CHAPTER 1
Introduction

➢ **Definitions**

➢ **International System of Units**

➢ **Newton's laws of motion**

➢ **Force Systems**

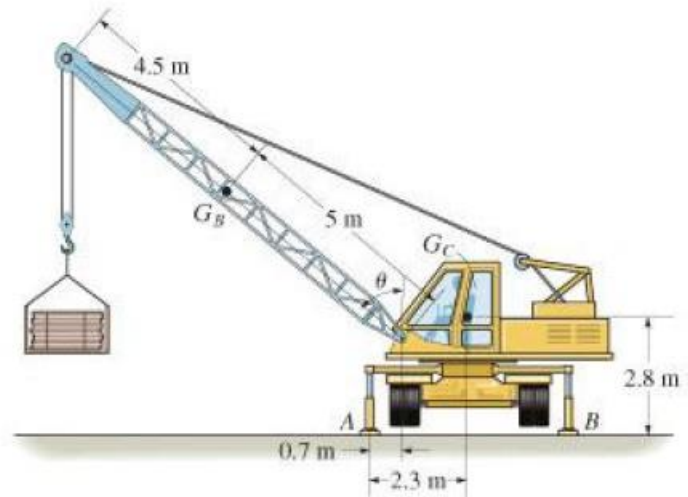# Mechanics is the branch of physical sciences

1-rigid body mechanics,

2-deformable body mechanics

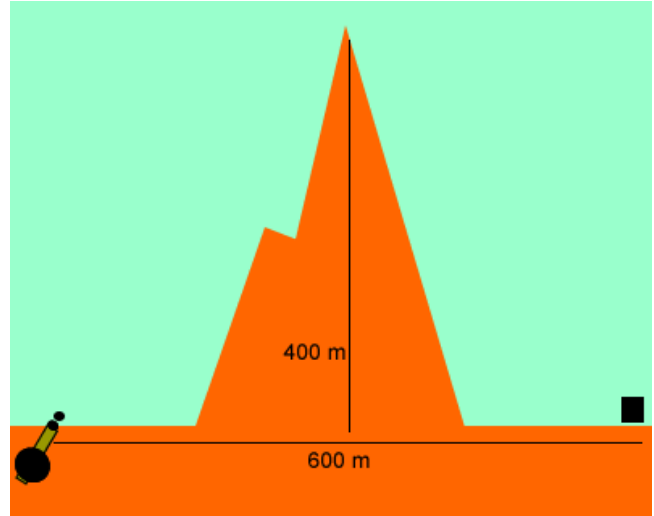3- fluid mechanics

Rigid body mechanics

1- Statics

deals with equilibrium of bodies under action of forces (bodies may be either at rest or move with a constant

Rigid-body Mechanics

2-Dynamic

deals with motion of bodies (accelerated motion)

Fundamental Concepts

1-Length (Space):

needed to locate position of a point in space, & describe size of the physical system.**(Distances, Geometric Properties)**

2-Time

measure of succession of events. **basic quantity in Dynamics**.

3-Mass

quantity of matter in a body. **measure of inertia of a body (its resistance to change in velocity)**

4- Force

represents the action of one body on another. **characterized by its magnitude, direction of its action, and its point of application**

Note : Force is a Vector quantity.

Fundamental Concepts

Newtonian Mechanics

➢Length, Time, and Mass are absolute concepts independent of each other.

➢Force is a derived concept not independent of the other fundamental concepts.

➢Force acting on a body is related to the mass of the body and the variation of its velocity with time.

➢Force can also occur between bodies that are physically separated (Ex: gravitational, electrical, and magnetic forces)

Fundamental Concepts

Remember:

➢ Mass is a property of matter that does not change from one location to another.

➢ Weight refers to the gravitational attraction of the earth on a body or quantity of mass. Its magnitude depends upon the elevation at which the mass is located

➢ Weight of a body is the gravitational force acting on it.

Idealizations

Particle: A body with mass but with dimensions that can be neglected

Rigid Body: A combination of large number of particles in which all particles remain at a fixed distance (practically) from one another before and after applying a load.

➢Material properties of a rigid body are not required to be considered when analyzing the forces acting on the body.

➢In most cases, actual deformations occurring in structures, machines, mechanisms, etc. are relatively small, and rigid body assumption is suitable for analysis

Idealizations

Concentrated Force: Effect of a loading which is assumed to act at a point (CG) on a body.

➤Provided the area over which the load is applied is very small compared to the overall size of the body.

Ex: Contact Force between a wheel and ground.

40 kN   160 kN

Force Systems

Force: Magnitude (*P*), direction (arrow) and point of application (point A) is important

Change in any of the three specifications will alter the effect on the bracket.

Force is a Fixed Vector

In case of rigid bodies, line of action of force is important (not its point of application if we are interested in only the resultant external effects of the force), we will treat most forces as

Cable Tension *P*
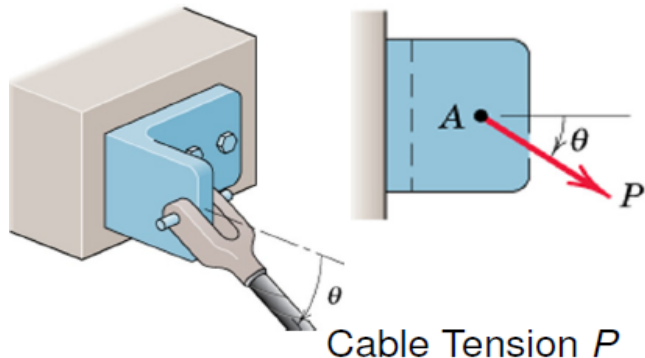
External effect: Forces applied (applied force); Forces exerted by bracket, bolts, Foundation (reactive force)

Internal effect: Deformation, strain pattern – permanent strain; depends on material properties of bracket, bolts, etc.

Force System

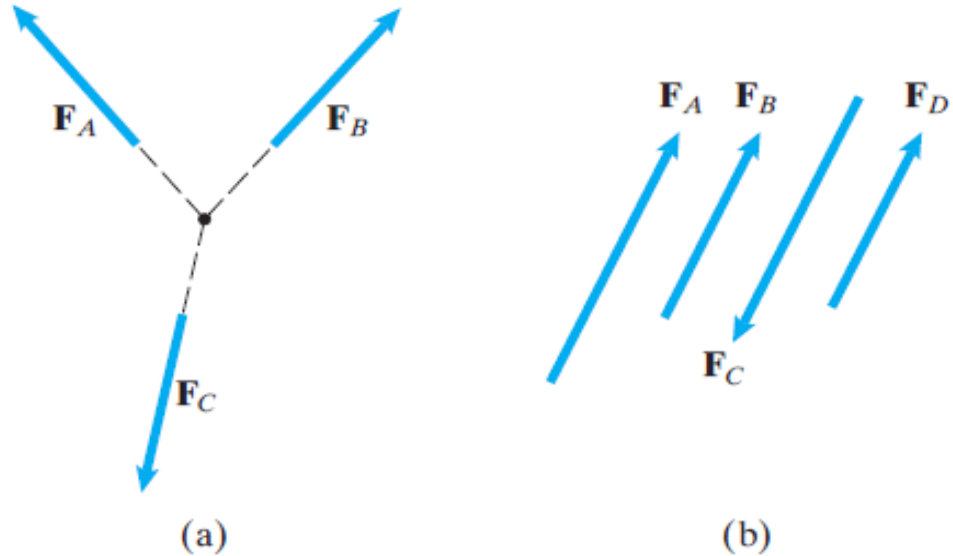A system of forces is simply a particular set of forces.


Fig. 1.5
(a) Concurrent forces.
(b) Parallel forces.

Moment of Force

When a force is applied to a body it will produce a tendency for the body to rotate about a point that is not on the line of action of the force. This tendency to rotate is sometimes called a ***torque, but most often it is called*** the moment of a force or simply the ***moment***

Magnitude.   The magnitude of M_O is

$$M_O = Fd$$

$M = F.d$

$M = F.d'$

$d' = d \sin \theta$

$M = 0$
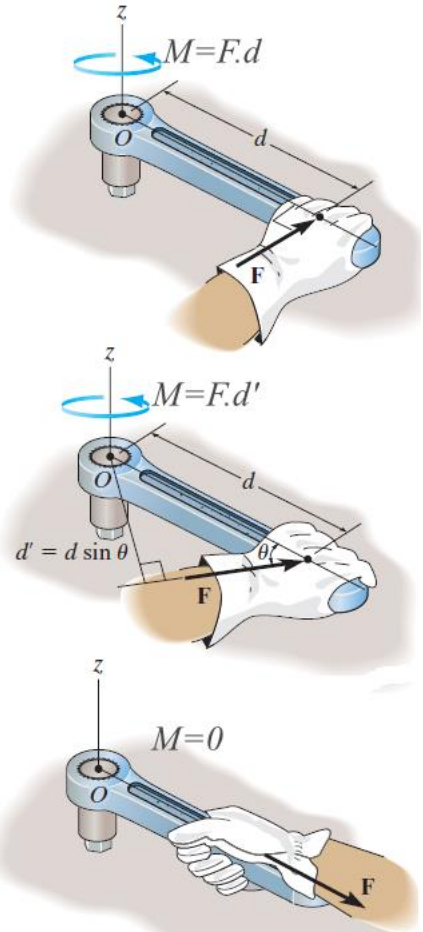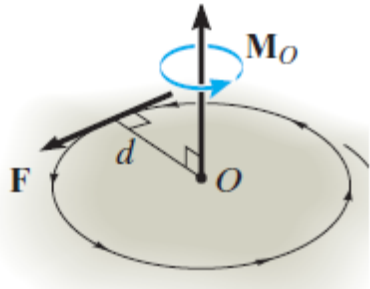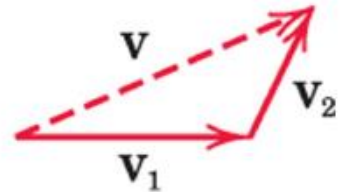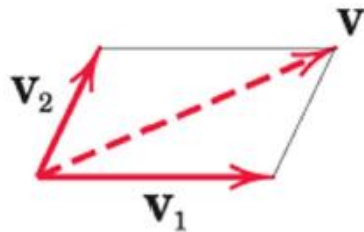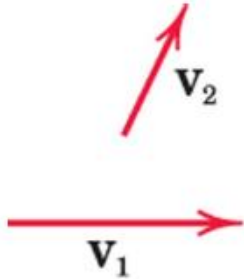
Scalars

only magnitude is associated.

Ex: time, volume, density, speed, energy, mass

Vectors

possess direction as well as magnitude, and must obey the parallelogram law of addition (and the triangle law).

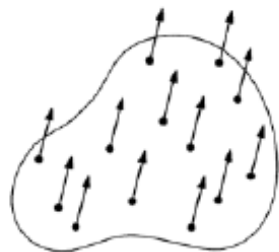Ex: displacement, velocity, acceleration, force, moment, momentum.

Equivalent Vector: $V = V_1 + V_2$ (Vector Sum)

Vectors

Free Vector: whose action is not confined to or associated with a unique line in space
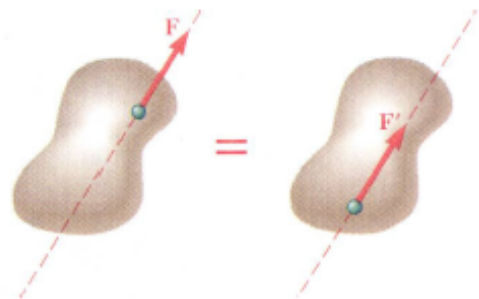Ex: Movement of a body without rotation.

Sliding Vector: has a unique line of action in space but not a unique point of application
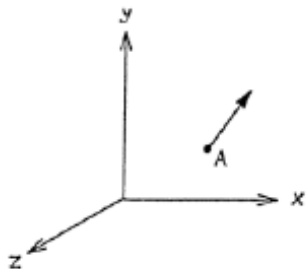Ex: External force on a rigid body
→ Principle of Transmissibility
→ Imp in Rigid Body Mechanics

Fixed Vector: for which a unique point of application is specified
Ex: Action of a force on deformable body

Newton's Law of Gravitational Attraction

➢Weight of a body (gravitational force acting on a body) is required to be computed in Statics as well as Dynamics.

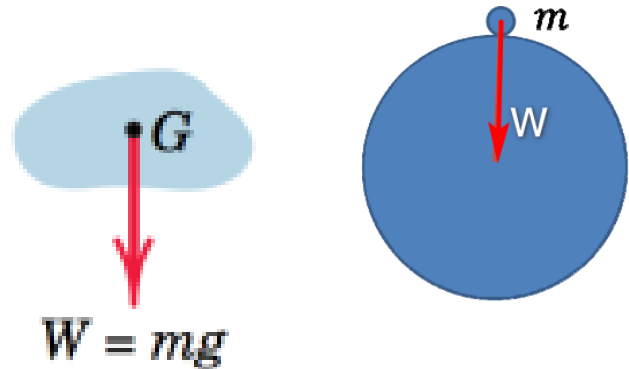➢Weight of a Body: If a particle is located at or near the surface of the earth, the only significant gravitational force is that between the earth and the particle

$$W = m . g$$

Where

m= mass of the body

g= acceleration due to gravity

$$W = mg$$

Free Body Diagram (FBD)

This diagram is a sketch of the outlined shape of the body, which represents it as being isolated or "free" from its surroundings, i.e., a "free body." On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied.
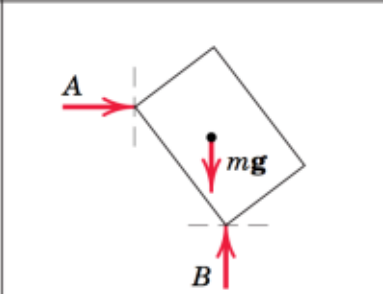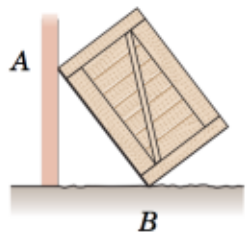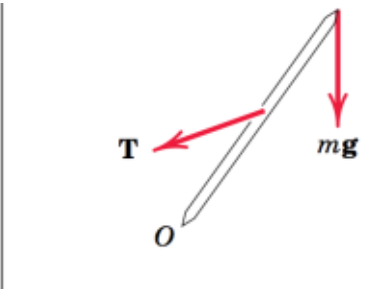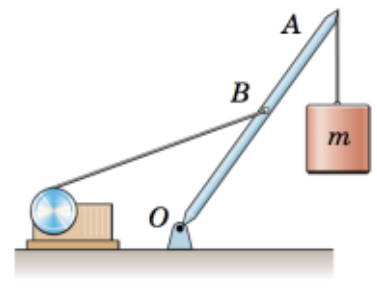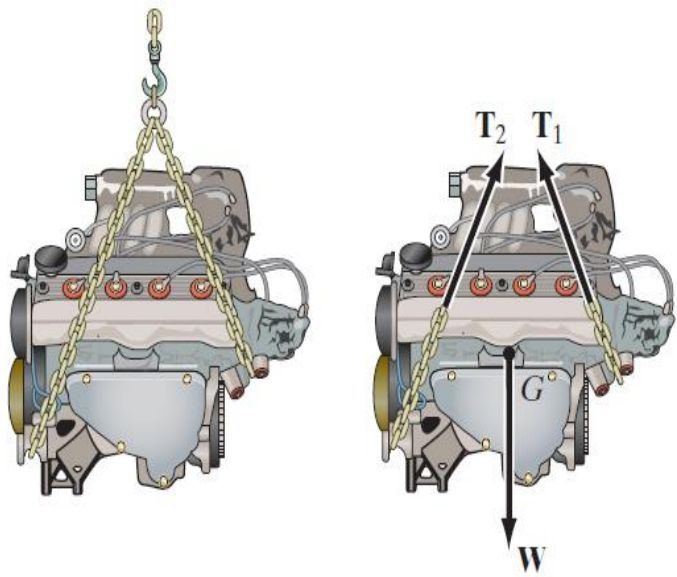
Newton's Three Laws of Motion

Basis of formulation of rigid body mechanics.

First Law: A particle originally at rest, or moving in a straight line with constant velocity, tends to remain in this state provided the particle is not subjected to an unbalanced force.

First law contains the principle of the equilibrium of forces → main topic of concern in Statics

F_1 F_2

v

F_3

Equilibrium
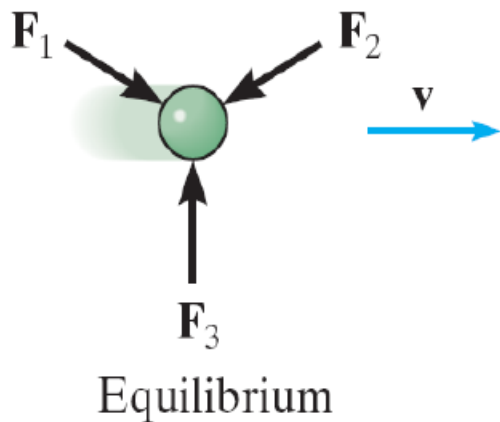
Newton's Three Laws of Motion

Second Law: A particle of mass "m" acted upon by an unbalanced force "F" experiences an acceleration "a" that has the same direction as the force and a magnitude that is directly proportional to the force.

$F \longrightarrow m$ $\xrightarrow{\;a\;}$ F = ma

Accelerated motion

Second Law forms the basis for most of the analysis in Dynamics

Newton's Three Laws of Motion

Third Law: The mutual forces of action and reaction between two particles are equal, opposite, and collinear.

force of A on B

F A B F

force of B on A
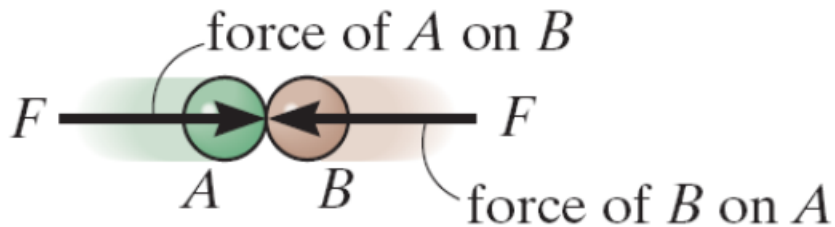
Action – reaction

Third law is basic to our understanding of Force → Forces always occur in pairs of equal and opposite forces.

International System of Units

Quantity	Dimensional symbol	SI units		U.S. customary units	
		Unit	Symbol	Unit	Symbol
mass	M	Kilogram	kg	Slug	-
length	L	Meter	m	Foot	ft
time	T	Second	s	Second	sec
force	F	Newton	N	pound	lb

Trigonometry

$$\sin \theta = \frac{Y}{S}$$

$$\cos \theta = \frac{X}{S}$$

$$\tan \theta = \frac{Y}{X}$$
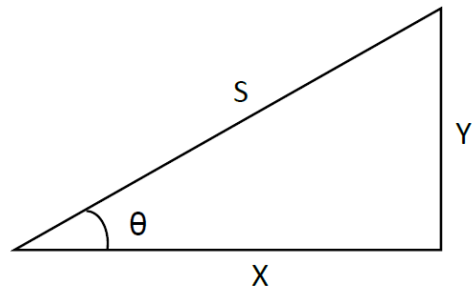
Table 1.1 The common prefixes used in SI units and the multiples they represent.

Prefix	Abbreviation	Multiple
nano-	n	10^{-9}
micro-	μ	10^{-6}
milli-	m	10^{-3}
kilo-	k	10^{3}
mega-	M	10^{6}
giga-	G	10^{9}

Table 1.3 Unit conversions.

Time	1 minute	=	60 seconds
	1 hour	=	60 minutes
	1 day	=	24 hours
Mass	1 slug	=	14.59 kilograms
Force	1 pound	=	4.448 newtons

Length	1 foot	=	12 inches
	1 mile	=	5280 feet
	1 inch	=	25.4 millimeters
	1 foot	=	0.3048 meters
Angle	2π radians	=	360 degrees

1–1.

What is the weight in newtons of an object that has a mass
of (a) 8 kg, (b) 0.04 kg, and (c) 760 Mg?

SOLUTION

(a) $W = 9.81(8) = 78.5 \text{ N}$ **Ans.**

(b) $W = 9.81(0.04)(10^{-3}) = 3.92(10^{-4}) \text{ N} = 0.392 \text{ mN}$ **Ans.**

(c) $W = 9.81(760)(10^{3}) = 7.46(10^{6}) \text{ N} = 7.46 \text{ MN}$ **Ans.**

1–2.

Represent each of the following combinations of units in the correct SI form: (a) /KN/μs, (b) Mg/mN, and (c) MN/(kg·ms).

