_{max} = -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{wL^4}{8}$$

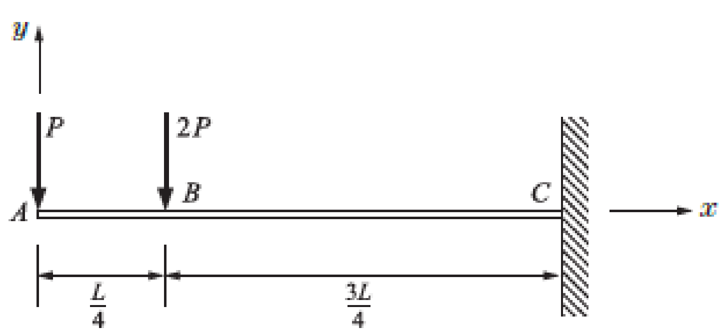
where the negative value denotes that this point on the deflection curve lies below the x-axis. The magnitude of the maximum deflection is

$$\Delta_{max} = \frac{wL^4}{8EI} \tag{6}$$

EXAMPLE (5):

Using singularity functions, determine the deflection curve of the cantilever beam subject to the loads shown

$$M = -P\langle x \rangle^1 - 2P\left\langle x - \frac{L}{4}\right\rangle^1 \tag{1}$$



$$EI\frac{d^2y}{dx^2} = -P\langle x\rangle^1 - 2P\left\langle x - \frac{L}{4}\right\rangle^1 \quad (2)$$

The first integration yields

$$EI\frac{dy}{dx} = -P\frac{\langle x\rangle^2}{2} - 2P\frac{\left\langle x - \frac{L}{4}\right\rangle^2}{2} + C_1 \quad (3)$$

where C_1 is a constant of integration. The next integration leads to

$$EIy = -\frac{P}{2}\frac{\langle x\rangle^3}{3} - 2P\frac{\left\langle x - \frac{L}{4}\right\rangle^3}{2(3)} + C_1\langle x\rangle + C_2 \quad (4)$$

where C_2 is a second constant of integration. These two constants may be determined from the two boundary conditions.

When $x = L$, $dy/dx = 0$, so from Eq. (3):

$$0 = -\frac{PL^2}{2} - P\left(\frac{3L}{4}\right)^2 + C_1 \quad (5)$$

When $x = L$, $y = 0$, so from Eq. (4):

$$0 = -\frac{PL^3}{6} - \frac{P}{3}\left(\frac{3L}{4}\right)^3 + C_1L + C_2 \quad (6)$$

Solving Eqs. (5) and (6),

$$C_1 = \frac{17}{16}PL^2 \qquad C_2 = -\frac{145}{192}PL^3 \quad (7)$$

The desired deflection curve is thus

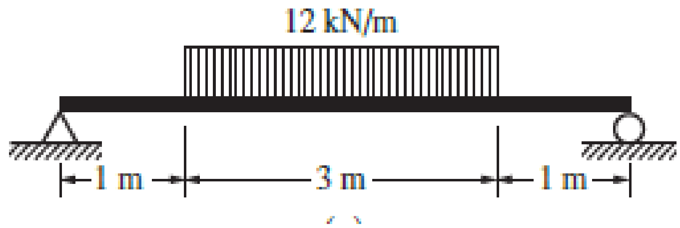
$$EIy = -\frac{P}{6}\langle x\rangle^3 - \frac{P}{3}\left\langle x - \frac{L}{4}\right\rangle^3 + \frac{17}{16}PL^2\langle x\rangle - \frac{145}{192}PL^3 \quad (8)$$

For example, the deflection at point B where $x = L/4$ is found from Eq. (8) to be

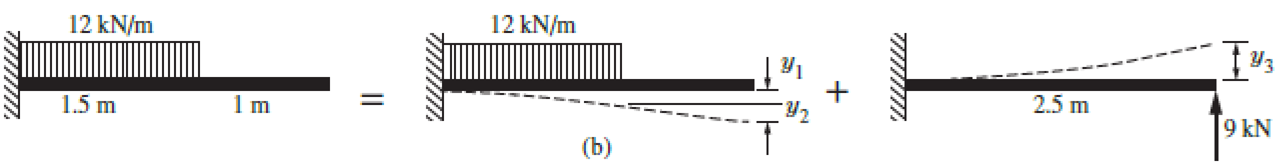
$$EIy]_{x=L/4} = -\frac{P}{6}\left(\frac{L}{4}\right)^3 - 0 + \frac{17}{16}PL^2\left(\frac{L}{4}\right) - \frac{145}{192}PL^3 \qquad \therefore\ y]_{x=L/4} = -\frac{94.5PL^3}{192EI}$$

Example (6):

Using superposition and the formulas of Table 1, calculate the maximum deflection of the beam shown. Use E = 200 GPa and I = 3500 cm4.



Solution:



observe that the beam is loaded symmetrically about its center where the slope is zero. So, the center of the beam can be considered to be the end of a cantilever beam. The right half of the beam is then composed of the two beams shown. Using superposition and Table 1, we have

$$\Delta_{max} = y_3 - y_1 - y_2$$

$$= \frac{PL_1^3}{3EI} - \frac{wL_2^4}{8EI} - \frac{wL_2^3}{6EI} \times L_3$$

$$= \frac{1}{(200 \times 10^9)(3500 \times 10^{-8})}\left(\frac{9000 \times 2.5^3}{3} - \frac{12\,000 \times 1.5^4}{8} - \frac{12\,000 \times 1.5^3}{6} \times 1\right)$$

$$= 0.00465 \text{ m} \quad \text{or} \quad 4.65 \text{ mm}$$

This displacement is upward but we know the right end does not deflect. Hence, this is the downward displacement of the center of the beam.

Buckling of Columns

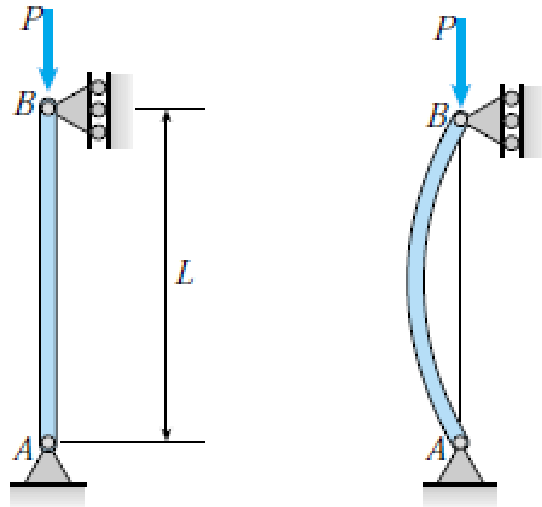
Introduction:

Columns : are long, slender structural members loaded axially in compression.