SOLUTION

(a) $kN/\mu s = 10^3 N/(10^{-6})\,s = GN/s$ **Ans.**

(b) $Mg/mN = 10^6 g/10^{-3} N = Gg/N$ **Ans.**

(c) $MN/(kg \cdot ms) = 10^6\,N/kg(10^{-3}\,s) = GN/(kg \cdot s)$ **Ans.**

1–5.

Represent each of the following as a number between 0.1 and 1000 using an appropriate prefix: (a) 45 320 kN, (b) 568(10^5) mm, and (c) 0.00563 mg.

SOLUTION

(a)   45 320 kN = 45.3 MN   **Ans.**

(b) $568(10^5)$ mm = 56.8 km   **Ans.**

(c)   0.00563 mg = 5.63 μg   **Ans.**

1–19.

A concrete column has a diameter of 350 mm and a length of 2 m. If the density (mass/volume) of concrete is 2.45 Mg/m^3, determine the weight of the column in pounds.

SOLUTION

$$V = \pi r^2 h = \pi \left(\frac{0.35}{2} \text{ m}\right)^2 (2 \text{ m}) = 0.1924 \text{ m}^3$$

$$m = \rho V = \left(\frac{2.45(10^3)\text{kg}}{\text{m}^3}\right)(0.1924 \text{ m}^3) = 471.44 \text{ kg}$$

$$W = mg = (471.44 \text{ kg})(9.81 \text{ m/s}^2) = 4.6248(10^3) \text{ N}$$

$$W = \left[4.6248(10^3) \text{ N}\right]\left(\frac{1 \text{ lb}}{4.4482 \text{ N}}\right) = 1.04 \text{ kip} \qquad \textbf{Ans.}$$

CHAPTER TWO
# RESULTANT OF FORCES SYSTEM

2.1 General

A **force** may be defined as the action of one body on another body which changes or tends to change the motion of the body acted on. Because of the inertia possessed by all material bodies, they react or oppose any force which acts on them (Newton's third law).

Note1: Forces may be considered as localized vectors and they cannot be defined unless all the following characteristics mentioned:
1- Magnitude,
2- Direction (sense and slope),
3- Location of any point on its line of action.
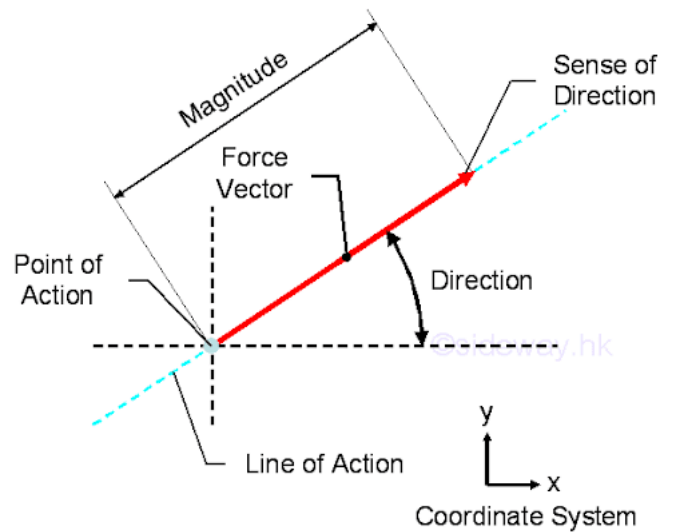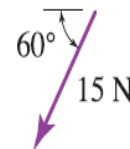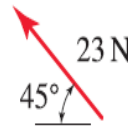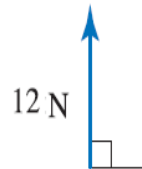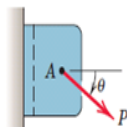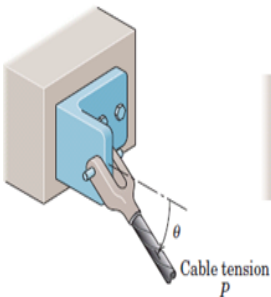
Note2: The third characteristic shows that if two forces have the same direction, they will produce the same external effect on a rigid body. This fact leads to the *principle of transmissibility* which states that the external effect of a force on a rigid body is independent of the point of application of the force along its line of action.
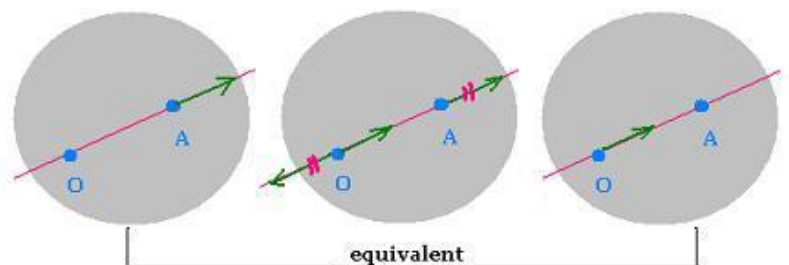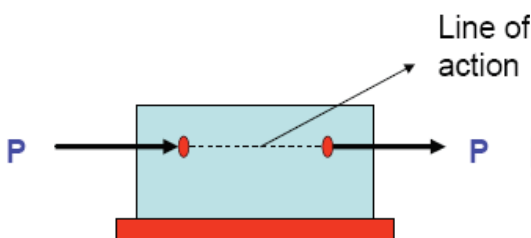
2.2 System of Forces

When several forces act in a given situation, they are called *system of forces* or *force system.* Force systems in 2D can be classified according to the arrangement of the lines of action of the forces of the system as follows:

1- **Concurrent, Coplanar:** The action lines of all the forces of the system are in the same plane and intersect at a common point.
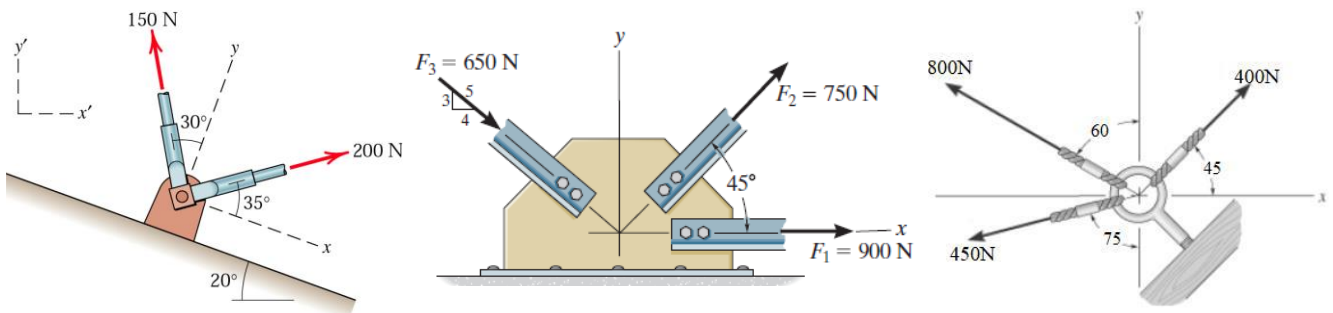
2- **Parallel, Coplanar:** The action lines of all the forces of the system are parallel and lie in the same plane.
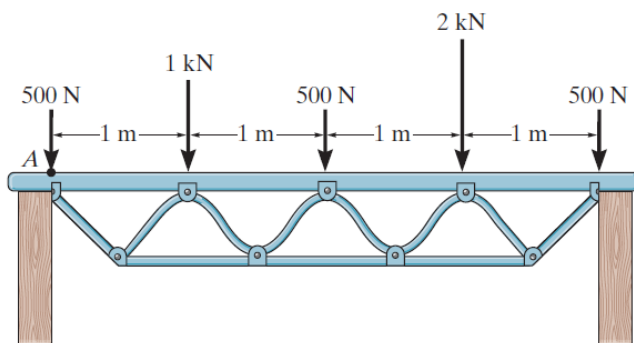
3- **Nonconcurrent, Nonparallel, Coplanar:** The action lines of all the forces of the system are in the same plane, but they are not all parallel and they do not intersect at a common point.
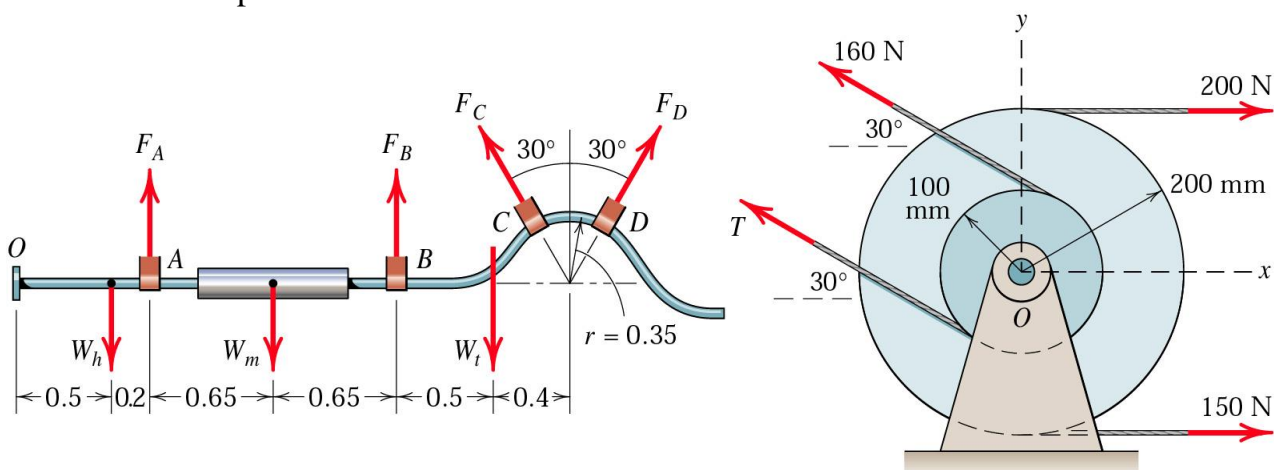
2.3 Composition and resolution of forces

- **Resolution of forces**: The process of replacing a force by its components. A *component of a force* is any one of two or more forces having the given forces as a resultant. So the term "***component***" is used to mean either one of two coplanar concurrent forces or any one of three noncoplanar concurrent forces having the given force as a resultant.
- **Composition of forces**: The process of replacing a force system by its resultant.

2.3.1 Resolution and Composition of Two **Concurrent, Coplanar** forces

1- **Parallelogram law of forces**

If two forces act simultaneously on a particle, the parallelogram law states that the resultant is equal to the diagonal of the parallelogram formed on the vectors of these forces.
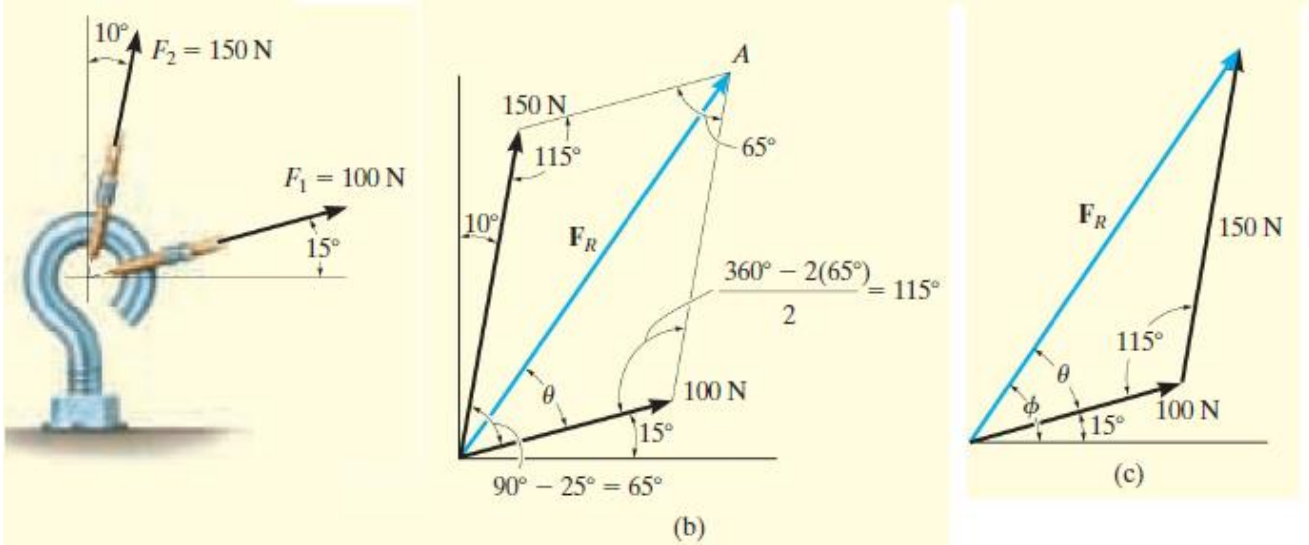
$$R = \sqrt{(F_1)^2 + (F_2)^2 - 2(F_1).(F_2)\cos\theta}$$
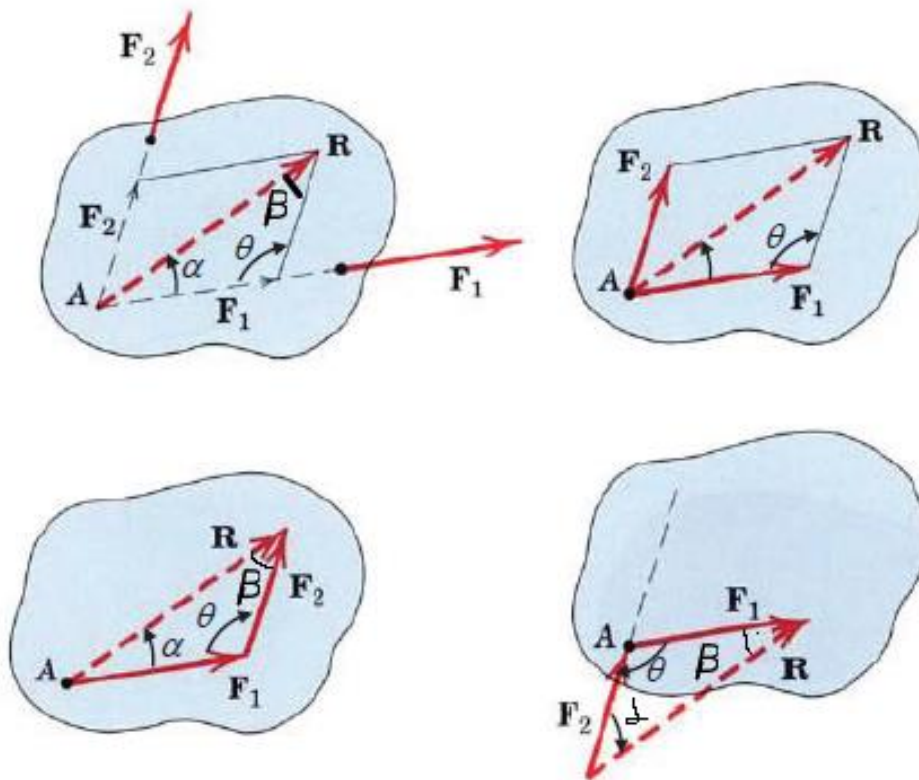
The angle the resultant makes with either force can be determined by the *law of sines*:

$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha} = \frac{R}{\sin\theta}$$

Example 1: The screw in Figure (a) is subjected to two forces, **F1** and **F2**. Determine the magnitude and direction of the resultant force.

(b)

(c)

Solution: By parallelogram, the resultant F_R is:

$$F_R = \sqrt{100^2 + 150^2 - 2(100)(150)\cos115°}$$

$$= \sqrt{10000 + 22500 - 30000(-0.4226)}$$

$$= 212.6\,N \cong 213\ N$$

Applying the law of sines to determine θ,

$$\frac{150}{\sin\theta} = \frac{212.6}{\sin115°} \implies \sin\theta = \frac{150}{212.6}(\sin115°)$$

$$\theta = 39.8°$$

Thus, the direction ϕ of F_R, measured from the horizontal, is

$$\phi = 39.8° + 15.0° = 54.8°$$

Example 2: Resolve the horizontal 600-N force in Figure (a) into components acting along u and v axes and determine the magnitude of these components.
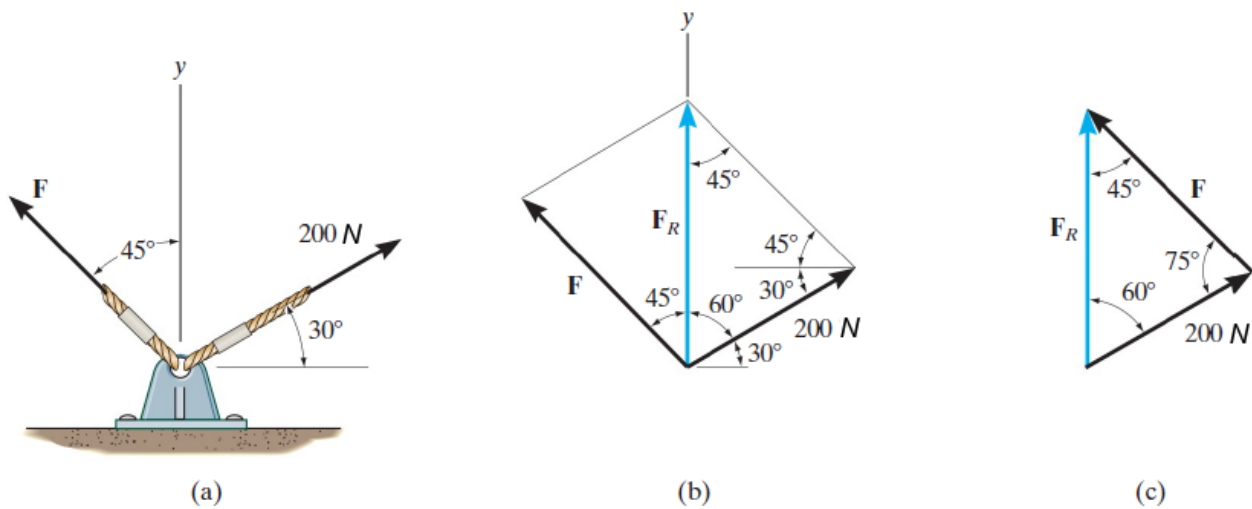

(a)

(b)

(c)
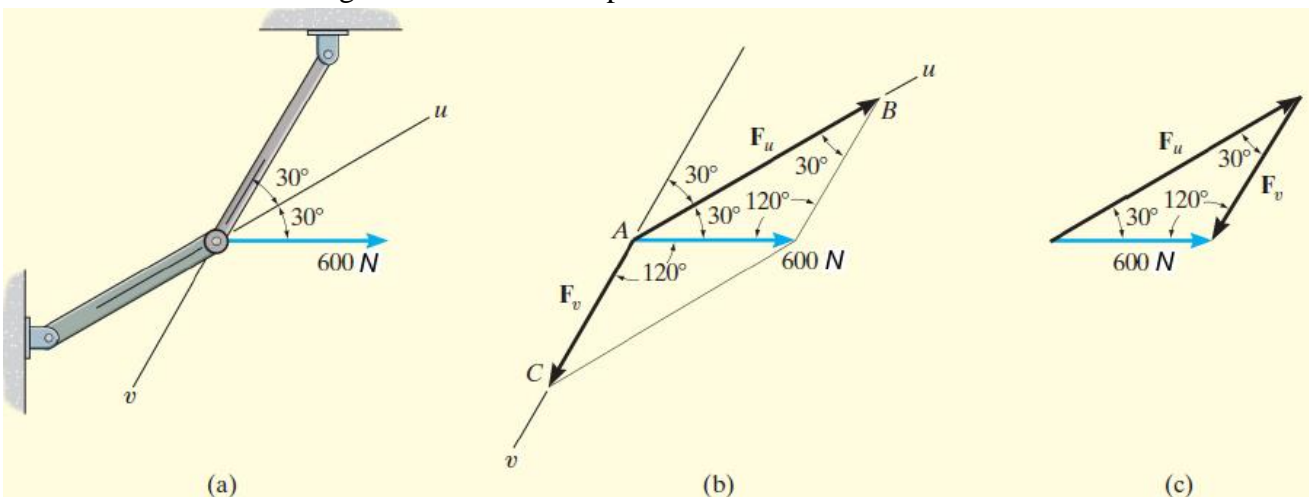
Solution: Applying the law of sines,

$$\frac{F_u}{\sin 120^\circ} = \frac{600}{\sin 30^\circ} \quad \Longrightarrow \quad F_u = 1039\ N$$

$$\frac{F_v}{\sin 30^\circ} = \frac{600}{\sin 30^\circ} \quad \Longrightarrow \quad F_v = 600\ N$$

<u>*Note:*</u> The result for F_u shows that sometimes a component can have a greater magnitude than the resultant.

Example 3: Determine the magnitude of the components force F in Figure (a) and the magnitude of the resultant force FR if FR is directed along the positive y-axis.

(a)   (b)   (c)

Solution: The magnitude of F_R and F can be determined by applying the law of sines,

$$\frac{F}{\sin 60^\circ} = \frac{200}{\sin 45^\circ} \quad \Longrightarrow \quad F = 245\ N$$

$$\frac{F_R}{\sin 75^\circ} = \frac{200}{\sin 45^\circ} \quad \Longrightarrow \quad F_R = 273\ N$$

Example 4: It is required that the resultant force acting on the eyebolt in Figure (a) be directed along the positive x-axis and that $F2$ has a *minimum* magnitude. Determine this magnitude, the angle θ, and the corresponding resultant force.

(a)

(b)

(c)
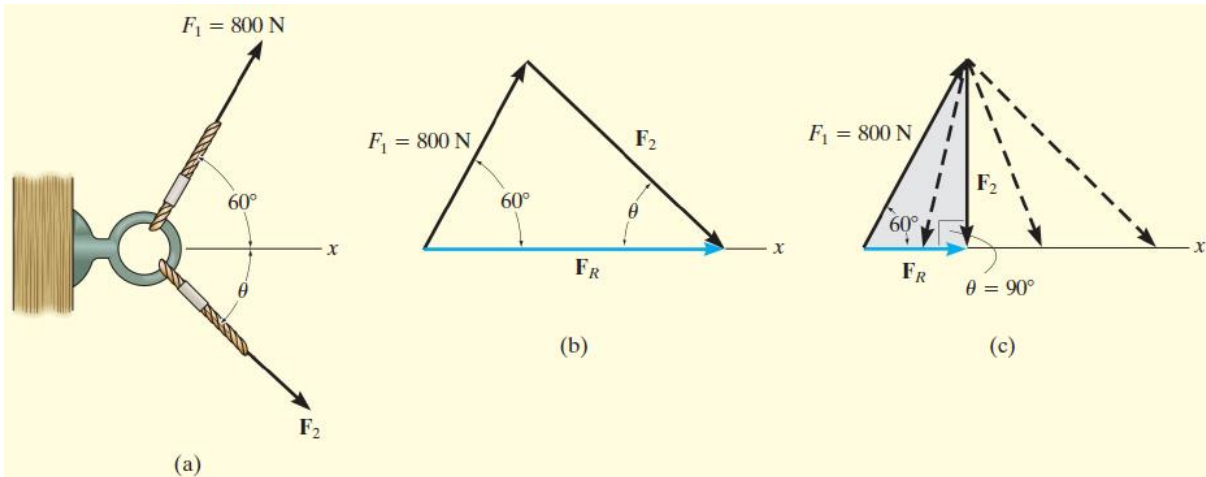
Solution: The magnitude of F_2 is a *minimum* or the shortest length when its line of action is *perpendicular* to the line of action of F_R, that is, when,

$$\theta = 90$$

Since the vector addition now forms a right triangle, the two unknown magnitudes can be obtained by trigonometry.
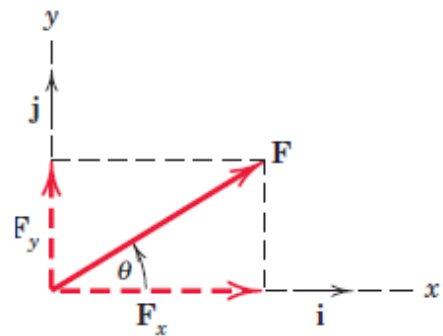
$$F_R = 800 \cos 60° = 400 \text{ N}$$

$$F_2 = 800 \sin 60° = 693 \text{ N}$$

2- Rectangular components

When a force is resolved into two components along x and y axes, the components are then called rectangular components.

$F_x = F \cos \theta$	$F = \sqrt{F_x^{\,2} + F_y^{\,2}}$	
$F_y = F \sin \theta$	$\theta = \tan^{-1}\dfrac{F_y}{F_x}$	-------- 2.4

Instead of using the angle 0, however, the direction of F can also be defined using a small "slope" triangle. Since this triangle and the larger triangle are similar, the proportional length of the sides gives.

- Horizontal component $Fx = F\left(\dfrac{a}{c}\right)$

- Vertical component $Fy = F\left(\dfrac{b}{c}\right)$

$$F_x = F \sin \beta$$
$$F_y = F \cos \beta$$

$$F_x = -F \cos \beta$$
$$F_y = -F \sin \beta$$

$$F_x = F \sin (\pi - \beta)$$
$$F_y = -F \cos (\pi - \beta)$$

$$F_x = F \cos (\beta - \alpha)$$
$$F_y = F \sin (\beta - \alpha)$$
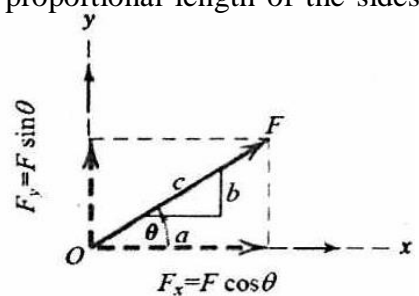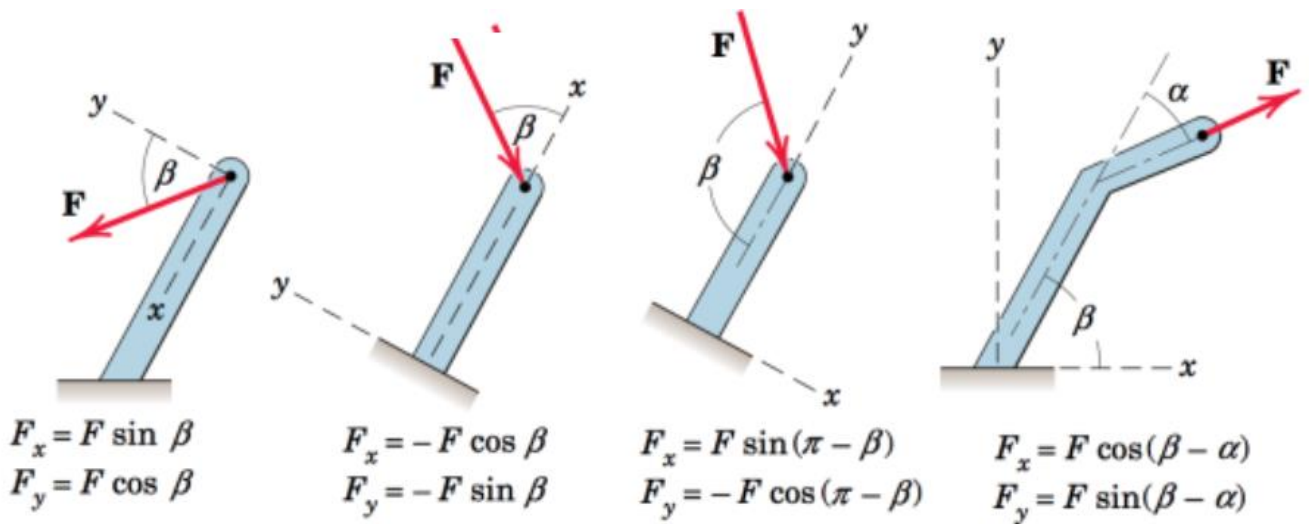
*Resultant of two Concurrent forces can be obtained by

$$\Sigma Fx = F1x + F2x$$
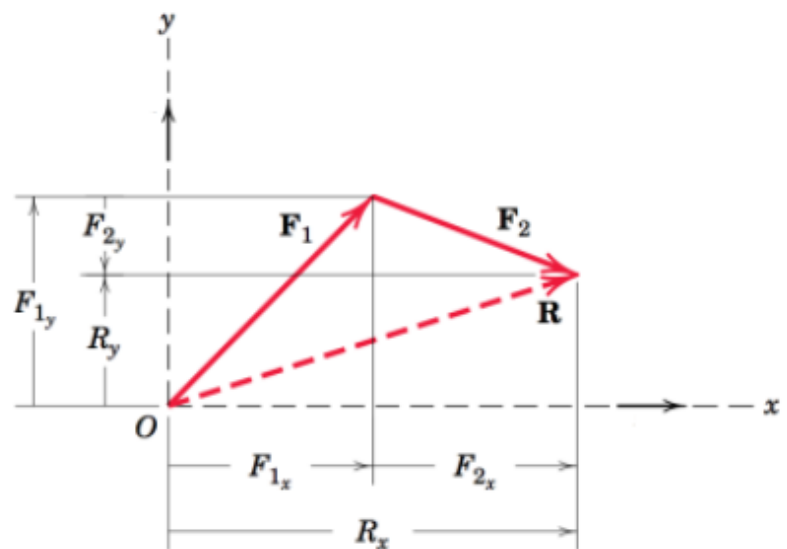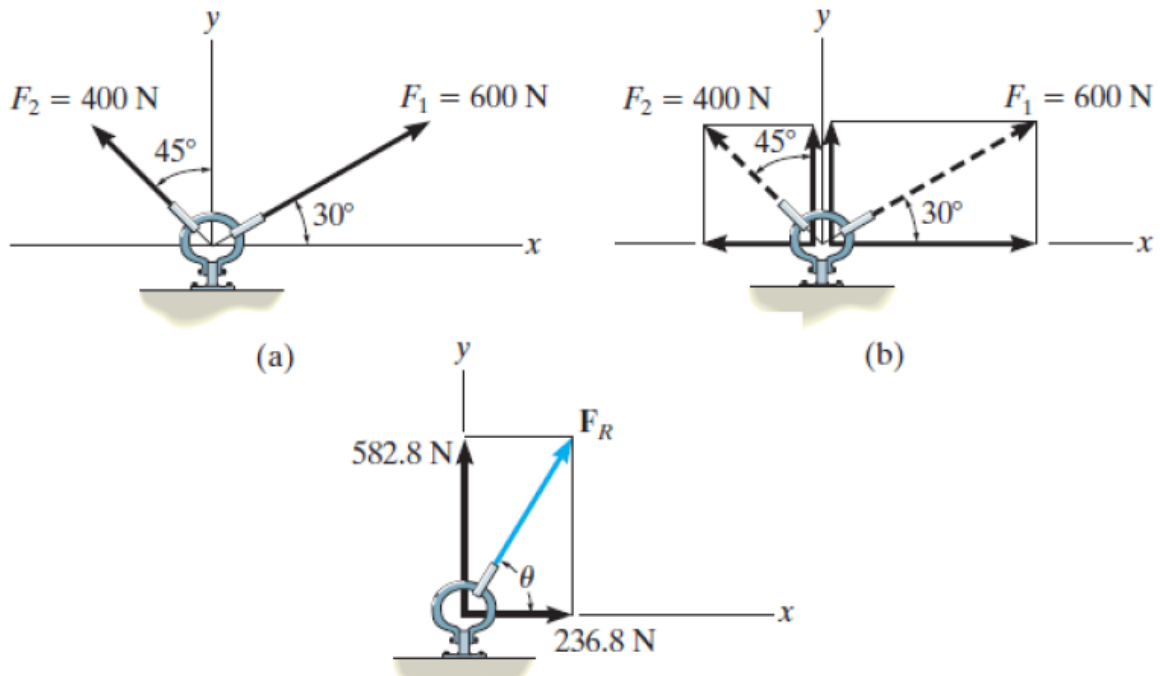$$\Sigma Fy = F1y + F2y$$

$$F = \sqrt{\Sigma F_x{}^2 + \Sigma F_y{}^2}$$

$$tan\,\theta = \frac{\Sigma Fy}{\Sigma Rx}$$

Example

The link in Fig. 2–19a is subjected to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude and direction of the resultant force .

(a)

(b)

SOLUTION I

Scalar Notation. First we resolve each force into its x and y components, Fig. 2–19b, then we sum these components algebraically.

$$\xrightarrow{+} (F_R)_x = \Sigma F_x; \qquad (F_R)_x = 600 \cos 30° \text{ N} - 400 \sin 45° \text{ N}$$
$$= 236.8 \text{ N} \rightarrow$$

$$+\uparrow (F_R)_y = \Sigma F_y; \qquad (F_R)_y = 600 \sin 30° \text{ N} + 400 \cos 45° \text{ N}$$
$$= 582.8 \text{ N} \uparrow$$

The resultant force, shown in Fig. 2–19c, has a *magnitude* of

$$F_R = \sqrt{(236.8 \text{ N})^2 + (582.8 \text{ N})^2}$$
$$= 629 \text{ N} \qquad\qquad\qquad\qquad Ans.$$

From the vector addition,

$$\theta = \tan^{-1}\left(\frac{582.8 \text{ N}}{236.8 \text{ N}}\right) = 67.9° \qquad\qquad Ans.$$

will show that Cartesian vector analysis is very beneficial for solving three-dimensional problems.

2.3.2 Resolution and Composition of three or more **Concurrent, Coplanar** forces

In determination of the resultant of several forces (more than two forces), using the rectangular component is more convenient than using the parallelogram rule more than once. Consider three forces as shown in figure below. So the resultant of these coplanar forces may be determined by the following steps:

1. Resolve each force into x and y components.
2. Add the respective using scalar algebra since they are collinear

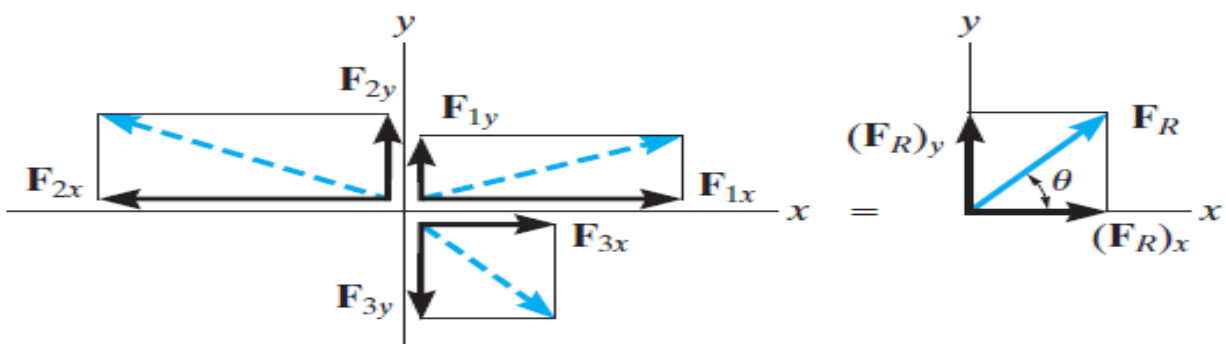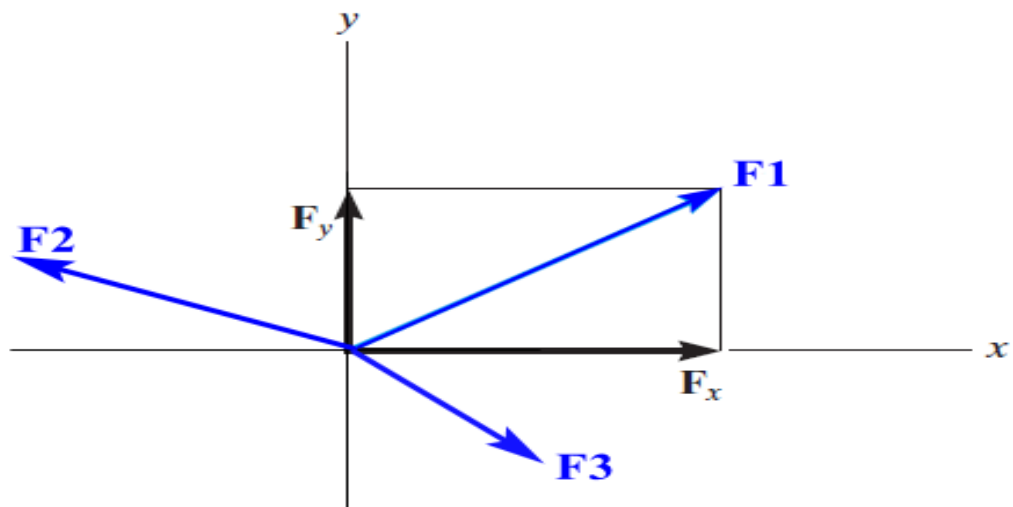$$FRx = \Sigma Fx = F1x + F2x + F3x + \cdots$$
$$FRy = \Sigma Fy = F1y + F2y + F3y + \cdots$$

3. The resultant force is then computed by using Pythagorean theorem,
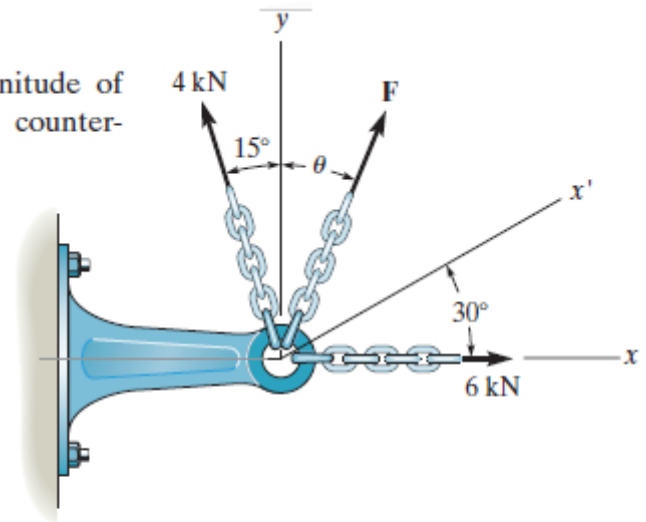
$$F = \sqrt{\Sigma F_x{}^2 + \Sigma F_y{}^2}$$

And the angle 0, which specifies the direction of resultant, is determined from trigonometry:

$$\tan \theta = \frac{FRy}{FRx}$$

Example

If $F = 5$ kN and $\theta = 30°$, determine the magnitude of the resultant force and its direction, measured counterclockwise from the positive x axis.

SOLUTION

Scalar Notation. Summing the force components along x and y axes algebraically by referring to Fig. a,

$\xrightarrow{+}(F_R)_x = \Sigma F_x;$ $(F_R)_x = 5 \sin 30° + 6 - 4 \sin 15° = 7.465$ kN $\rightarrow$

$+\uparrow(F_R)_y = \Sigma F_y;$ $(F_R)_y = 4 \cos 15° + 5 \cos 30° = 8.194$ kN $\uparrow$

By referring to Fig. b, the magnitude of the resultant force is

$$F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2} = \sqrt{7.465^2 + 8.194^2} = 11.08 \text{ kN} = 11.1 \text{ kN} \quad \textbf{Ans.}$$

And its directional angle θ measured counterclockwise from the positive x axis is

$$\theta = \tan^{-1}\left[\frac{(F_R)_y}{(F_R)_x}\right] = \tan^{-1}\left(\frac{8.194}{7.465}\right) = 47.67° = 47.7° \qquad \textbf{Ans.}$$

Example Determine the x and y components of each force acting on the *gusset plate* of a bridge truss. Show that the resultant force is zero.
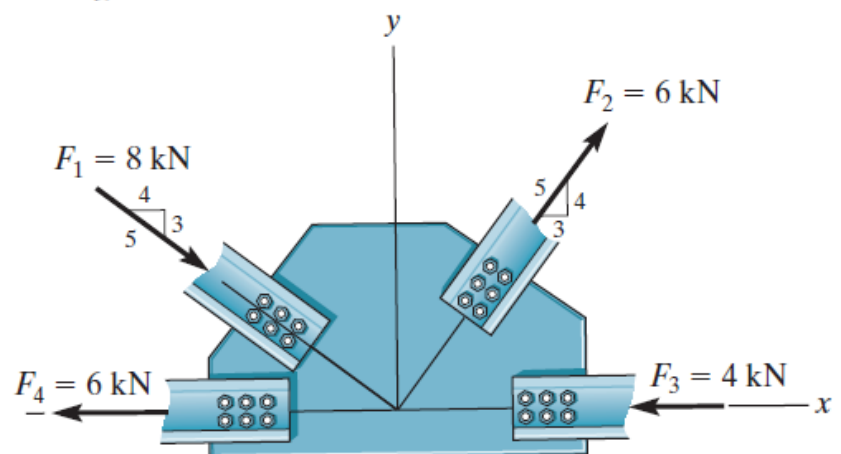
Example If the magnitude of the resultant force acting on the bracket is to be 450 N directed along the positive u axis, determine the magnitude of $\mathbf{F}_1$ and its direction ϕ.
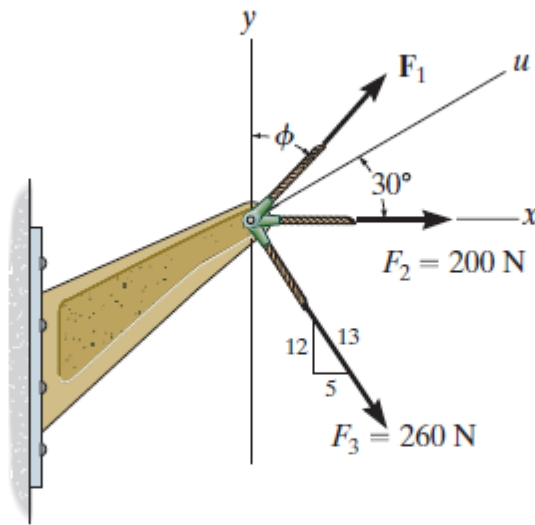

SOLUTION

Rectangular Components: By referring to Fig. *a*, the x and y components of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_R$ can be written as

$(F_1)_x = F_1 \sin \phi$ $\qquad\qquad\qquad (F_1)_y = F_1 \cos \phi$

$(F_2)_x = 200$ N $\qquad\qquad\qquad\qquad (F_2)_y = 0$

$(F_3)_x = 260\left(\dfrac{5}{13}\right) = 100$ N $\qquad\qquad (F_3)_y = 260\left(\dfrac{12}{13}\right) = 240$ N

$(F_R)_x = 450 \cos 30° = 389.71$ N $\qquad (F_R)_y = 450 \sin 30° = 225$ N

Resultant Force: Summing the force components algebraically along the x and y axes,

$\xrightarrow{+} \Sigma (F_R)_x = \Sigma F_x;\quad 389.71 = F_1 \sin \phi + 200 + 100$

$$F_1 \sin \phi = 89.71 \qquad\qquad\qquad (1)$$

$+\uparrow \Sigma (F_R)_y = \Sigma F_y;\quad 225 = F_1 \cos \phi - 240$

$$F_1 \cos \phi = 465 \qquad\qquad\qquad (2)$$

Solving Eqs. (1) and (2), yields

$\qquad\qquad\qquad \phi = 10.9° \qquad\qquad\qquad F_1 = 474$ N $\qquad\qquad\qquad$ **Ans.**

2.4 Concept of moment

2.4.1 Moment about Point

$M=F.d$
F: the magnitude of the force,
d: the perpendicular distance from the
axis to the line of action of the force.