Buckling : Is a type of failure that occurs in Load –carrying structures.

Note : In this chapter, the buckling of columns will consider specifically.

Buckling of Columns



(a) (b)

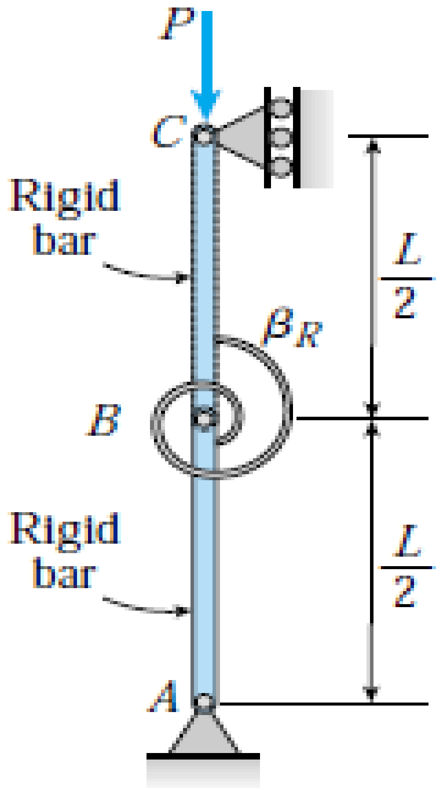
- When a compression member is relatively slender, it may deflect laterally and fail by bending rather than failing by direct compression of the material.

Buckling of Columns

Buckling and Stability

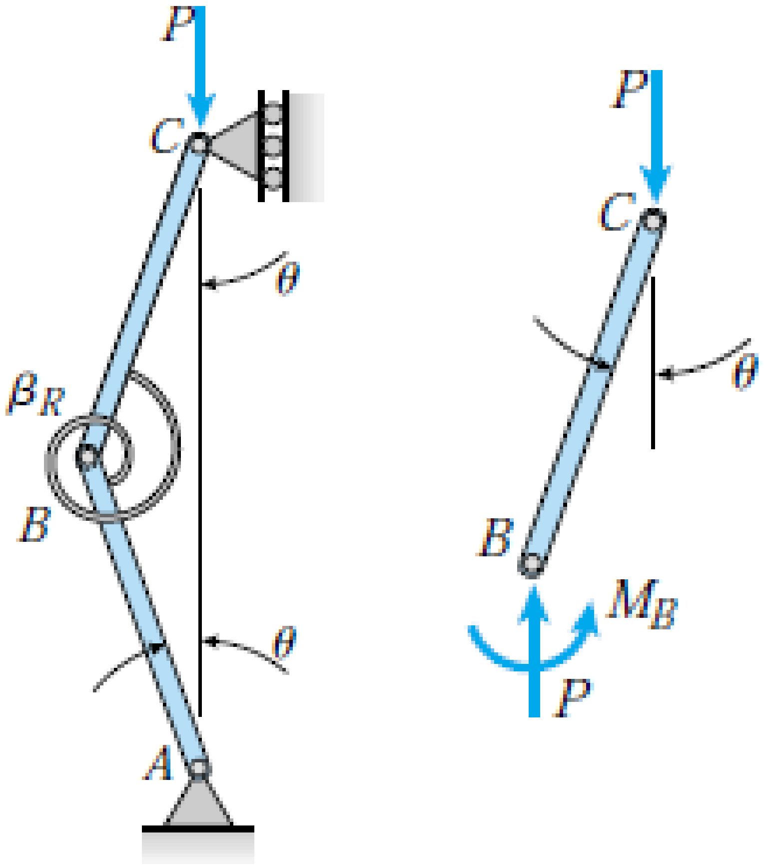
Consider the idealized structure, or buckling model, shown in Fig. below

Note : The two rigid bars **AB** and **BC** are joined at ***B*** *by a pin connection and held in a* vertical position by a rotational spring having stiffness β_R.



Buckling of Columns

- **Under the compressive load P, the structure is disturbed.**
- **The disturbance of structure is represented by the movement of pin B by a small distance laterally.**



- ***The movement of B cause* :**

  1. the rigid bars to rotate through small angles θ;

  2. and a moment to develop in the spring (Restoring Moment).

Buckling of Columns

Notes :

1. The direction of restoring moment is such that it tends to return the structure to its original straight position.
2. Thus, the axial compressive load and restoring moment have opposite effects—the restoring moment tends to decrease the displacement and the axial force tends to increase it.
3. **If** the axial force ***P*** *is relatively small, the action of the restoring moment* will predominate over the action of the axial force and the structure will return to its initial straight position. So, the structure is said to be **stable**.

Buckling of Columns

4. **If** the axial force **P** is large, the lateral displacement of point B will increase and the bars will rotate through larger and larger angles until the structure collapses. Here, the structure is **unstable and fails by lateral buckling.**

Critical Load

Critical load is a special value of the axial load at which the transition between the stable and unstable condition will occur.

Critical load is denoted by P_{cr} .

Buckling of Columns

Consider the disturbed buckling model:

$M_B = \beta_R \,.\, 2\theta = 2\,\theta\,\beta_R$

The lateral displacement of B = $(L/2)\tan\theta$

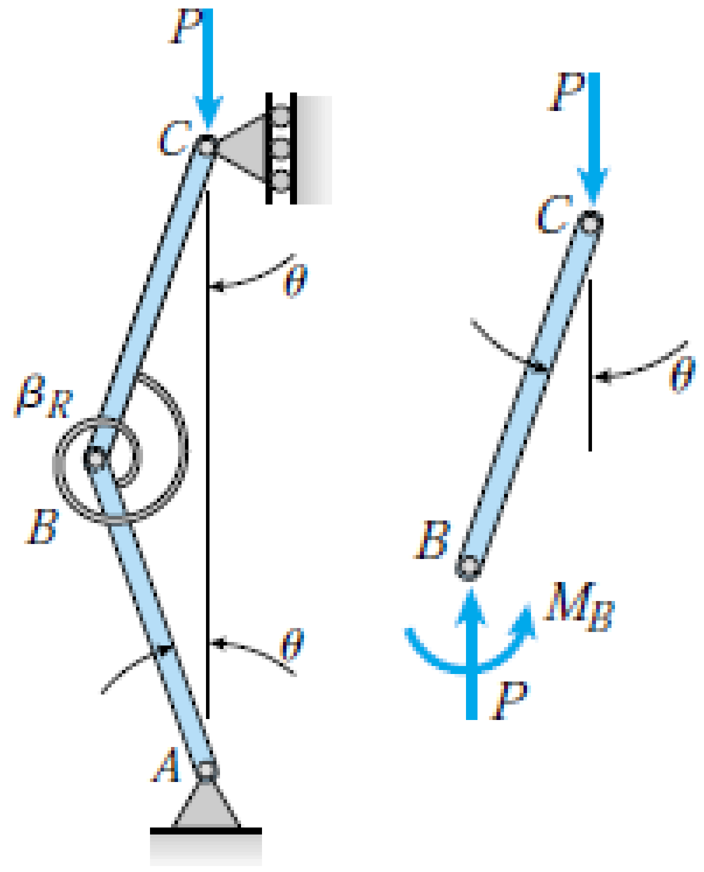
Since θ is very small; then

The lateral displacement of B = $\theta\, L/2$

$\sum M_B = 0 \qquad M_B - P\,(\theta\, L/2) = 0$

Or; $\quad 2\,\theta\,\beta_R - P\,(\theta\, L/2) = 0$

$[\,2\,\beta_R - (PL/2)\,]\,\theta = 0$



Buckling of Columns

$$[\, 2\,\beta_R - (PL/2)\,]\,\theta = 0$$

Either $\theta = 0$ **the structure is in equilibrium**

Or $2\,\beta_R - (PL/2) = 0 \;\Rightarrow\; 2\,\beta_R - (P_{cr}\,L/2) = 0$

$$P_{cr} = 4\,\beta_R / L$$

Notes:

1. At the critical value of the load the structure is in equilibrium regardless of the magnitude of the angle θ (at any value of disturbance, θ remains small).

Buckling of Columns

2. The critical load is the only load for which the structure will be in equilibrium in the disturbed position.

3. The critical load represents the boundary between the stable and unstable conditions.

4. If the axial load is less than P_{cr} , the effect of the moment in the spring predominates and the structure returns to the vertical position after a slight disturbance; if the axial load is larger than P_{cr} , the effect of the axial force predominates and the structure buckles:

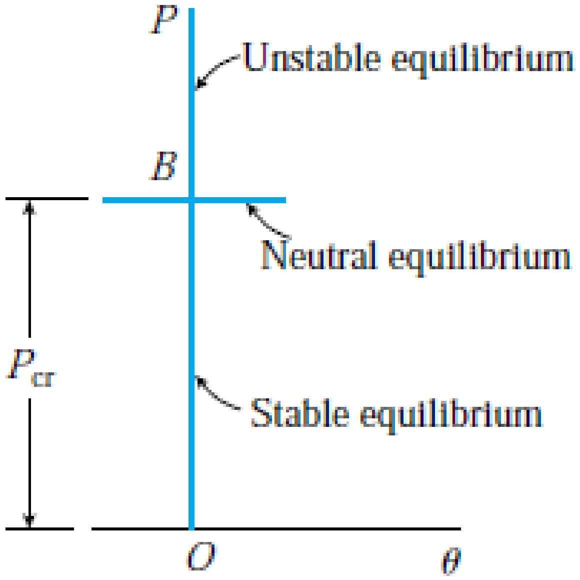
If $P < P_{cr}$ the structure is stable;

If $P > P_{cr}$ the structure is unstable.

Buckling of Columns

Stable, Neutral and Unstable Equilibrium

Consider the diagram shown below:



1. $0 < P < P_{cr}$

The equilibrium of structure is **stable**. The structure is in equilibrium only when it is Perfectly straight (**$\theta = 0$**).

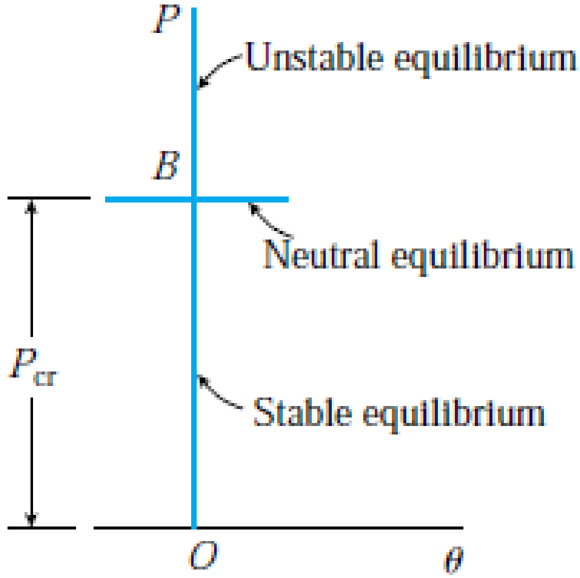
2. $P > P_{cr}$

The equilibrium of structure is unstable and cannot be maintained. The slightest disturbance will cause the structure to buckle.

Buckling of Columns

3. $P = P_{cr}$

The structure is neither stable nor unstable — it is at the boundary between stability and instability. This condition is referred to as **neutral equilibrium**.



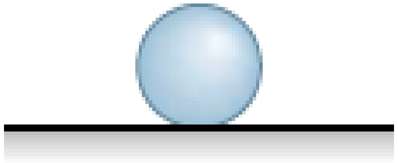
Buckling of Columns

To explain the above three equilibrium conditions; Consider the following :

1. If the surface is concave upward, like the inside of a dish, the equilibrium is stable and the ball always returns to the low point when disturbed.

2. If the surface is convex upward, like a dome, the ball can theoretically be in equilibrium on top of the surface, but the equilibrium is unstable and in reality the ball rolls away.

3. If the surface is perfectly flat, the ball is in neutral equilibrium and remains wherever it is placed.

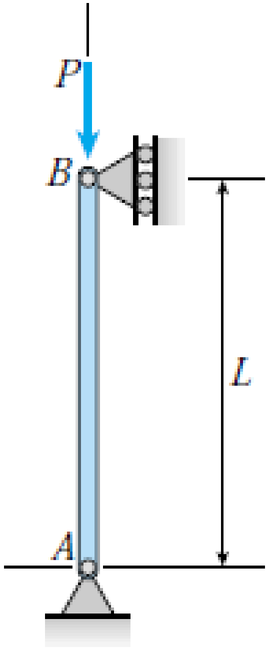


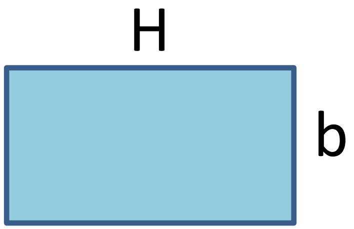
Buckling of Columns

Definitions and Assumptions

1. A compression member is generally considered to be a column when its unsupported length is more than 10 times its least dimension.

$L > 10\ b$





Buckling of Columns

2. **Columns are usually divided into** :

   a. **long column** : fail by buckling or excessive lateral bending;

   b. **intermediate column** : fail by a combination of crushing and buckling;

   c. **short column** (**short compression blocks)** : fail by crushing.

3. An **ideal column** is assumed to be a homogeneous member of constant cross section that is initially straight and is subjected to axial compressive loads.