(a)

Direction. The direction of **M**O is defined by its *moment axis*, which is perpendicular to the plane that contains the force **F** and its moment arm d. The right-hand rule is used to establish the sense of direction of **M**O.


(b)

(c)

$M = Fd$

(d)
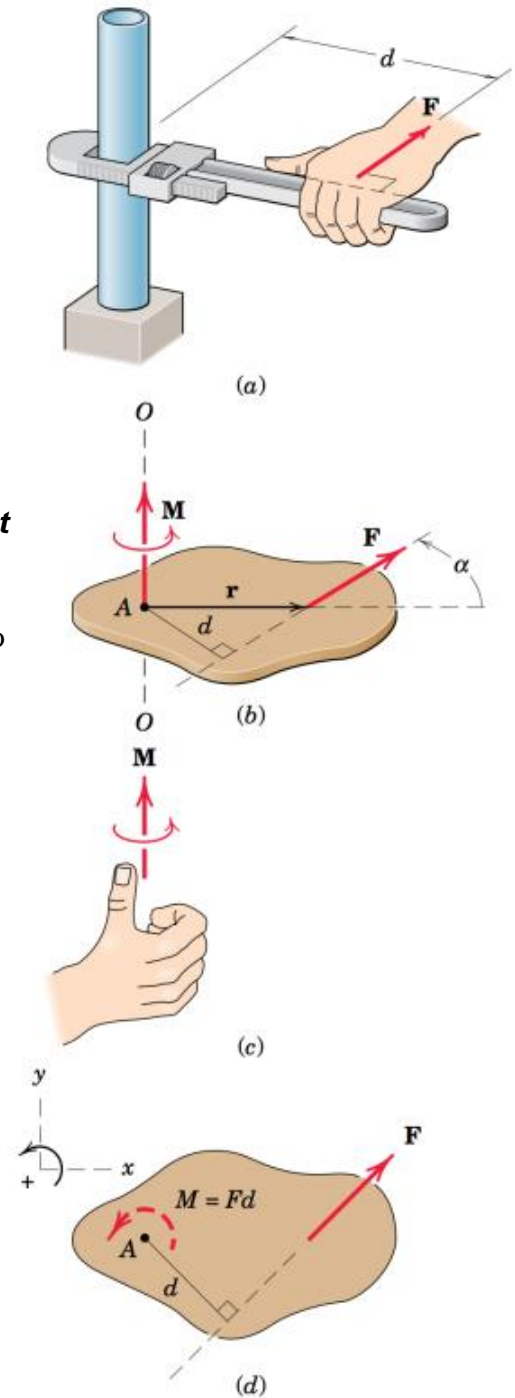
2.3.2 Varignon's Theorem

One of the most useful principles of mechanics is Varignon's theorem, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

$$M = M_x + M_y = (F_x * d_y) + (F_y * d_x)$$
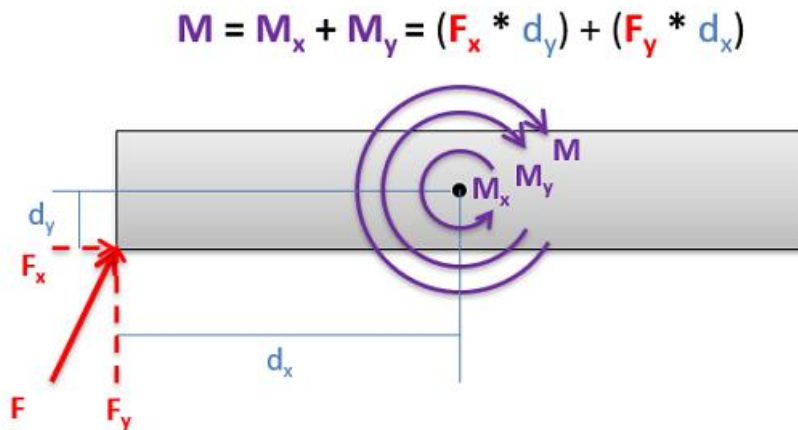
2.3.3 Resultant Moment

The resultant moment $(MR)O$ about point O (the z axis) can be determined by *finding the algebraic sum* of the moments caused by all the forces in the system
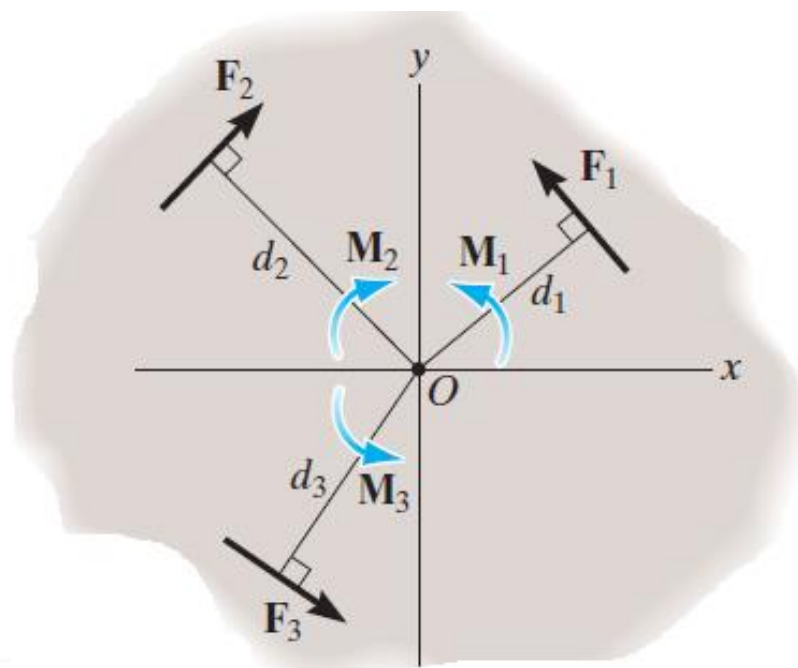
$$\zeta + (M_R)_o = \Sigma Fd; \qquad (M_R)_o = F_1d_1 - F_2d_2 + F_3d_3$$

Example

For each case illustrated in Fig. 4–4, determine the moment of the force about point O.

solution
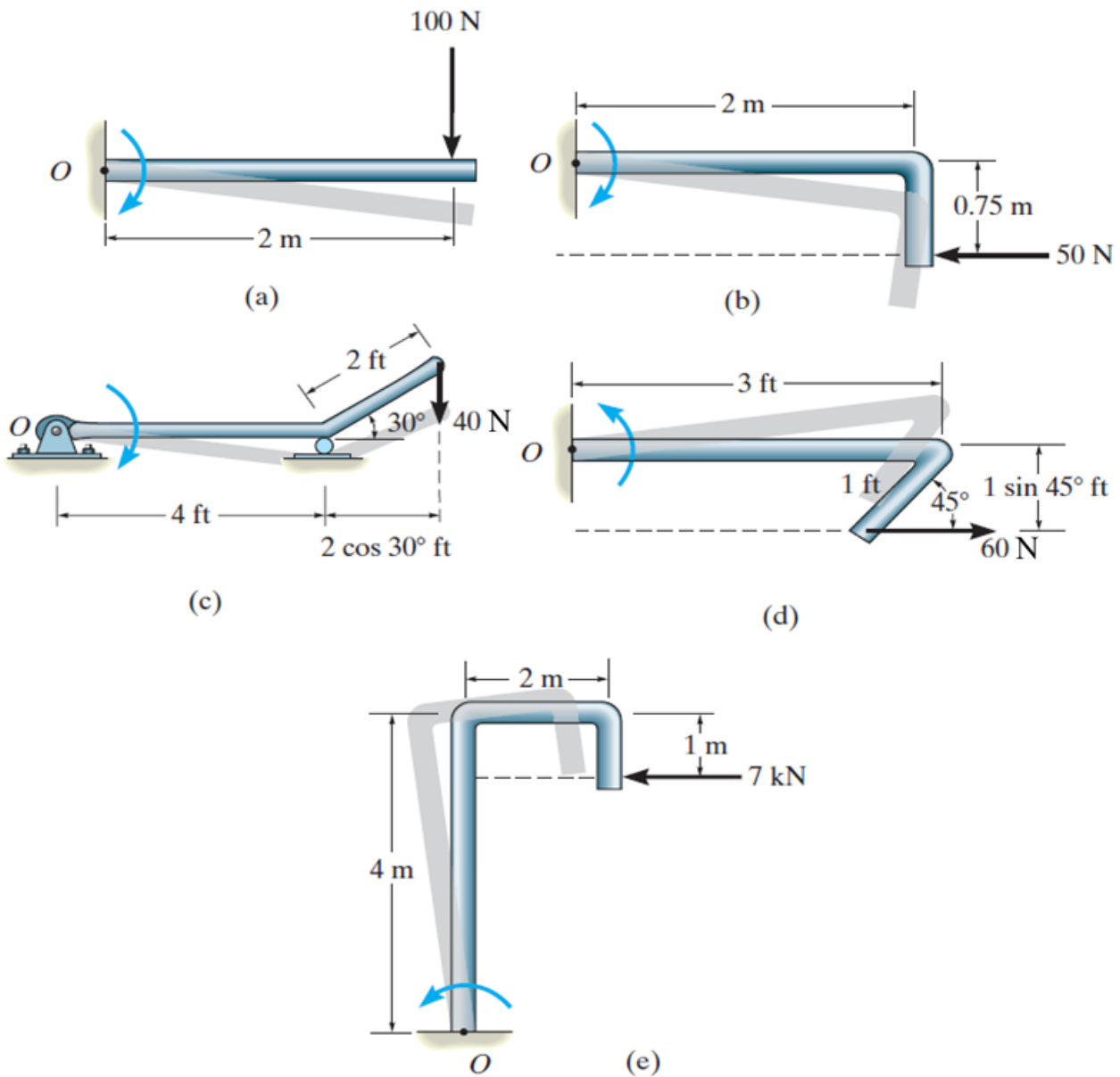
Fig. 4–4a $M_O = (100 \text{ N})(2 \text{ m}) = 200 \text{ N} \cdot \text{m} \, \circlearrowright$ Ans.

Fig. 4–4b $M_O = (50 \text{ N})(0.75 \text{ m}) = 37.5 \text{ N} \cdot \text{m} \, \circlearrowright$ Ans.

Fig. 4–4c $M_O = (40 \text{ N})(4 \text{ m} + 2 \cos 30° \text{ m}) = 229 \text{ N} \cdot \text{m} \, \circlearrowright$ Ans.

Fig. 4–4d $M_O = (60 \text{ N})(1 \sin 45° \text{ ft}) = 42.4 \text{ N} \cdot \text{m} \, \circlearrowright$ Ans.

Fig. 4–4e $M_O = (7 \text{ kN})(4 \text{ m} - 1 \text{ m}) = 21.0 \text{ kN} \cdot \text{m} \, \circlearrowright$ Ans.

(a)

(b)

(c)

(d)

(e)

Example

Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways.

Solution. **(I)** The moment arm to the 600-N force is

$$d = 4 \cos 40° + 2 \sin 40° = 4.35 \text{ m}$$

By $M = Fd$ the moment is clockwise and has the magnitude

$$M_O = 600(4.35) = 2610 \text{ N·m} \qquad \text{Ans.}$$

(II) Replace the force by its rectangular components at A,

$$F_1 = 600 \cos 40° = 460 \text{ N}, \qquad F_2 = 600 \sin 40° = 386 \text{ N}$$

By Varignon's theorem, the moment becomes

$$M_O = 460(4) + 386(2) = 2610 \text{ N·m} \qquad \text{Ans.}$$

(III) By the principle of transmissibility, move the 600-N force along its line of action to point B, which eliminates the moment of the component F_2. The moment arm of F_1 becomes

$$d_1 = 4 + 2 \tan 40° = 5.68 \text{ m}$$

and the moment is
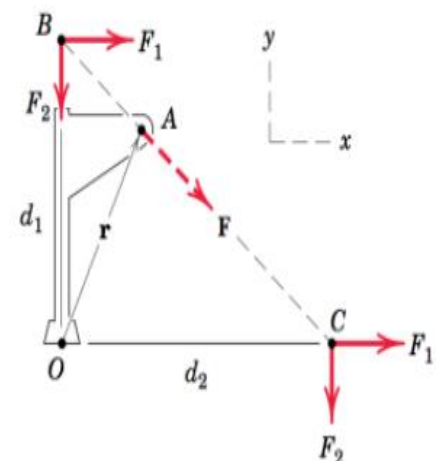
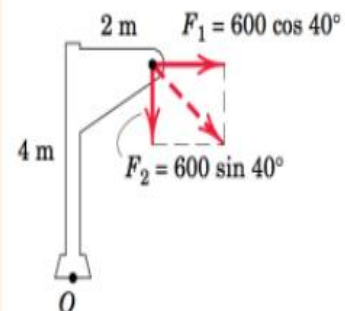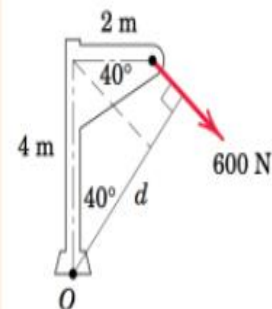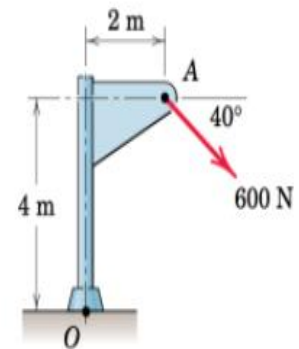$$M_O = 460(5.68) = 2610 \text{ N·m} \qquad \text{Ans.}$$

(IV) Moving the force to point C eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40° = 6.77 \text{ m}$$

and the moment is

$$M_O = 386(6.77) = 2610 \text{ N·m} \qquad \text{Ans.}$$

Example

An experimental device imparts a force of magnitude $F = 225$ N to the front edge of the rim at A to simulate the effect of a slam dunk. Determine the moments of the force F about point O and about point B. Finally, locate, from the base at O, a point C on the ground where the force imparts zero moment.
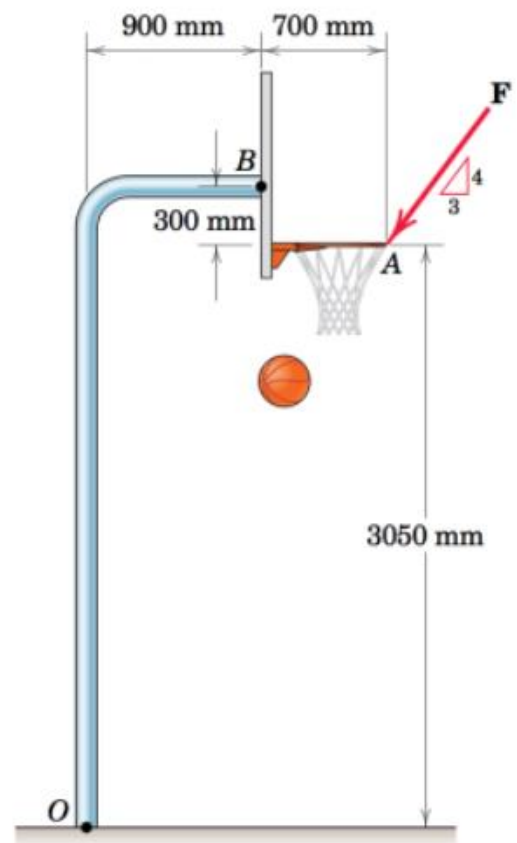
Example

The 30-N force P is applied perpendicular to the portion BC of the bent bar. Determine the moment of P about point B and about point A.
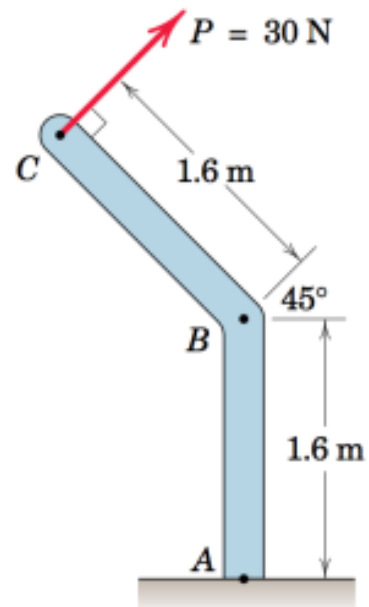
2.4.2 Couple

A couple consists of two forces which have equal magnitudes and parallel but opposite in sense. The moment of the couple is the algebraic sum of the moments of its forces about any axis perpendicular to the plane of the couple.

(a)

$$M = F(a + d) - Fa$$

$$M = Fd$$

(b)

Note : Because the total force exerted by a couple is zero, a couple is often represented by the moment it exerts. When the lines of action of the forces of a couple lie in the x y plane, the couple can be represented by its magnitude and a circular arrow that indicates its direction
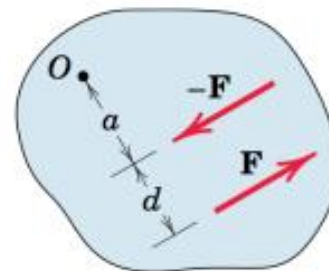
(c)

| Counterclockwise couple | Clockwise couple |

(d)

2.4.3 Equivalent couple

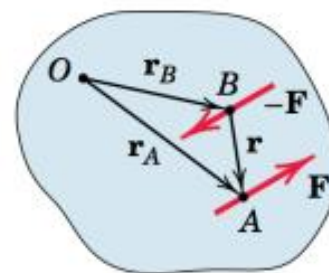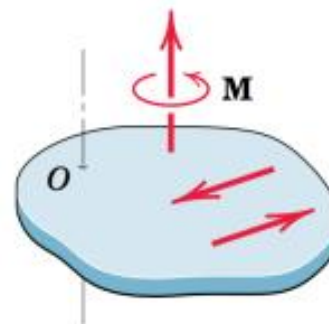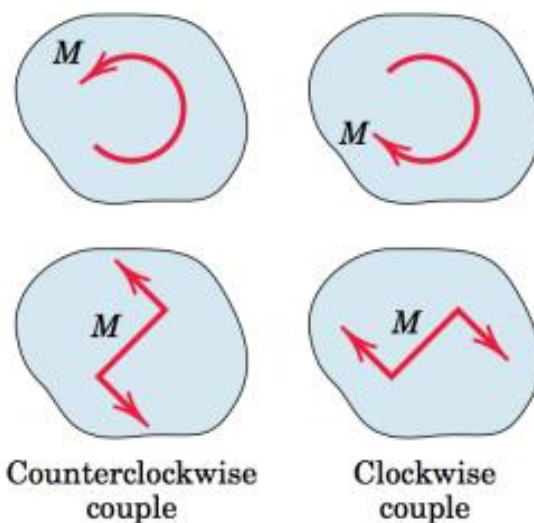If two couples produce a moment with the same magnitude and direction, then these two couples are equivalent

$(P = F)$
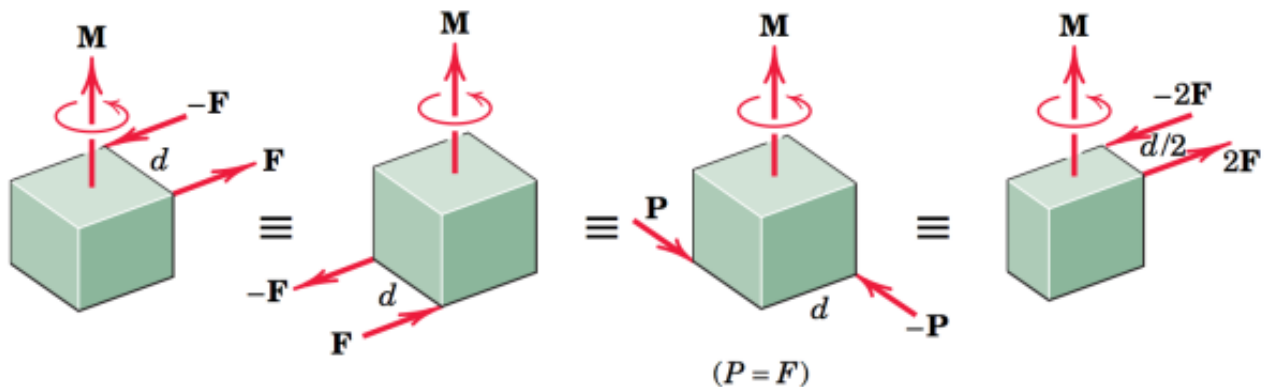
2.4.4 Force –couple system

When a force is moved to another point P that is not on its line of action, it will create the same external effects on the body if a couple moment is also applied to the body. The couple moment is determined by taking the moment of the force about point P.
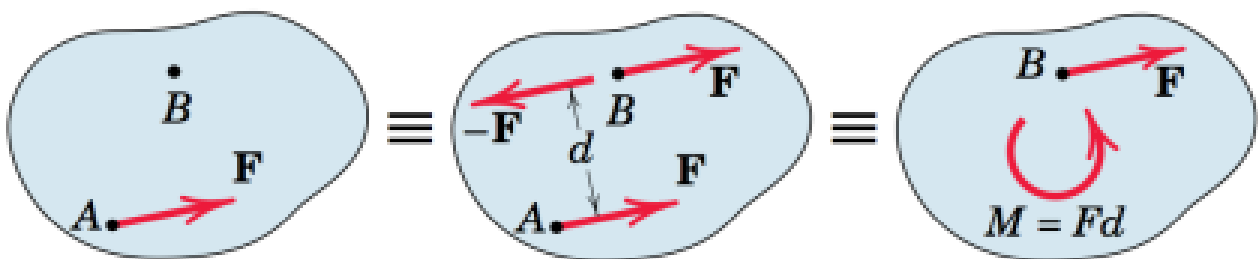
Example

Replace the horizontal 400-N force acting on the lever by an equivalent system consisting of a force at O and a couple.

Solution. We apply two equal and opposite 400-N forces at O and identify the counterclockwise couple

$[M = Fd]$ $M = 400(0.200 \sin 60°) = 69.3$ N·m   *Ans.*

> Thus, the original force is equivalent to the 400-N force at O and the 69.3-N·m couple as shown in the third of the three equivalent figures.

400 N · 400 N · 400 N · 400 N · 400 N · 400 N

69.3 N·m

400 N · 200 mm · 60° · O
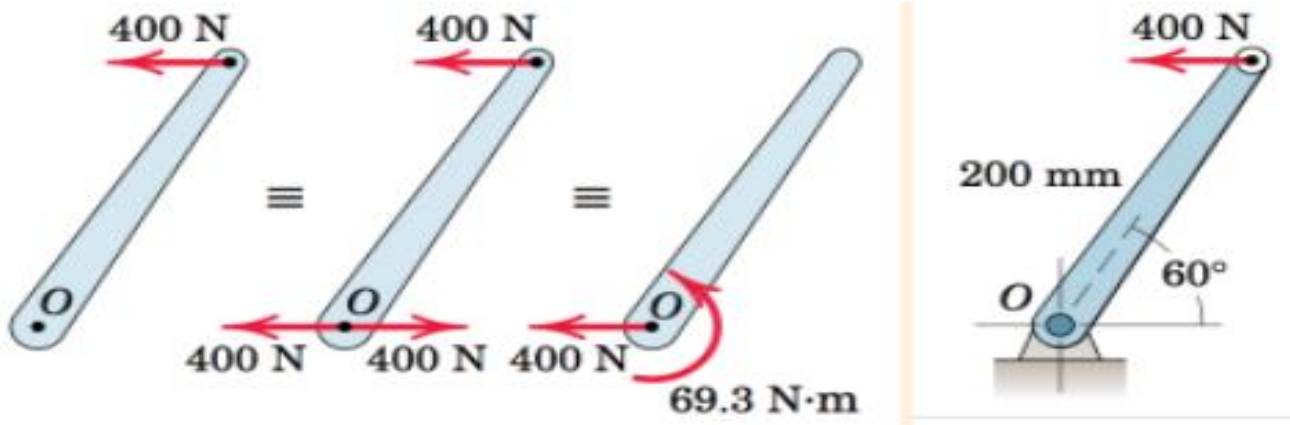
Example

Determine the resultant couple moment of the three couples acting on the plate in Fig. 4–30.

$F_1 = 200$ N

$F_3 = 300$ N

$d_1 = 4\,m$

$F_2 = 450$ N A

$d_3 = 5\,m$

$d_2 = 3\,m$

Fig. 4–30

$F_2 = 450$ N

$F_1 = 200$ N

$F_3 = 300$ N

B
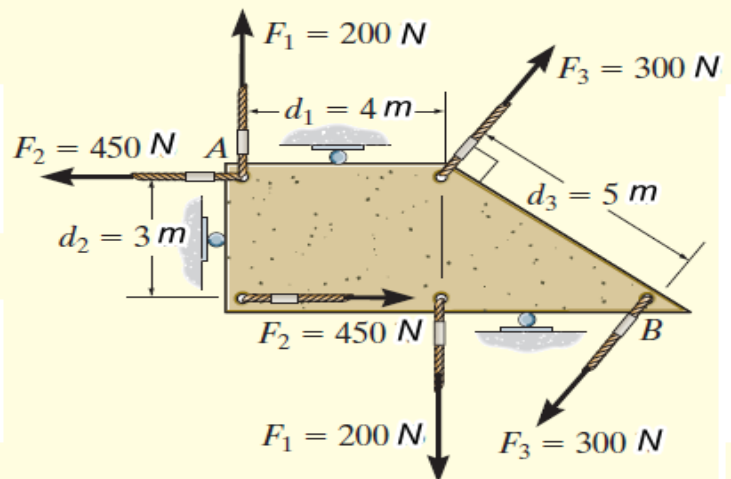
SOLUTION

As shown the perpendicular distances between each pair of couple forces are $d_1 = 4\,m$, $d_2 = 3\,m$, and $d_3 = 5\,m$. Considering counterclockwise couple moments as positive, we have

$\circlearrowleft + M_R = \Sigma M; \quad M_R = -F_1 d_1 + F_2 d_2 - F_3 d_3$

$\qquad = -(200\ N)(4\ m) + (450\ N)(3\ m) - (300\ N)(5\ m)$

$\qquad = -950\ \text{lb}\cdot\text{ft} = 950\ N\cdot m \circlearrowright \qquad\qquad$ *Ans.*

The negative sign indicates that $\mathbf{M}_R$ has a clockwise rotational sense.

Example
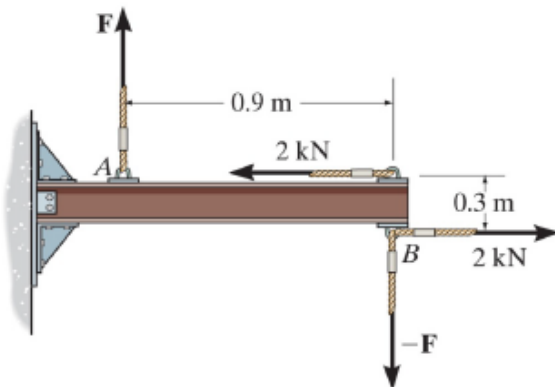
Given: Two couples act on the beam with the geometry shown.

Find: The magnitude of F so that the resultant couple moment is 1.5 kN·m clockwise.

Solution:

The net moment is equal to:

$$+ \Sigma M = -F (0.9) + (2) (0.3)$$

$$= -0.9 F + 0.6$$

$$-1.5 \text{ kN·m} \quad = -0.9 F + 0.6$$

F = 2.33 kN

Example

The device shown is a part of an automobile seat-back-release mechanism. The part is subjected to the 4-N force exerted at A and a 300-N·mm restoring moment exerted by a hidden torsional spring. Determine the y-intercept of the line of action of the single equivalent force.
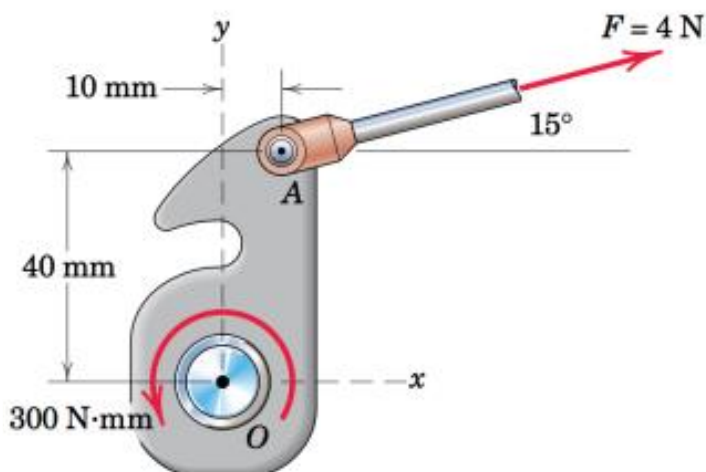
2.5 - Resultant of Parallel, Coplanar forces

Parallel forces can be in the same or in opposite directions. The sign of the direction can be chosen arbitrarily, meaning, taking one direction as positive makes the opposite direction negative. The complete definition of the resultant is according to its magnitude, direction, and line of action.

= 90N to the right

= 20N to the right
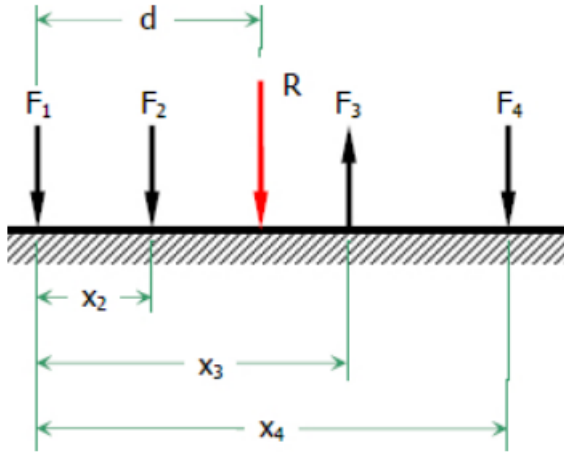
$$R = \Sigma F = F_1 + F_2 + F_3 + \ldots$$
$$Rd = \Sigma Fx = F_1 x_1 + F_2 x_2 + F_3 x_3 + \ldots$$

Example

$$R = F_1 + F_2 + F_3 + F_4$$

$$R = 150 - 600 + 100 - 250 = -600 \ N$$

$$Rd = F_1 d_1 + F_2 d_2 + F_3 d_3 + F_4 d_4$$

$$d = 3.13 \ m$$
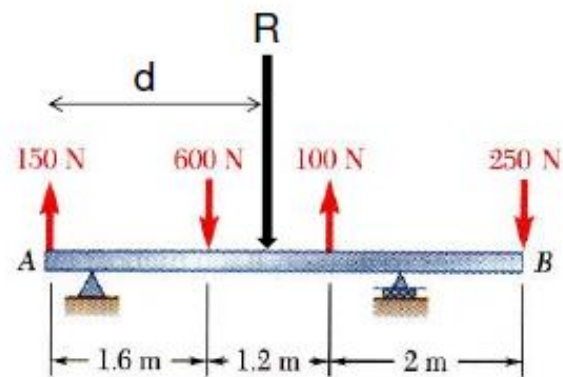
*Two parallel equal forces act in opposite direction
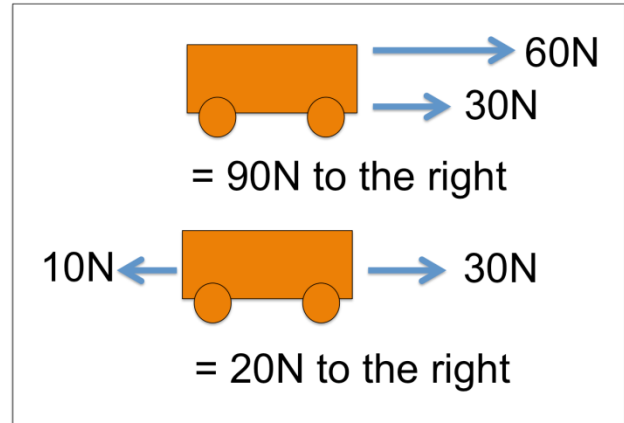
Two parallel forces that are equal in magnitude, opposite in direction, and not colinear will create a rotation effect. This type of pair is called a Couple. The placement of a couple in the plane is immaterial, meaning, its rotational effect to the body is not a function of its placement. The magnitude of the couple is given by
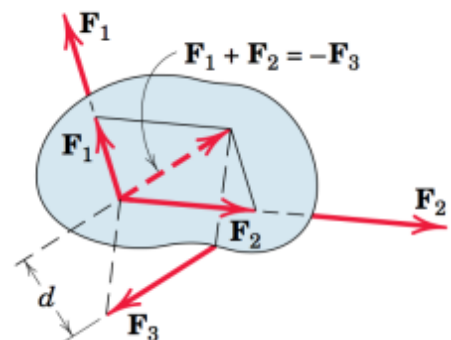
2.6 Resultant of Distributed Loads

The resultant of a distributed load is equal to the area of the load diagram. It is acting at the centroid of that area as indicated. The figure below shows the three common distributed loads namely; rectangular load, triangular load, and trapezoidal load.
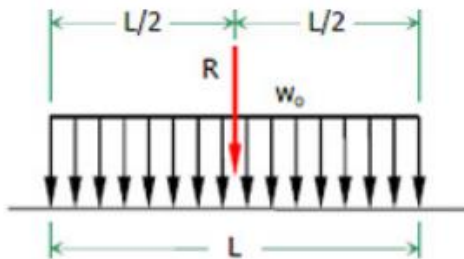
Rectangular Load

$$R = w_o L$$
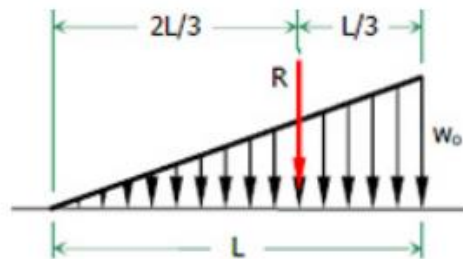
Triangular Load

$$R = \tfrac{1}{2} w_o L$$
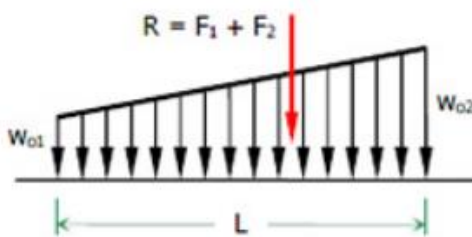
Trapezoidal Load

$$R = w_{o1} L + \tfrac{1}{2}(w_{o2} - w_{o1})L$$

Resolution of trapezoidal load into rectangular and triangular loads
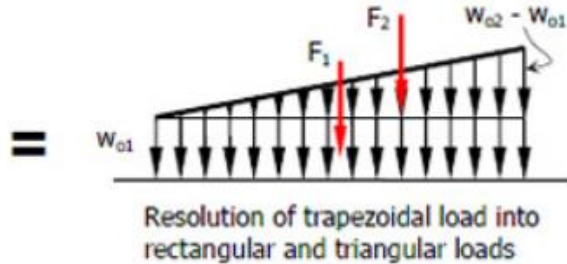
Example
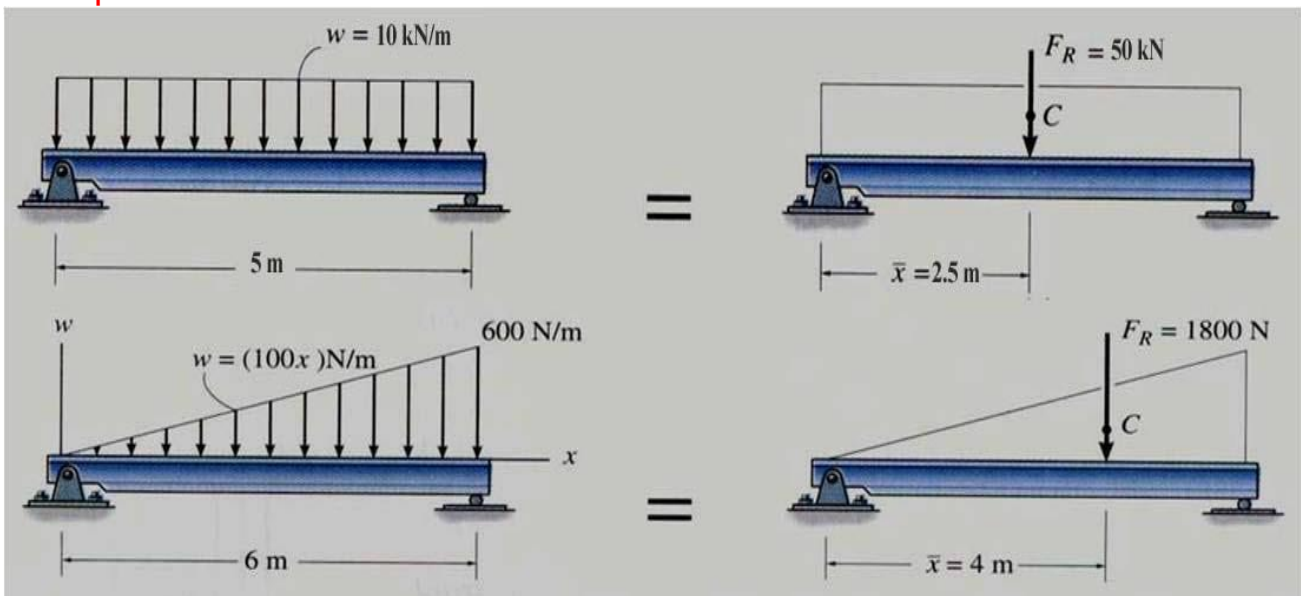
2.7 Resultant of Nonconcurrent, Nonparallel, Coplanar forces

The resultant of a force system is the simplest force system which can replace the original system without changing its external effect on a rigid body. The resultant of a force system can be:
1- a single force,
2- a pair of parallel forces having the same magnitudes but opposite sense (called a *couple*),or
3- a force and a couple.
If the resultant is a force and a couple, the force will not be parallel to the plane containing the couple.

(a)

(b)

(c)

(d)
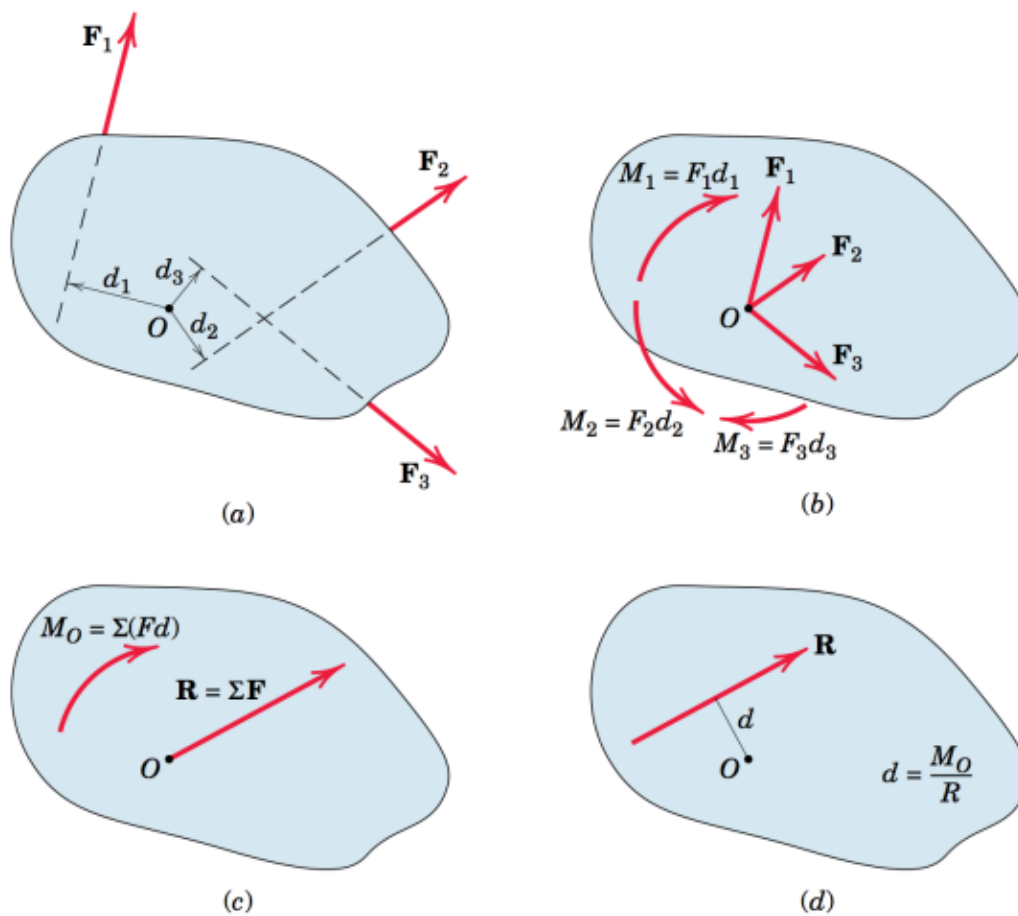
This process is summarized in equation form by

$$\mathbf{R} = \Sigma\mathbf{F}$$
$$M_O = \Sigma M = \Sigma(Fd)$$
$$Rd = M_O$$

Sample Problem 2/9

Determine the resultant of the four forces and one couple which act on the plate shown.