Buckling of Columns

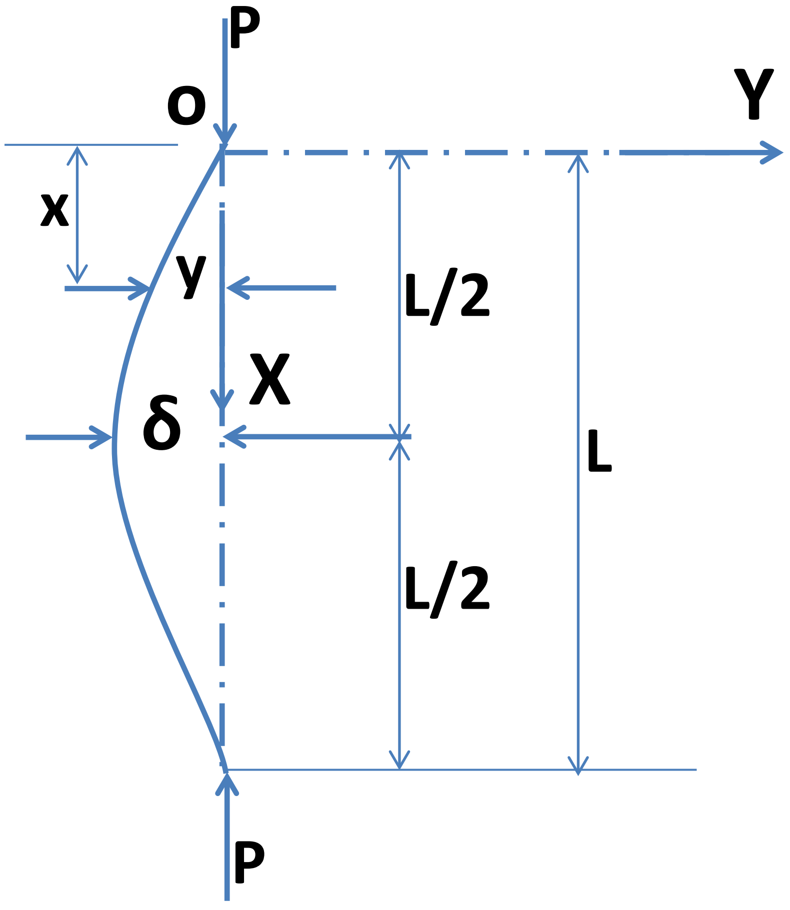
Long Columns by Euler's Formula

Figure below shows the center line of a column in equilibrium under the action of its critical load P.

The column is assumed to have hinged ends restrained against lateral movement.

The maximum deflection **δ** is so small that there is no appreciable difference between the original length of the column and its projection on a vertical plane

(**i.e. the slope dy/dx is so small**).



Buckling of Columns

The differential equation of the elastic curve of a beam:

$$EI\ (d^2y / dx^2) = M = P\ (-y) = -P\ y \qquad \text{.........} \quad (1)$$

Note **: 1. The above – ve sign is for deflection (y) not for moment.**

2. If the deflection is in the +ve direction, the result is still –ve because the moment become –ve.

3. Eq. (1) cannot be integrated directly because M is not a function of x.

Buckling of Columns

Rewrite Eq. (1) in the form :

$EI\ (d / dx)\ (dy / dx) = -\ P\ y$ ...... (2)

Multiply Eq.(2) by **2dy** yields:

$2\ EI\ (d)\ (dy / dx)^2 = -\ 2\ P\ y\ dy$ ...... (3)

Integrate Eq. (3):

$2\ EI\ (dy / dx)^2 = -\ P\ y^2 + C_1$ or;

$EI\ (dy / dx)^2 = -\ P\ y^2 + C_1$ ... (4) [**the constants becomes within C_1**]

Buckling of Columns

$y = \delta$ when $dy/dx = 0$ substituting in Eq. (4):

$0 = -P\delta^2 + C_1$ $C_1 = P\delta^2$

So;

$EI\,(dy/dx)^2 = -P y^2 + P\delta^2$

$EI\,(dy/dx)^2 = P(\delta^2 - y^2)$ …… (5)

Or;

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\,\sqrt{\delta^2 - y^2}$$

Buckling of Columns

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\ \sqrt{\delta^2 - y^2}$$

And;

$$\frac{dy}{\sqrt{\delta^2 - y^2}} = \sqrt{\frac{P}{EI}}\ dx \qquad \text{...... (6)}$$

Integrating of Eq. (6) yields:

$$sin^{-1}\frac{y}{\delta} = x\sqrt{\frac{P}{EI}} + C_2$$

Buckling of Columns

At $x = 0$ → $y = 0$ → $C_2 = 0$

So;

$$sin^{-1}\frac{y}{\delta} = x\sqrt{\frac{P}{EI}} \quad \text{or} \quad y = \delta \sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \ldots\ldots(7)$$

Eq. (7) indicates that the shape of a sine curve.

Substituting $y = 0$ at $x = L$ into Eq. (7) yields:

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Buckling of Columns

Or;