(b) R = 148.3 N, 1.600 m, A, 63.2°, O, B, x

(c) 132.4x − 66.9y = −237, C, y, O, R, b, x

Top right figure: 2 m, 5 m, 60 N, 50 N, 45°, 2 m, 2 m, 40 N, 1 m, 140 N·m, 80 N, O, 30°, x

(a) y, R = 148.3 N, θ = 63.2°, |M_O| = 237 N·m, O, x
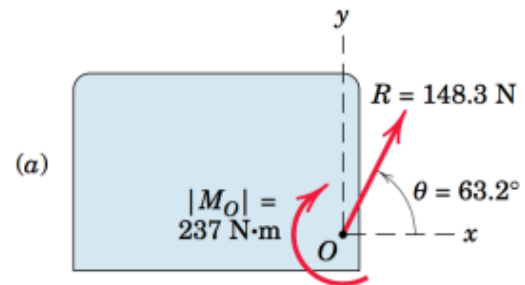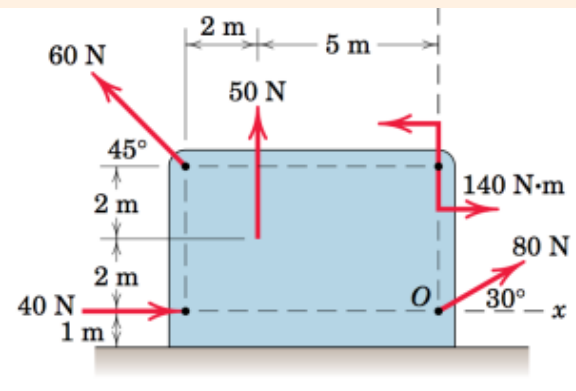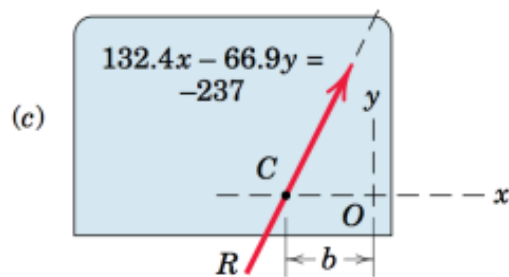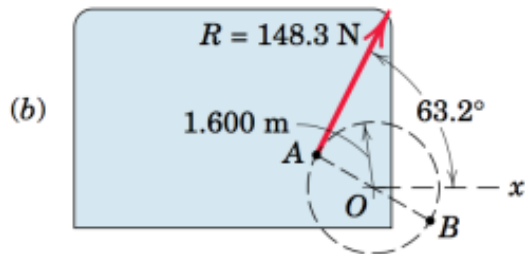
$[R_x = \Sigma F_x]$ $\qquad\qquad R_x = 40 + 80 \cos 30° - 60 \cos 45° = 66.9 \text{ N}$

$[R_y = \Sigma F_y]$ $\qquad\qquad R_y = 50 + 80 \sin 30° + 60 \cos 45° = 132.4 \text{ N}$

$[R = \sqrt{R_x^2 + R_y^2}]$ $\qquad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3 \text{ N}$

$\left[\theta = \tan^{-1} \dfrac{R_y}{R_x} \right]$ $\qquad \theta = \tan^{-1} \dfrac{132.4}{66.9} = 63.2°$

$[M_O = \Sigma(Fd)]$ $\qquad M_O = 140 - 50(5) + 60 \cos 45°(4) - 60 \sin 45°(7)$

$\qquad\qquad\qquad\qquad\qquad = -237 \text{ N·m}$

$[Rd = |M_O|]$ $\qquad\qquad 148.3d = 237 \qquad d = 1.600 \text{ m}$

$R_y b = |M_O|$ $\qquad$ and $\qquad b = \dfrac{237}{132.4} = 1.792 \text{ m}$

Example

Replace the loading acting on the beam by a single resultant force. Specify where the force acts, measured from end A.
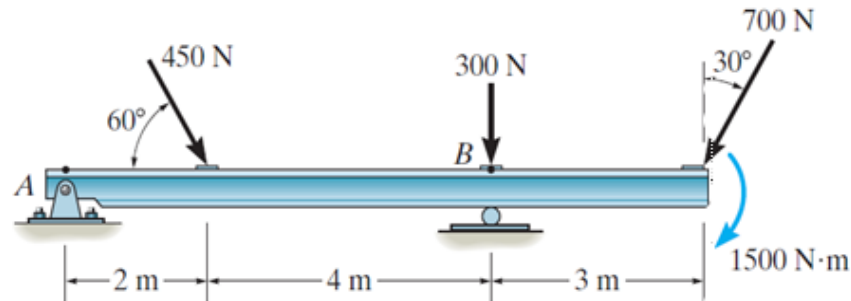
SOLUTION

$$\xrightarrow{+} F_{Rx} = \Sigma F_x; \qquad F_{Rx} = 450 \cos 60° - 700 \sin 30° = -125 \text{ N} = 125 \text{ N} \quad \leftarrow$$

$$+\uparrow F_{Ry} = \Sigma F_y; \qquad F_{Ry} = -450 \sin 60° - 700 \cos 30° - 300 = -1296 \text{ N} = 1296 \text{ N} \quad \downarrow$$

$$F = \sqrt{(-125)^2 + (-1296)^2} = 1302 \text{ N} \qquad \qquad \textbf{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{1296}{125}\right) = 84.5° \qquad \nearrow \qquad \qquad \textbf{Ans.}$$

$$\zeta + M_{RA} = \Sigma M_A; \qquad 1296(x) = 450 \sin 60°(2) + 300(6) + 700 \cos 30°(9) + 1500$$

$$x = 7.36 \text{ m} \qquad \qquad \textbf{Ans.}$$

2/86 If the resultant of the loads shown passes through point B, determine the equivalent force–couple system at O.
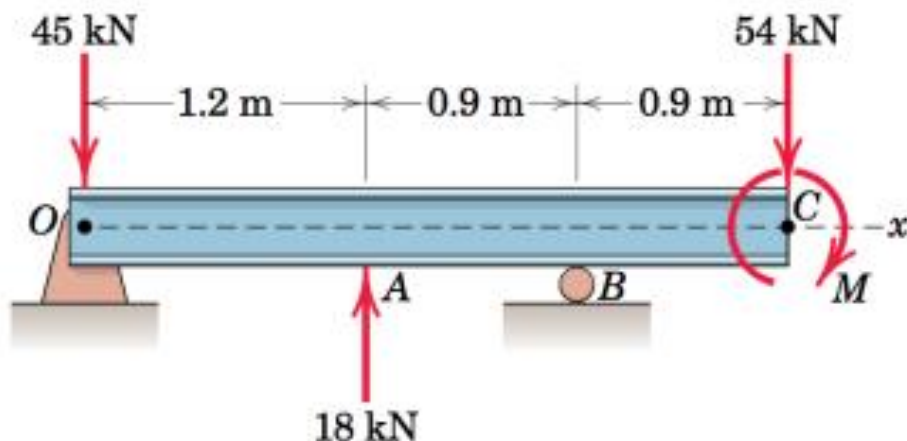
Replace the force and couple moment system acting on the beam in Fig. 4–44a by an equivalent resultant force, and find where its line of action intersects the beam, measured from point O.

(a)

(b)
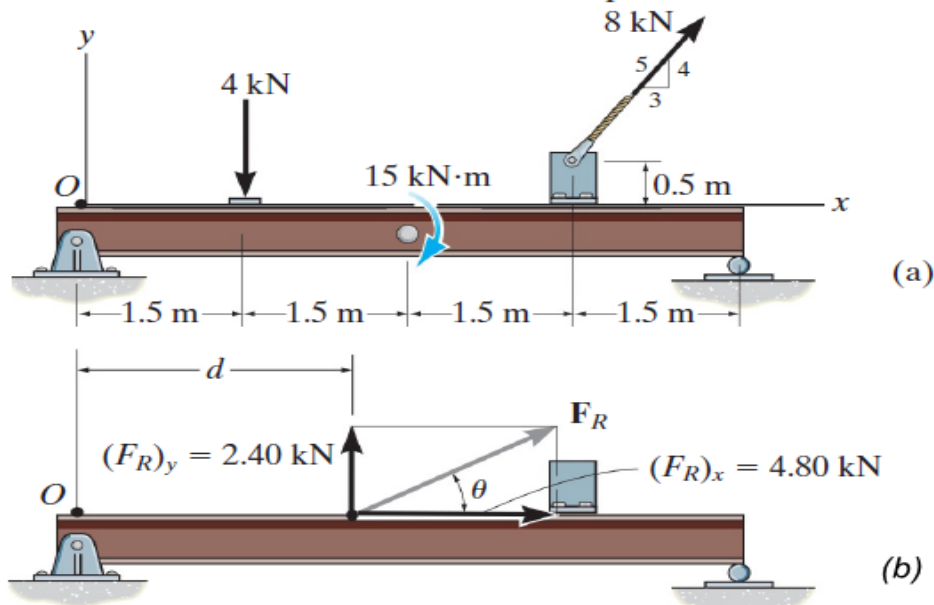
SOLUTION

Force Summation. Summing the force components,

$$\xrightarrow{+}(F_R)_x = \Sigma F_x; \qquad (F_R)_x = 8 \text{ kN}\left(\tfrac{3}{5}\right) = 4.80 \text{ kN} \rightarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \qquad (F_R)_y = -4 \text{ kN} + 8 \text{ kN}\left(\tfrac{4}{5}\right) = 2.40 \text{ kN}\uparrow$$

From Fig. 4–44b, the magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(4.80 \text{ kN})^2 + (2.40 \text{ kN})^2} = 5.37 \text{ kN} \qquad \textit{Ans.}$$

The angle θ is

$$\theta = \tan^{-1}\left(\frac{2.40 \text{ kN}}{4.80 \text{ kN}}\right) = 26.6° \qquad \textit{Ans.}$$

Moment Summation. We must equate the moment of $\mathbf{F}_R$ about point O in Fig. 4–44b to the sum of the moments of the force and couple moment system about point O in Fig. 4–44a. Since the line of action of $(\mathbf{F}_R)_x$ acts through point O, *only* $(\mathbf{F}_R)_y$ *produces a moment* about this point. Thus,

$$\zeta + (M_R)_O = \Sigma M_O; \qquad 2.40 \text{ kN}(d) = -(4 \text{ kN})(1.5 \text{ m}) - 15 \text{ kN} \cdot \text{m}$$

$$-\left[8 \text{ kN}\left(\tfrac{3}{5}\right)\right](0.5 \text{ m}) + \left[8 \text{ kN}\left(\tfrac{4}{5}\right)\right](4.5 \text{ m})$$

$$d = 2.25 \text{ m} \qquad \textit{Ans.}$$

Example

Replace the force system acting on the post by a resultant force, and specify where its line of action intersects the post AB measured from point A.
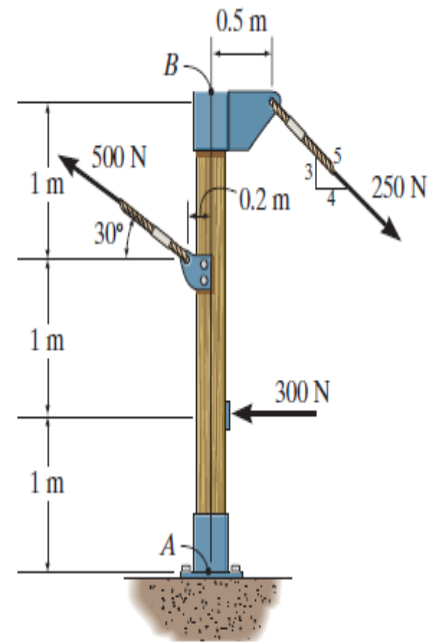
SOLUTION

Equivalent Resultant Force: Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are resolved into their x and y components, Fig. a. Summing these force components algebraically along the x and y axes,

$$\xrightarrow{+}(F_R)_x = \Sigma F_x; \quad (F_R)_x = 250\left(\frac{4}{5}\right) - 500\cos 30° - 300 = -533.01 \text{ N} = 533.01 \text{ N} \leftarrow$$

$$+\uparrow(F_R)_y = \Sigma F_y; \quad (F_R)_y = 500\sin 30° - 250\left(\frac{3}{5}\right) = 100 \text{ N}\uparrow$$

The magnitude of the resultant force $\mathbf{F}_R$ is given by

$$F_R = \sqrt{(F_R)_x{}^2 + (F_R)_y{}^2} = \sqrt{533.01^2 + 100^2} = 542.31 \text{ N} = 542 \text{ N} \qquad \textbf{Ans.}$$

The angle θ of $\mathbf{F}_R$ is

$$\theta = \tan^{-1}\left[\frac{(F_R)_y}{(F_R)_x}\right] = \tan^{-1}\left[\frac{100}{533.01}\right] = 10.63° = 10.6° \;\searrow \qquad \textbf{Ans.}$$
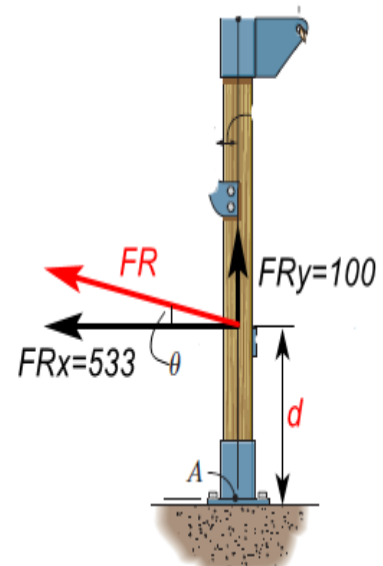
Location of the Resultant Force: Applying the principle of moments, Figs. a and b, and summing the moments of the force components algebraically about point A,

$$\zeta+(M_R)_A = \Sigma M_A; \quad 533.01(d) = 500\cos 30°(2) - 500\sin 30°(0.2) - 250\left(\frac{3}{5}\right)(0.5) - 250\left(\frac{4}{5}\right)(3) + 300(1)$$

$$d = 0.8274 \text{ mm} = 827 \text{ mm} \qquad \textbf{Ans.}$$

2/88 If the resultant of the forces shown passes through point A, determine the magnitude of the unknown tension T_2 which acts on the braked pulley.

275 N
1600 N
1.5 m
0.5 m
1.8 m
O
300 N
A
1.25 m
B
50°
0.6 m
T_2
650 N
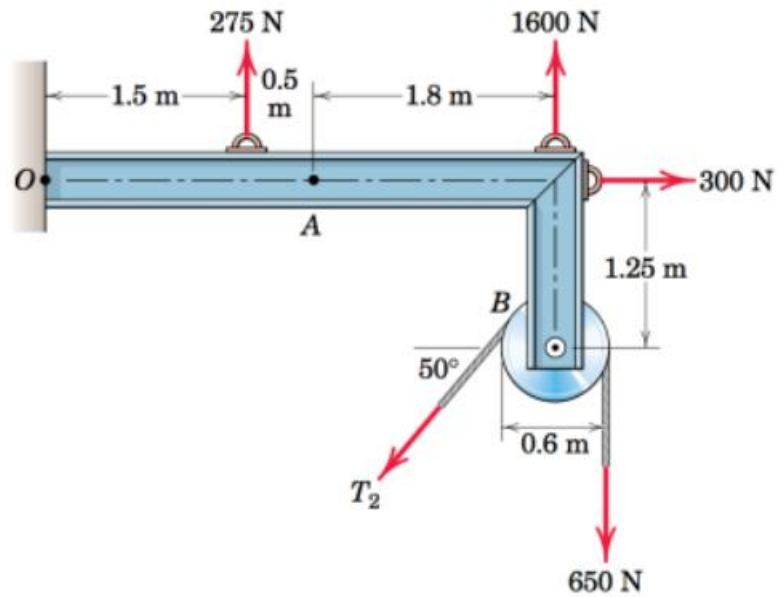
2/87 If the resultant of the two forces and couple M passes through point O, determine M.
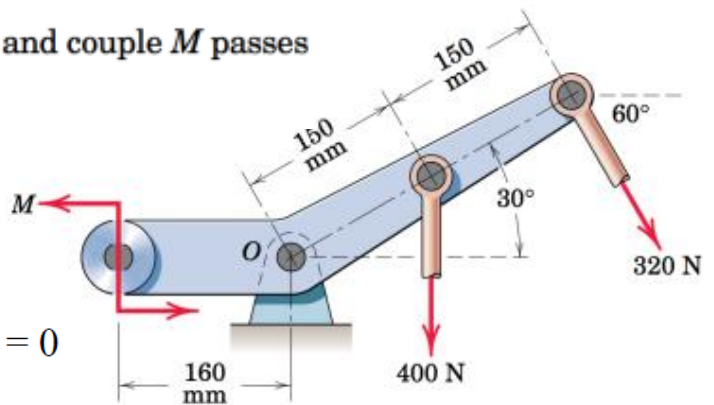
150 mm
60°
150 mm
30°
M
O
320 N
$M_O = M - 400 \times 0.15\cos30 - 320 \times 0.3 = 0$
160 mm
400 N
$M = 148 \text{ Nm CCW}$

Example : Replace the three forces acting on the bent beam by a single equivalent force R. Specify the distance x from the point O in which the line of action of R passes.

y
200 N
160 N
250 mm
O
x
250 mm
250 mm
240 N
125 mm

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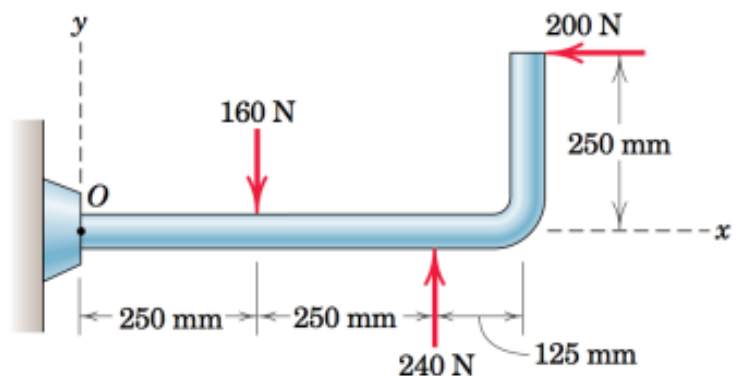
CHAPTER THREE
EQUILIBRIUM

</div>

3.1 General

the term "equilibrium" or, more specifically, "static equilibrium" is used to describe an object at rest. To maintain equilibrium, it is necessary to satisfy Newton's first law of motion, which requires the resultant force acting on a particle to be equal to zero. This condition may be stated mathematically as:

"When a body is in equilibrium, the resultant of all forces acting on it is zero" which leads to:

(Summation of internal and external forces and moments equals to zero)

3.2 Free Body diagram (F.B.D)

To apply the equation of equilibrium, we must account for all the known and unknown forces (F) which act on the body. The best way to do this is to think of the body as isolated and "free" from its surroundings .A drawing that shows the particle with all the forces that act on it is called a ***free-body diagram*** (FBD)

Before presenting a formal procedure as to how to draw a free-body diagram, we will first consider three types of supports often encountered in particle equilibrium problems.