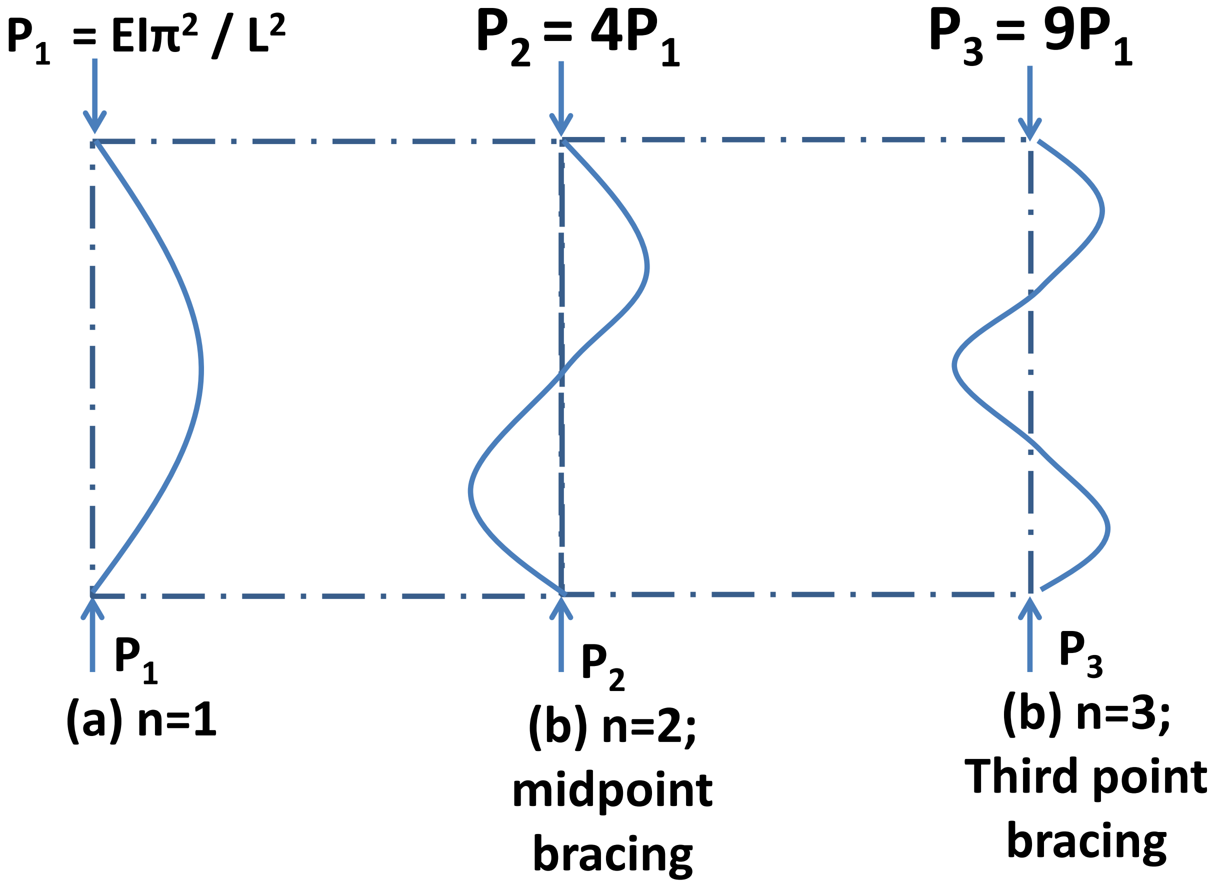
$$\left(L\sqrt{\frac{P}{EI}}\right) = n\pi \qquad (n = 0,1,2,3 \ldots \ldots .)$$

From which;

$$P = n^2 \frac{EI\pi^2}{L^2} \qquad \text{......(8)}$$

Buckling of Columns

$$P = n^2 \frac{EI\pi^2}{L^2} \qquad \text{......(8)}$$



Buckling of Columns

Notes: **1.** The value **n=0** is meaningless because then **P=0**.

2. For any value of **n** (except **n =0**) the column bends into the shapes shown above.

3. The most important case is in **Fig. (a);** while the others occur with larger **P** and are possible only if the column is braced at the middle or third points respectively.

4. Bracing reduces the effective length from **L** at case **a** to (**L/2**) and **(** **L/3**) at cases **b** and **c** respectively.

Buckling of Columns

The critical load for a hinged ended column is therefore:

$$P = \frac{EI\pi^2}{L^2}$$

Critical Load for different end conditions:

The critical load for columns with different end conditions can be expressed in terms of the critical load for a hinged column, which is taken as the fundamental case.

1. ***Column with Fixed ends*** :

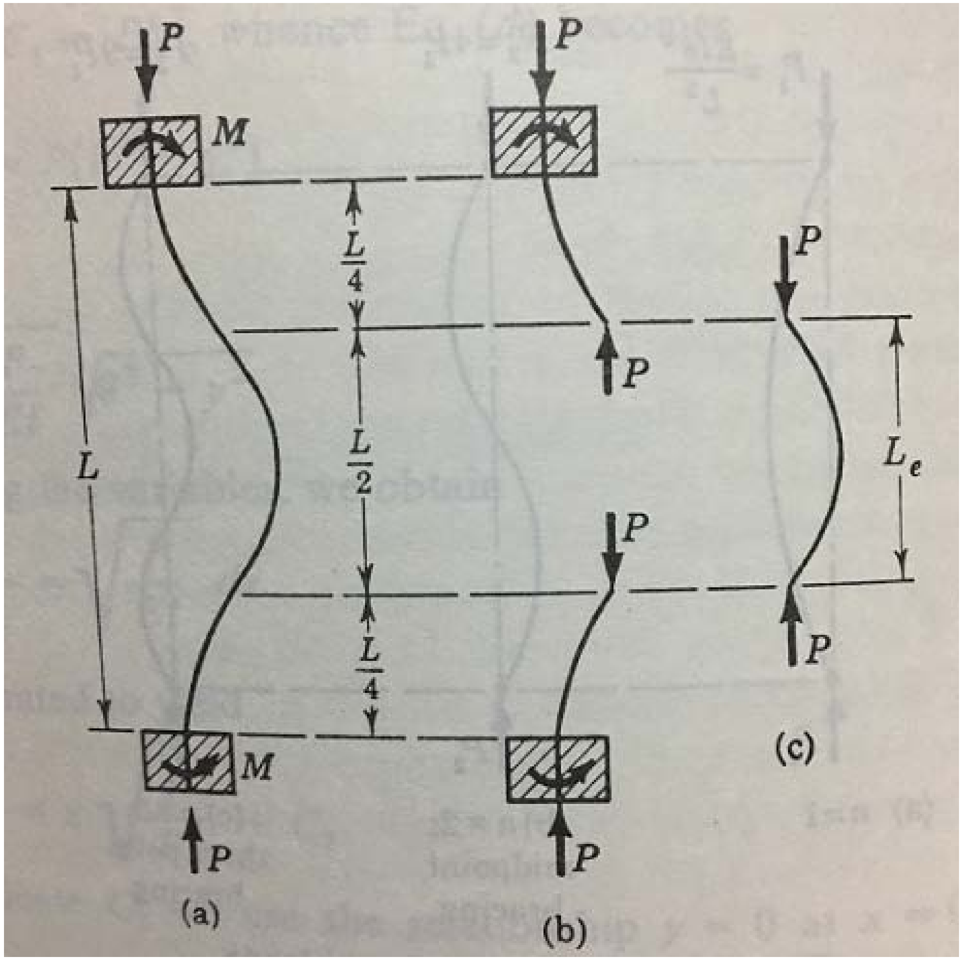
Due to symmetry, the inflection points at the quarter points of its unsupported length. This case is equivalent to a hinged column having ($L_e = L/2$).

Buckling of Columns

So;

$$P = \frac{EI\pi^2}{L_e^{\ 2}} = \frac{EI\pi^2}{\left(\frac{L}{2}\right)^2} = 4\,\frac{EI\pi^2}{L^2}$$

Therefore; the buckling Strength of the fixed ends column is 4 times that of the hinged ends column.



Buckling of Columns

Fig. b above shown the case of a column built in at one end and free at the other (flagpole column).