1-Spring

$$F = k\,x$$

k = stiffness of spring   (N/m)
x = distance measured from its *unloaded* position (m)

$$F_{spring} = kx$$
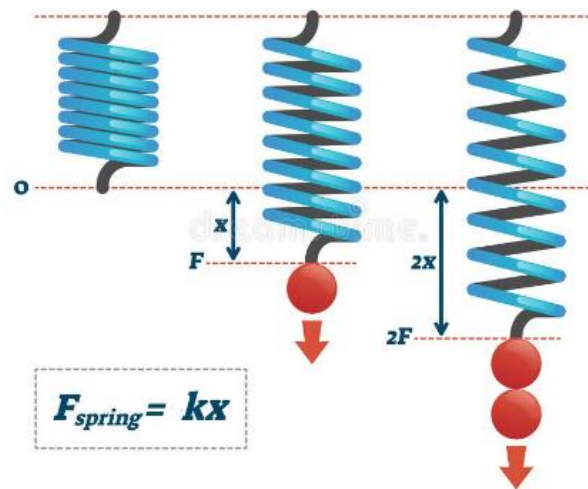
2-Smooth Contact

If an object rests on a *smooth surface*, then the surface will exert a force on the object that is normal to the surface at the point of contact.

(a)   (b)

(b)

(c)

Effect of B acting on A

Effect of sloped blade acting on A

Effect of gravity (weight) acting on A

Effect of sloped fork acting on A

(e)
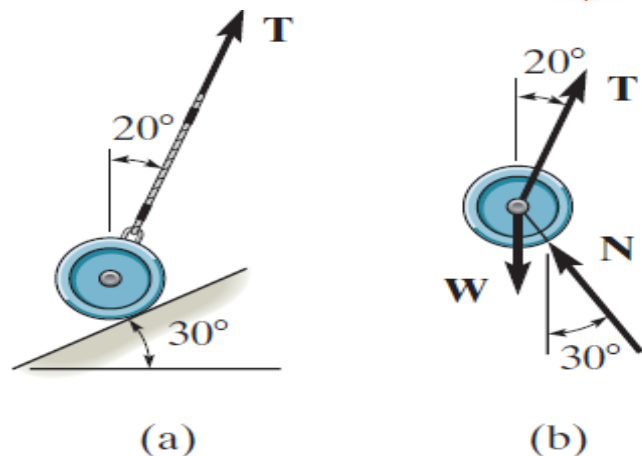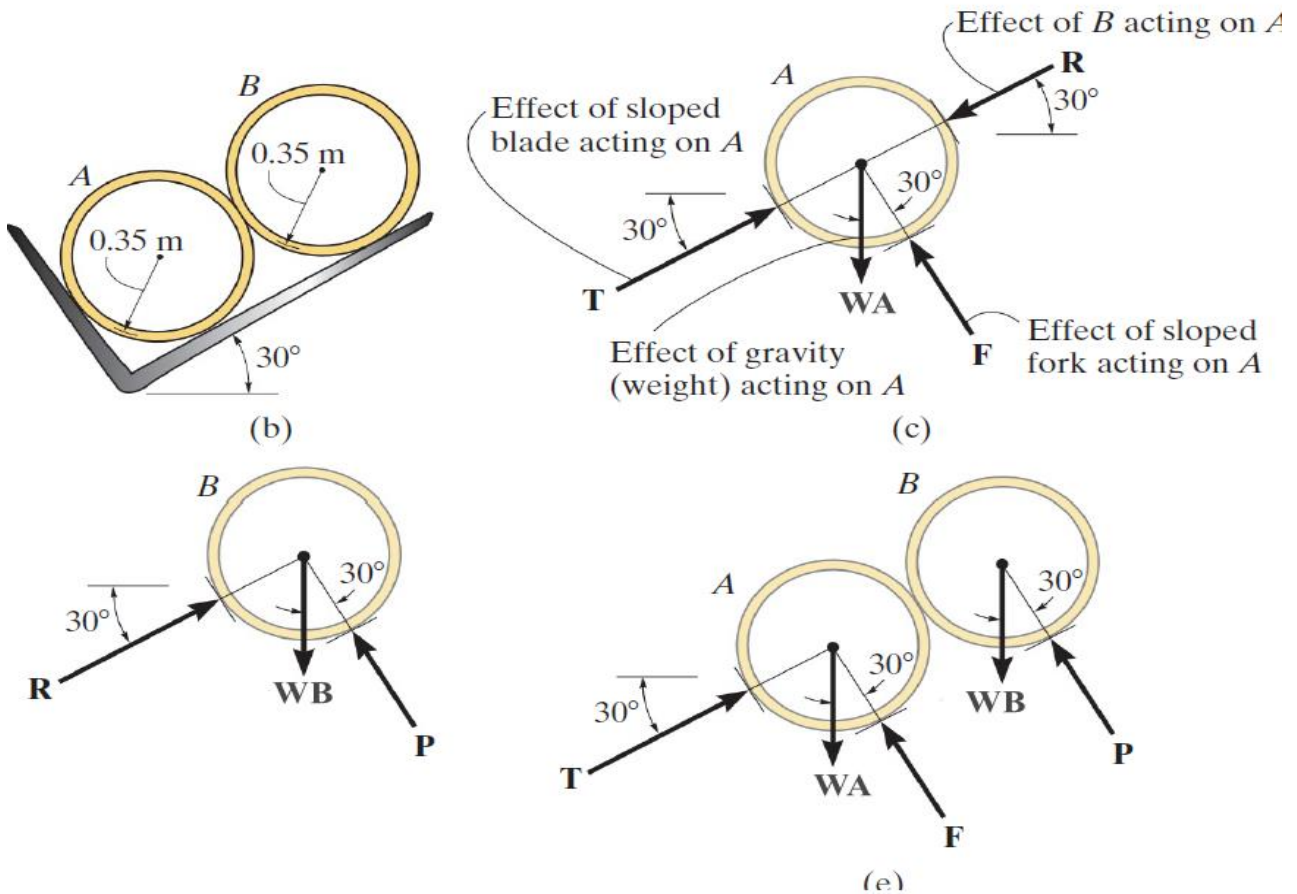
3-Cables and Pulleys. Unless otherwise stated throughout this book, except in Sec. 7.4, all cables (or cords) will be assumed to have negligible weight and they cannot stretch. Also, a cable can support *only* a tension or "pulling" force, and this force always acts in the direction of the cable. The tension force developed in a continuous cable which passes over a frictionless pulley must have a *constant* magnitude to keep the cable in equilibrium. Hence, for an angle θ, shown in figure, the cable is subjected to a constant tension T throughout its length.

W=T

W=2T

W=T+Tsin θ

MA ≈ 2

MA ≈ 3

MA ≈ 4

(a)

(b)

(c)
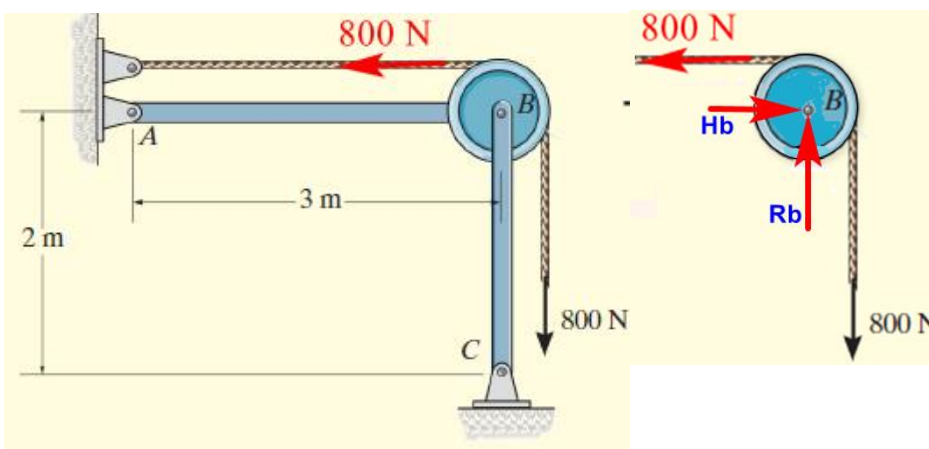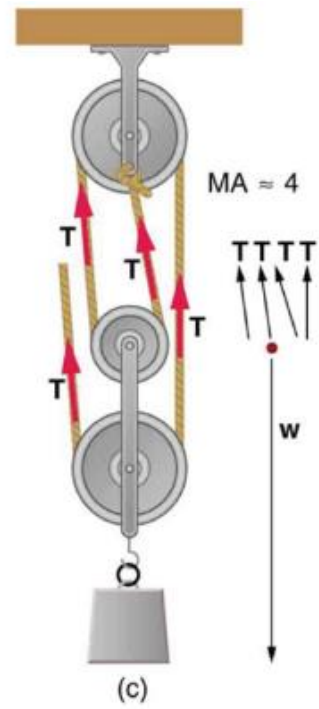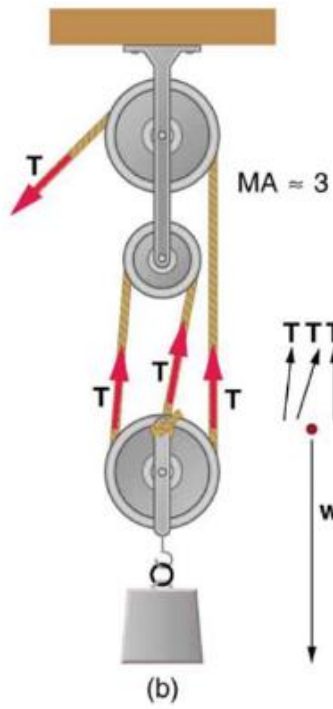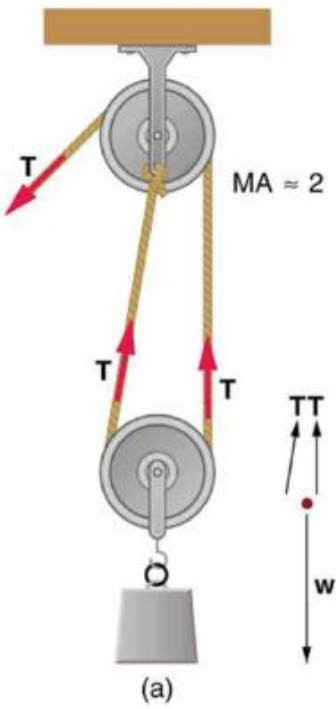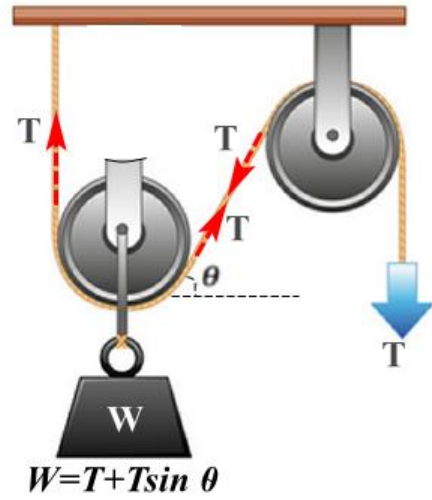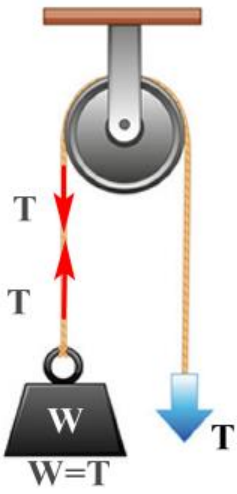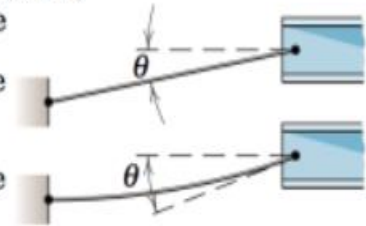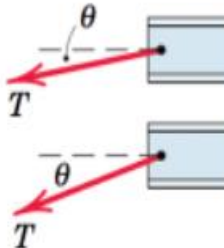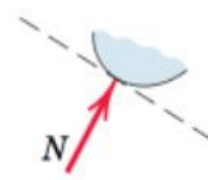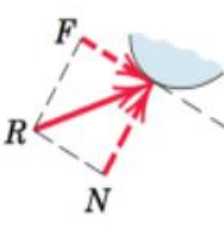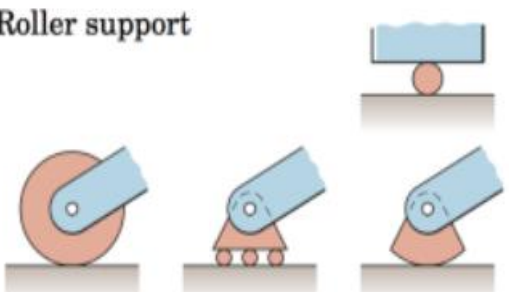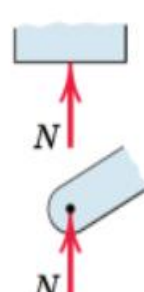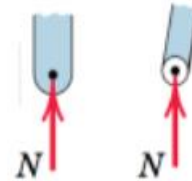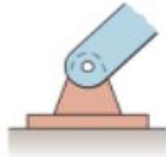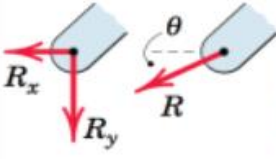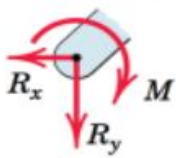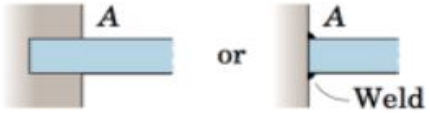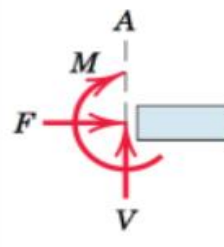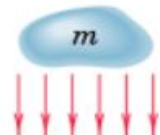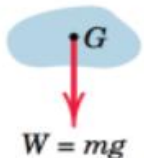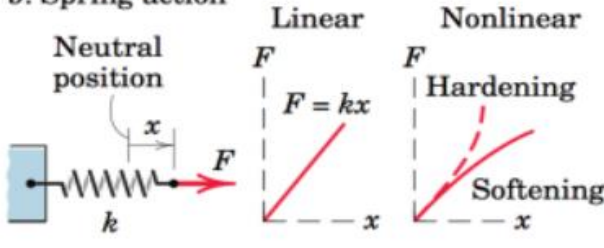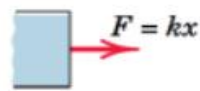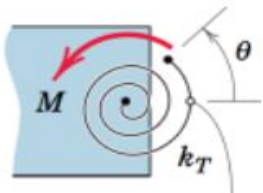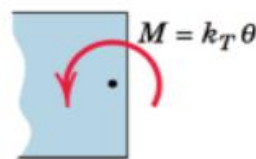
Example: Determine the value of P, if the load W= 800 N.
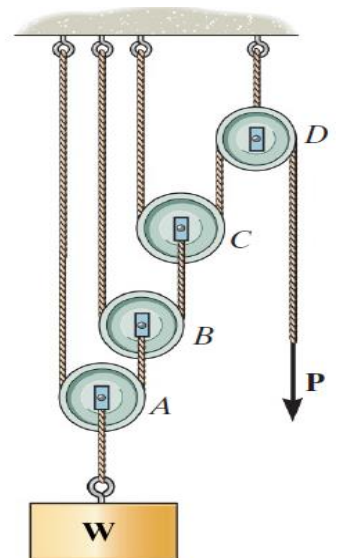
3.2.1 Modeling the Action of Forces in 2D Analysis

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	Action on Body to Be Isolated
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible	θ T θ T Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces	N Contact force is compressive and is normal to the surface.
3. Rough surfaces	F R N Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support	N N Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide	N N Collar or slider free to move along smooth guides; can support force normal to guide only.

Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection	**Pin free to turn** A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M. **Pin not free to turn**
7. Built-in or fixed support A or A —Weld	A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction m	The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of gravity G. $W = mg$
9. Spring action Neutral position **Linear** **Nonlinear** F F $F = kx$ Hardening Softening k	Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance. $F = kx$
10. Torsional spring action M θ k_T Neutral position	For a linear torsional spring, the applied moment M is proportional to the angular deflection θ from the neutral position. The stiffness k_T is the moment required to deform the spring one radian. $M = k_T\theta$