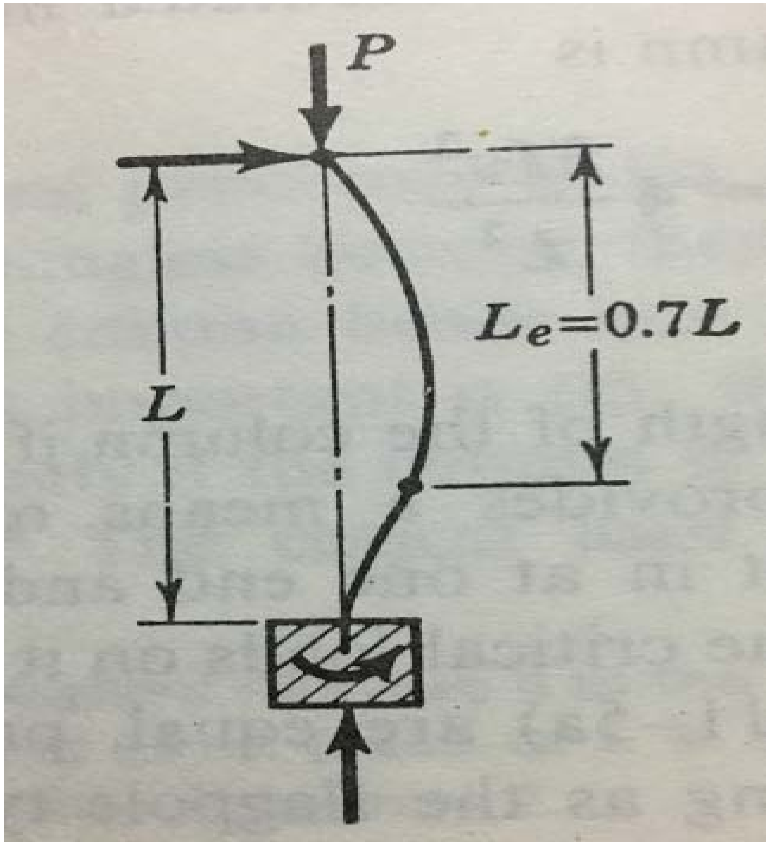
$$P = \frac{4EI\pi^2}{L_e{}^2} = \frac{4EI\pi^2}{(4L)^2} = \frac{1}{4}\,\frac{EI\pi^2}{L^2}$$

Therefore; the buckling strength of the flagpole column is one quarter that of the hinged ends column.

Buckling of Columns

2. Column hinged at one end and fixed at the other

$$P = \frac{EI\,\pi^2}{L_e^2} = \frac{EI\,\pi^2}{(0.7\,L^2)} \approx 2\,\frac{EI\,\pi^2}{L^2}$$



Therefore; all the above conditions are expressed in terms of the critical load of the hinged ends column multiplied by a factor N.

Buckling of Columns

Limitations of Euler's Formula

In order for Euler's Formula to be applicable, the stress accompanying the bending which occurs during buckling must not exceed the proportional limit : So; for hinged ends column :

$$P = \frac{EI\pi^2}{L^2} \qquad \text{since :} \qquad r = \sqrt{\frac{I}{A}} \;\rightarrow\; I = A\,r^2$$

So;

$$P = \frac{E\,A\,r^2\,\pi^2}{L^2} \qquad \text{or} \qquad \frac{P}{A} = \frac{E\,\pi^2}{\left(\frac{L}{r}\right)^2}$$

Buckling of Columns

Where : **r** = least radius of gyration

A = cross – sectional area.

$$\frac{P}{A} = \frac{E\ \pi^2}{\left(\frac{L}{r}\right)^2}$$

P/A = average stress in column when carrying its

critical load = ***critical stress*** $\leqslant$ ***stress at proportional limit.***

L / r = slenderness ratio of the column

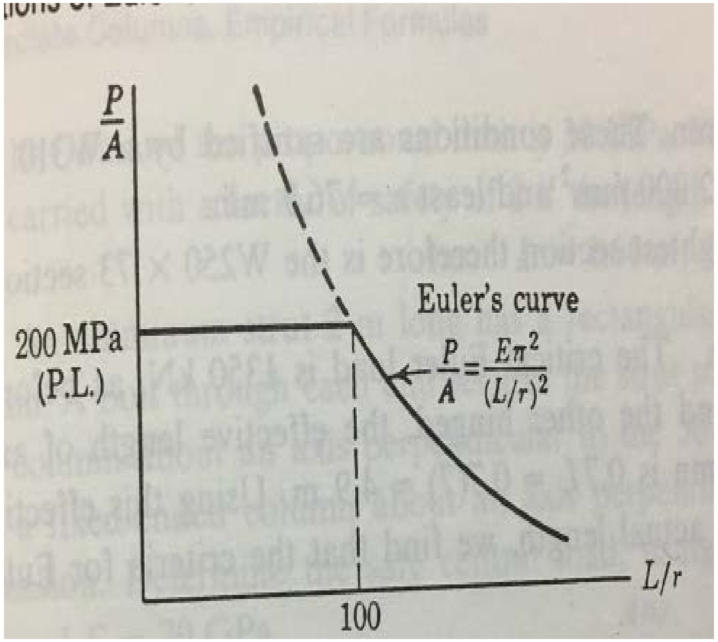
Ex: for a steel section has E= 200 GPa and proportional limit of 200 MPa

$(L / r)^2 = (200 \times 10^9 \times \pi^2) / (200 \times 10^6) = 10000$

Buckling of Columns

So; **L / r = 100**

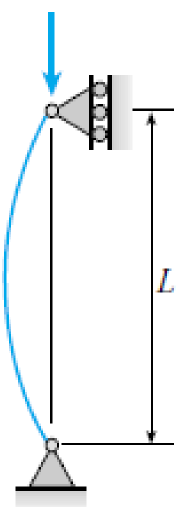
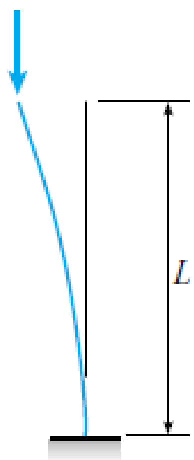
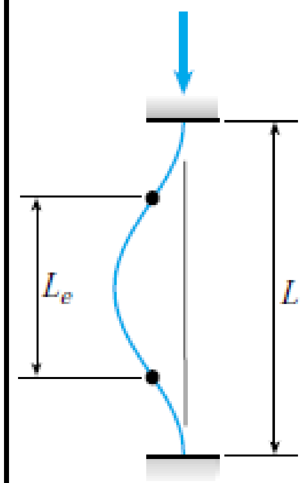
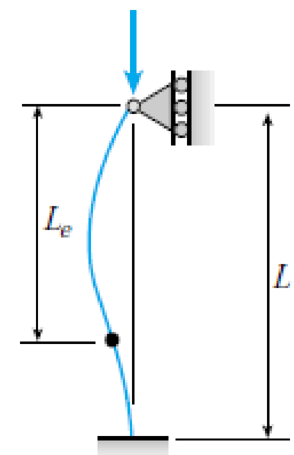
1. Below this value (**dashed Line**), Euler's unit load Exceeds the proportional Limit.
(i.e. for **L/r < 100**; Euler's Formula is not valid and the **P. L.** is taken as the **critical stress**)



2. Also; from the curve; the critical or allowable stress decreases rapidly as (**L/r**) increases, and for good design choose (**L/r**) as small as possible.

Buckling of Columns

Summary

(a) Pinned-pinned column	(b) Fixed-free column	(c) Fixed-fixed column	(d) Fixed-pinned column
$P_{cr} = \dfrac{\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{\pi^2 EI}{4L^2}$	$P_{cr} = \dfrac{4\pi^2 EI}{L^2}$	$P_{cr} = \dfrac{2.046\,\pi^2 EI}{L^2}$
			
$L_e = L$	$L_e = 2L$	$L_e = 0.5L$	$L_e = 0.699L$

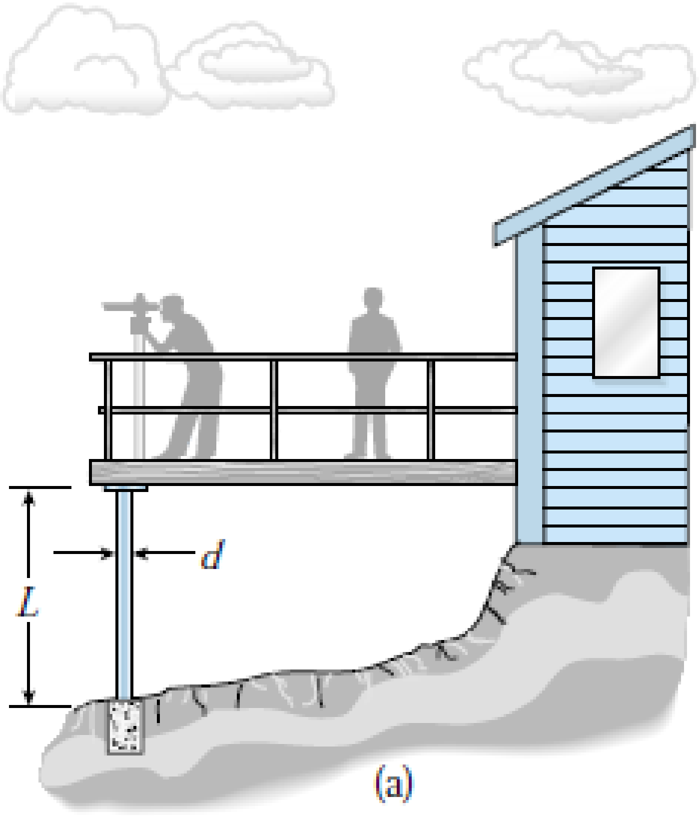
Buckling of Columns

Ex.1 : A viewing platform in a wild-animal park (**Fig. a below**) is supported by a row of aluminum pipe columns having length (***L* = *3.25 m***) *and outer diameter (* ***d* = 100 mm**). The bases of the columns are set in concrete footings and the tops of the columns are supported laterally by the platform. The columns are being designed to support compressive loads (***P = 100 kN*** *).*

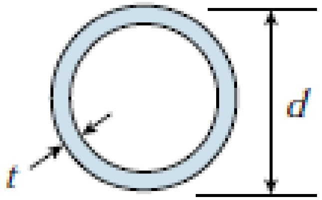
Determine the minimum required thickness ***t*** *of the columns (**Fig. b below**)* if a factor of safety *(**S.F.= 3**) is required with respect to Euler buckling. (For the* aluminum, use **72 GPa** for the modulus of elasticity and use **480 MPa** for the proportional limit.)

Buckling of Columns

Ex. 1 (Continued)



(a)



(b)

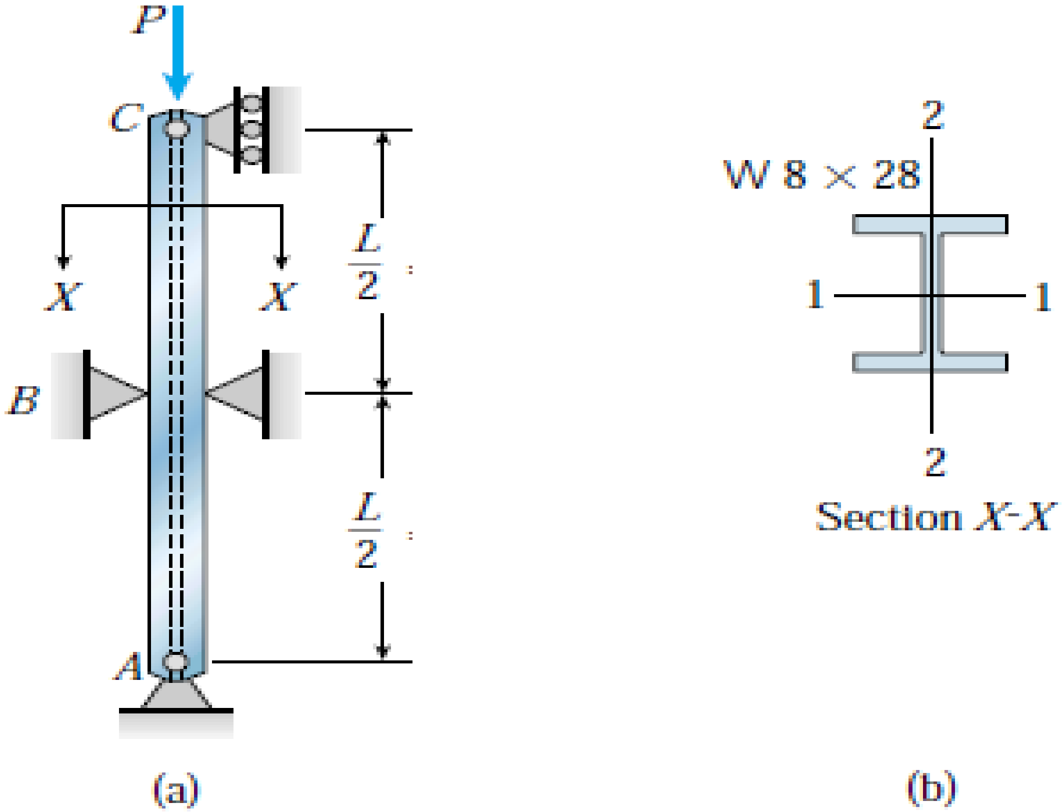
Buckling of Columns

Ex. 2: A long, slender column ***ABC*** *is pin-supported at the ends and compressed by an* axial load ***P*** *(**Fig. below**). Lateral support is provided at the midpoint* ***B*** *in the* plane of the figure, while that perpendicular to the plane of the figure is provided at the support only.

The column is constructed of a steel wide-flange section (**W 8 × 28**) having modulus of elasticity ($E = 2 \times 10^5$ MPa) *and proportional limit (* $\sigma_{PL} = 290$ MPa *).* The total length of the column is ($L = 7.6$ m*).* Determine the allowable load P_{allow} *using a factor of safety (* $S.F. = 2.5$ *) with* respect to Euler buckling of the column. Using $I_1 = 40.8 \times 10^6$ mm^4 and $I_2 = 9.0 \times 10^6$ mm^4 , $A = 5323$ mm^2.

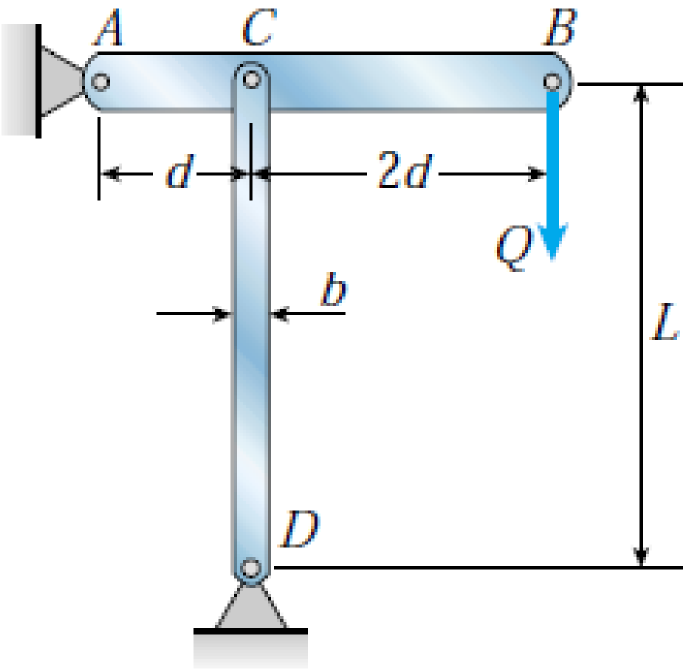
Buckling of Columns

Ex. 2 (Continued)



Buckling of Columns

Ex. 3: A horizontal beam ***AB*** *is pin-supported at end* ***A*** *and* carries a load ***Q*** *at end* ***B****, as shown in the figure. The beam* is supported at ***C*** *by a pinned-end column. The column is a* solid steel bar (***E = 200 GPa***) *of square cross section having* length (***L = 1.8 m***) *and side dimensions (* ***b = 60 mm*** *).* Based upon the critical load of the column, determine the allowable load ***Q*** *if the factor of safety with respect to* buckling is (***S.F. = 2.0*** *).*



Buckling of Columns

Ex: (**Prob. 1102, Singer Page 450**)

A **50 mm** by **100 mm** timber is used as a column with fixed ends. Determine the minimum length at which the Euler's formula can be used if **E = 10 GPa** and the proportional limit is **30 MPa**. What central load can be carried with a factor of safety of **2** if the length is **2.5m**?

Buckling of Columns

Solution:

A = $50 \times 100 = 5 \times 10^3\ mm^2$

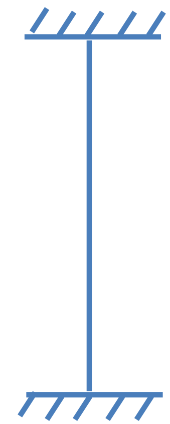
Iy = $100 \times 50^3 / 12 = 1.042 \times 10^6\ mm^4$

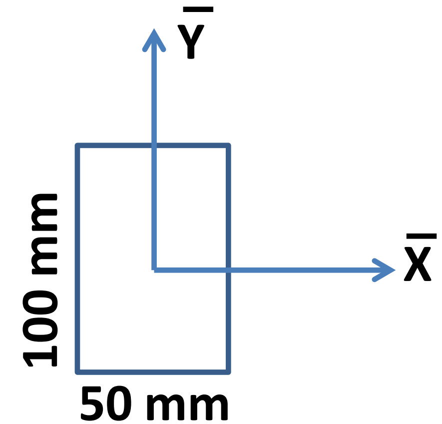
$P_{cr} = \sigma_{PL} \times A = 30 \times 5 \times 10^3 = 15 \times 10^4\ N$

$$P_{cr} = 4\frac{EI\,\pi^2}{L^2} \qquad \textit{For Fixed Ends}$$

$$15 \times 10^4 = 4\,\frac{10 \times 10^3 \times 1.042 \times 10^6 \times \pi^2}{L^2}$$

$L_{min.} = 1.656\ m$





Buckling of Columns

$$P_{cr} = 4\,\frac{10 \times 10^3 \times 1.042 \times 10^6 \times \pi^2}{2500^2} = 65818\ N = 65.818\ kN$$

$P_{working}$ or P_{safe} = P_{cr} / 2 = 65.818 / 2 = 32.9 kN

Buckling of Columns

Ex: (Prob. 1109 Singer Page 451)

Select the **lightest section** that will act as a column **12m** long with fixed ends and support an axial load of **700 kN** with a factor of safety of **2.0**. Assume that the proportional limit is **200 MPa** and **E = 200 GPa**.

Buckling of Columns

Solution:

$$P_{cr} = 4\frac{EI\,\pi^2}{L^2} \qquad \textit{For Fixed Ends}$$

1. If the critical load is govern:

P_{cr} = **700 × 2 = 1400 kN**

$$1400 \times 10^3 = 4\,\frac{200 \times 10^3 \times I \times \pi^2}{12000^2}$$

$I = 25.532 \times 10^6 \ mm^4$

Buckling of Columns

$$\sigma_{PL} = 4\frac{E \times \pi^2}{(\frac{L}{r})^2} \Rightarrow 200 = 4\frac{200 \times 10^3 \times \pi^2}{(\frac{L}{r})^2}$$

$$\frac{L}{r} \geq 198.7 \Rightarrow r \leq \frac{12000}{198.7} = 60.4\,mm$$

Choose a section of at least $I \geq 25.532 \times 10^6\ mm^4$ and at least $r \leq 60.4\ mm$. Say W1.

2. **If σ_{PL}**

$A_{min.} = P_{cr} / \sigma_{PL} = 1400 \times 10^3 / 200 = 7000\ mm^2$

Buckling of Columns

Choose a section of at least $A \geq 7000 \text{ mm}^2$ and at least $r \geq 60.4 \text{ mm}$. Say W2.

Choose the lightest one between W1 and W2.