CONSTRUCTION OF FREE-BODY DIAGRAMS

The full procedure for drawing a free-body diagram which isolates a body or system consists of the following steps.
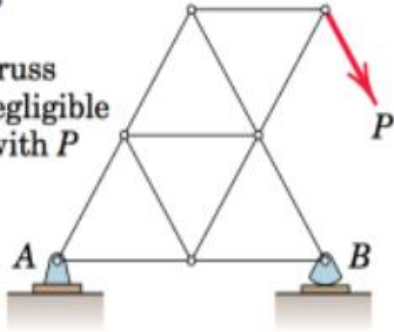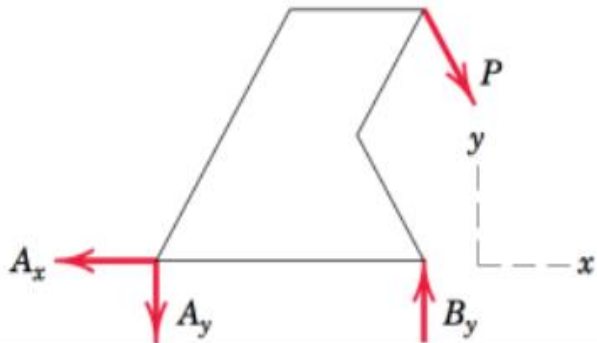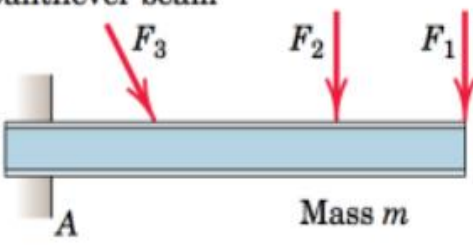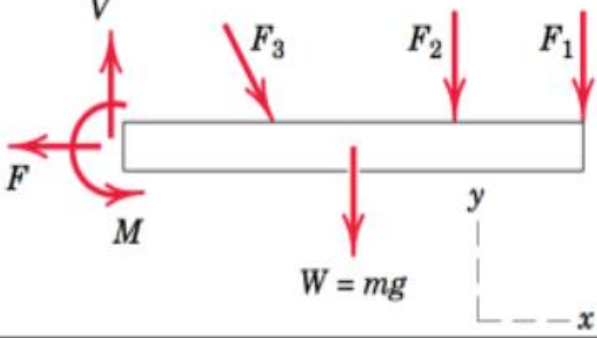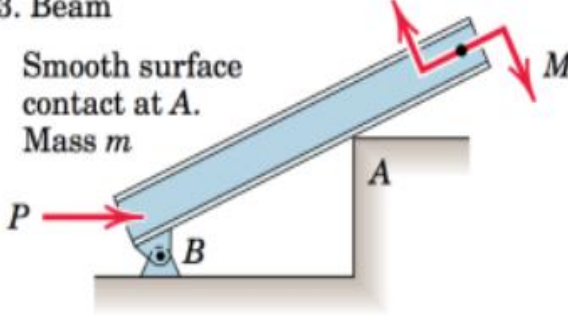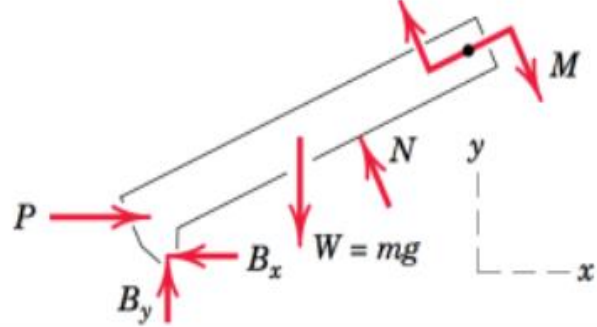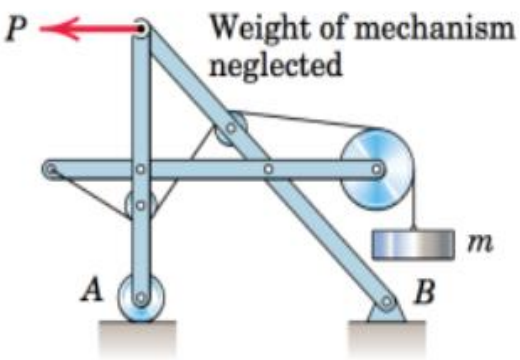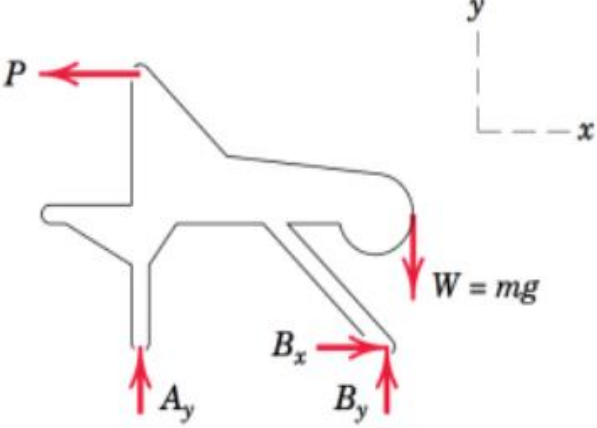
Step 1. Decide which system to isolate. The system chosen should usually involve one or more of the desired unknown quantities.
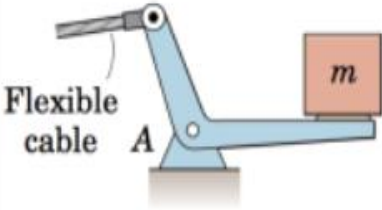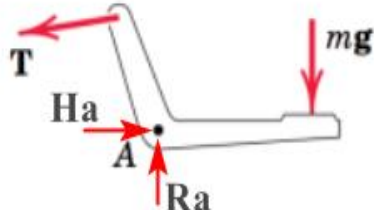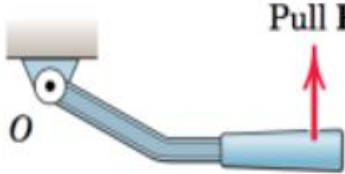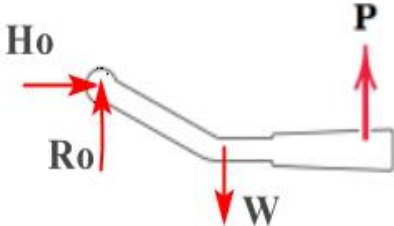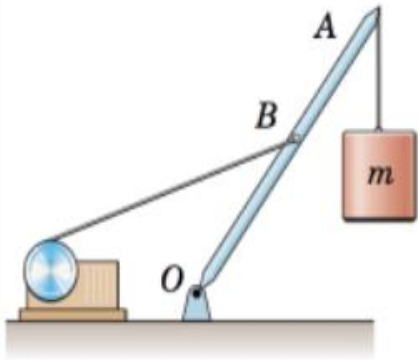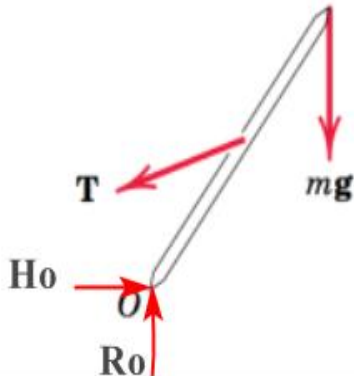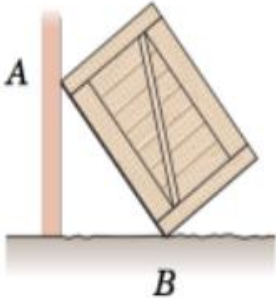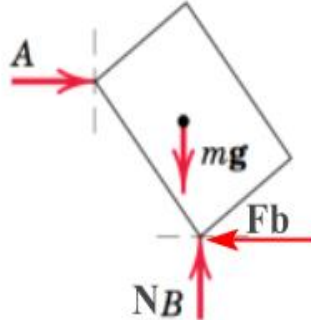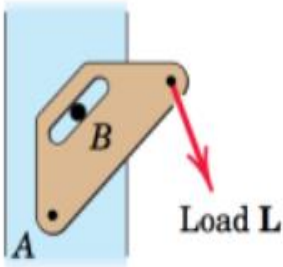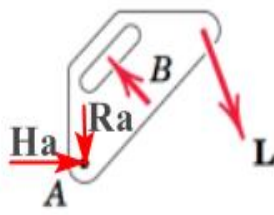
Step 2. Next isolate the chosen system by drawing a diagram which represents its *complete external boundary*. This boundary defines the isolation of the system from *all* other attracting or contacting bodies, which are considered removed. This step is often the most crucial of all. Make certain that you have *completely isolated* the system before proceeding with the next step.
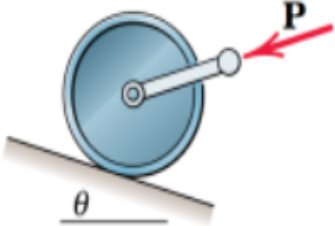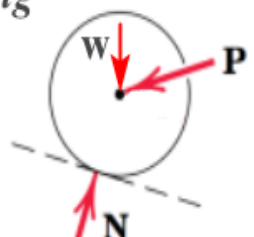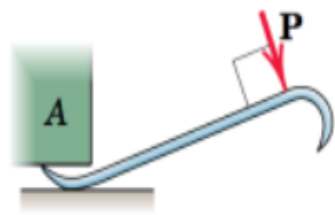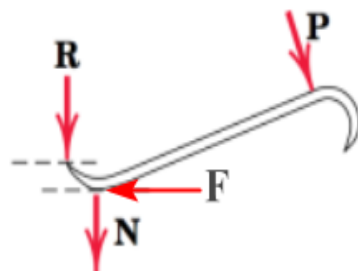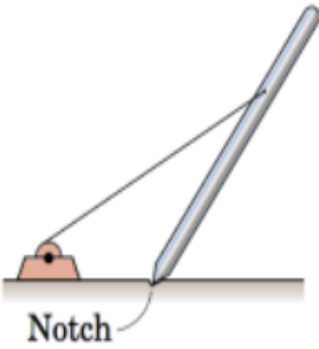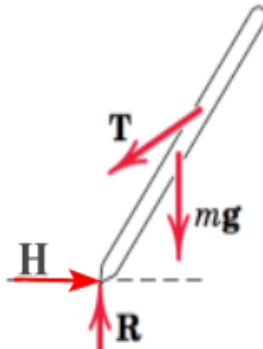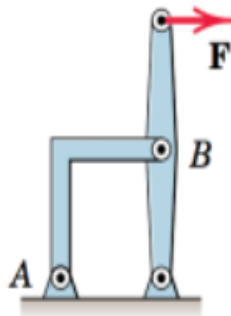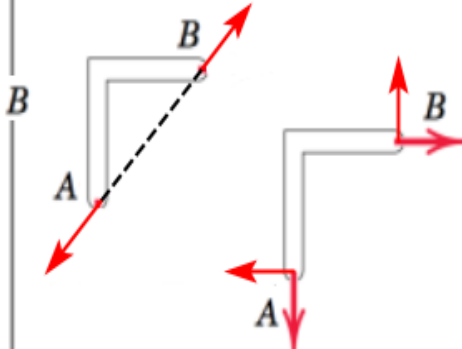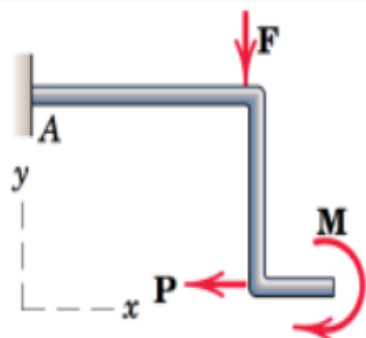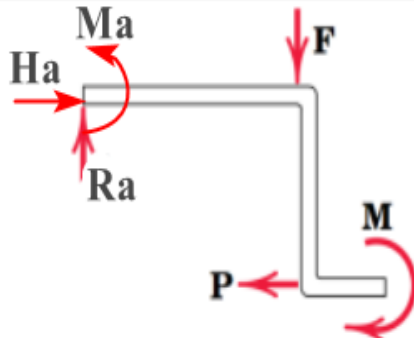
Step 3. Identify all forces which act *on* the isolated system as applied *by* the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system. Make a systematic traverse of the entire boundary to identify all contact forces. Include body forces such as weights, where appreciable. Represent all known forces by vector arrows, each with its proper magnitude, direction, and sense indicated. Each unknown force should be represented by a vector arrow with the unknown magnitude or direction indicated by symbol. If the sense of the vector is also unknown, you must arbitrarily assign a sense. The subsequent calculations with the equilibrium equations will yield a positive quantity if the correct sense was assumed and a negative quantity if the incorrect sense was assumed. It is necessary to be *consistent* with the assigned characteristics of unknown forces throughout all of the calculations. If you are consistent, the solution of the equilibrium equations will reveal the correct senses.

Step 4. Show the choice of coordinate axes directly on the diagram. Pertinent dimensions may also be represented for convenience. Note, however, that the free-body diagram serves the purpose of focusing attention on the action of the external forces, and therefore the diagram should not be cluttered with excessive extraneous information. Clearly distinguish force arrows from arrows representing quantities other than forces. For this purpose a colored pencil may be used.

3.2.3 Examples of F.B.D

SAMPLE FREE-BODY DIAGRAMS	
Mechanical System	Free-Body Diagram of Isolated Body
1. Plane truss Weight of truss assumed negligible compared with P	
2. Cantilever beam Mass m	
3. Beam Smooth surface contact at A. Mass m	
4. Rigid system of interconnected bodies analyzed as a single unit Weight of mechanism neglected	

	Body	complete FBD
1. Bell crank supporting mass m with pin support at A.	Flexible cable A, mass m	T, Ha, A, Ra, mg
2. Control lever applying torque to shaft at O.	O, Pull P	Ho, Ro, W, P
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.	A, B, mass m, O	T, mg, Ho, O, Ro
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.	A, B	A, mg, Fb, NB
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.	B, A, Load L	B, Ra, Ha, A, L

	Body	Wrong or Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		$w = mg$
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.	Notch	
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

3.3 Equilibrium conditions

we defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium in two dimensions may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0$$

3.3.1 Categories of Equilibrium

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

3.3.2 Two- and Three-Force Member

❖ Two-Force Members.

for any two-force member to be in equilibrium, the two forces acting on the member *must have the same magnitude, act in opposite directions, and have the same line of action, directed along the line joining the two points where these forces act.*

❖ Three-Force Members.

If a member is subjected to only *three forces*, it is called a ***three-force member***. Moment equilibrium can be satisfied only if the three forces form a *concurrent* or *parallel* force system

(a) Three-force member

Two-force members

(b) Closed polygon
satisfies $\Sigma F = 0$

(a)

(b)

Three-force member
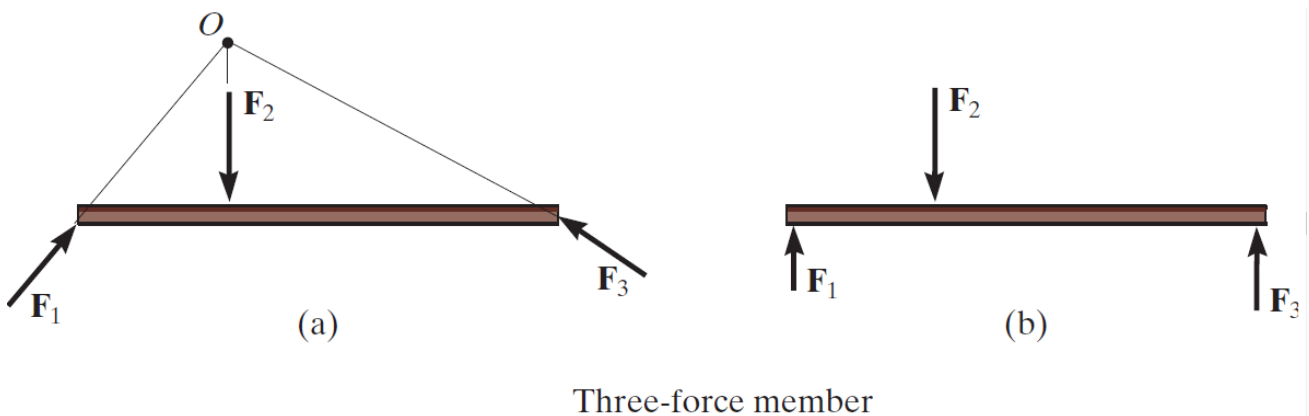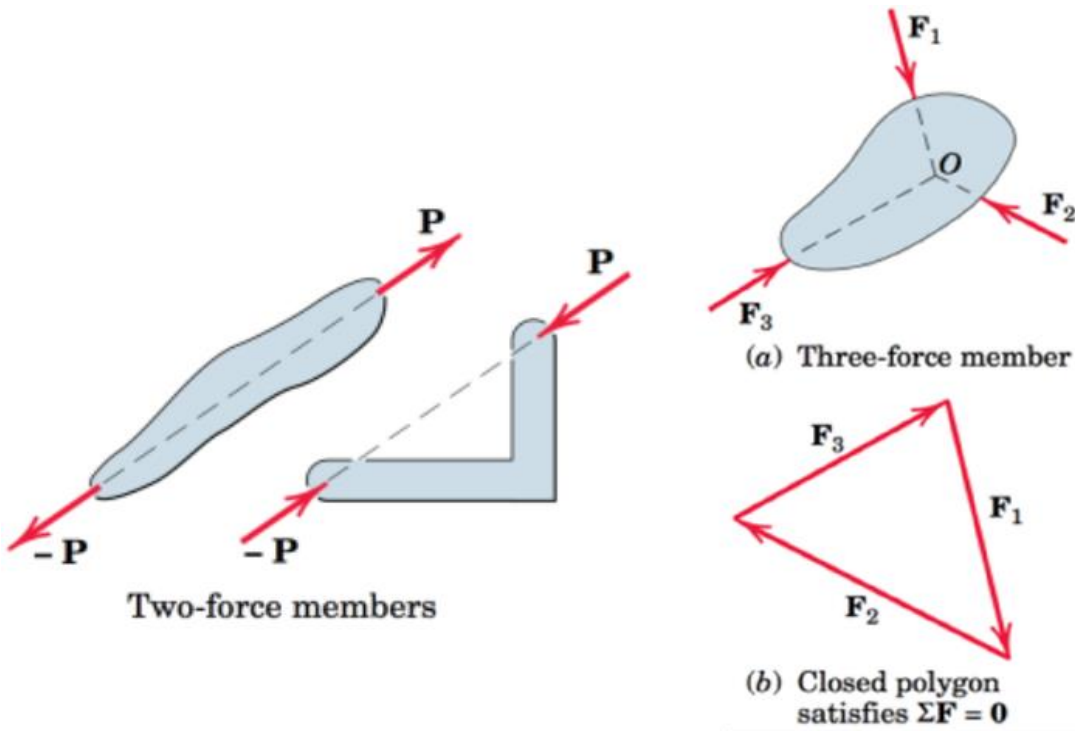
❖ The FPD of pin between two members

(a)

Example1

$T = 50\,(9.81)\ \text{N}$

$D_y = 490.5\ \text{N}$

$D_x = 490.5\ \text{N}$

F_{BC}

50 kg

$50\,(9.81)\ \text{N}$

$0.9\ \text{m}$

$0.6\ \text{m}$

$1.20\ \text{m}$

(a)

(b)

Example 2

F_{BA}

800 N

F_{BA}

800 N

800

800 N

F_{BC}

800 N

F_{BC}

3 m

2 m

C
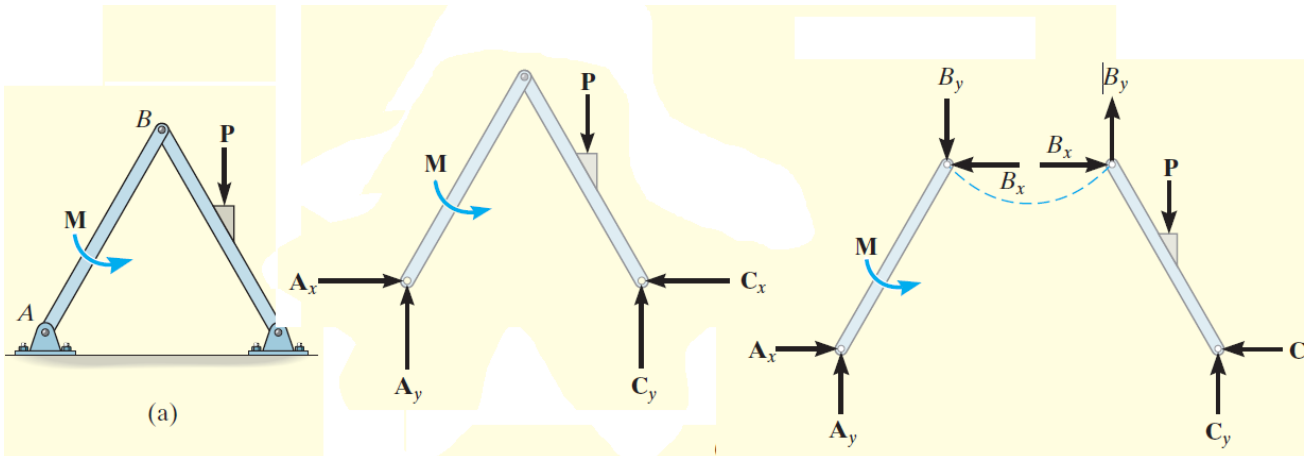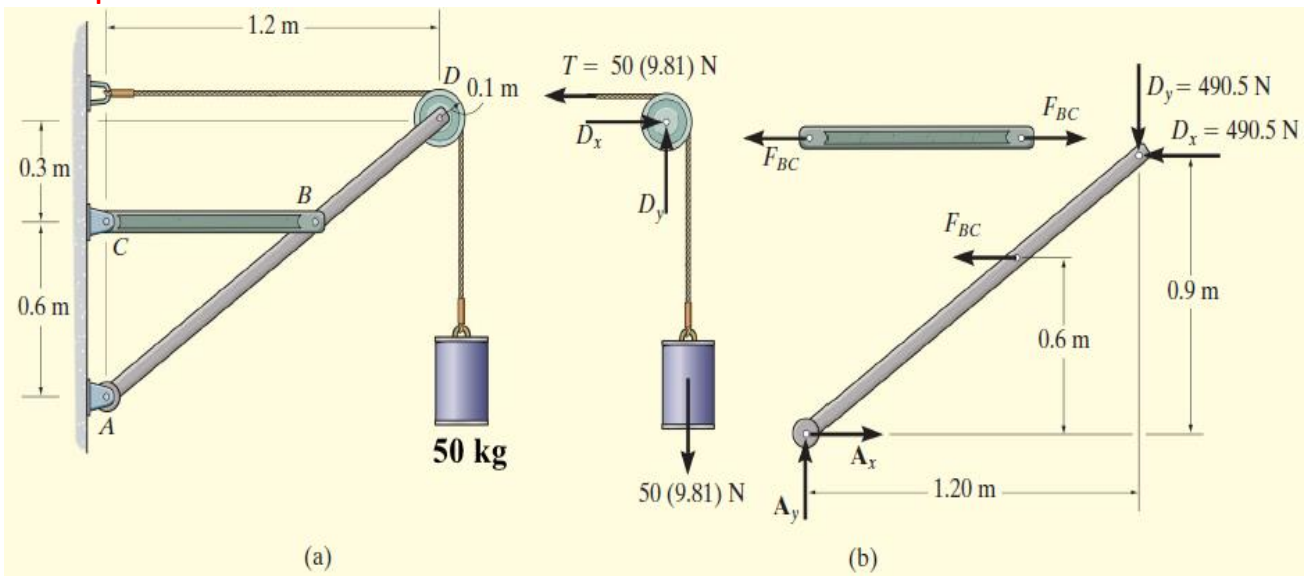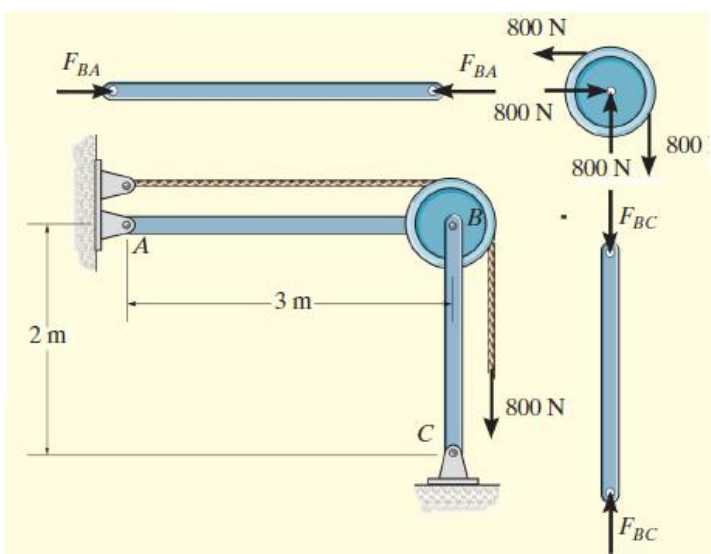
3.3.3 Alternative equilibrium Equations

In addition to the three general equilibrium equation there are two other ways to express general conditions for the equilibrium of forces in two ways dimensions. If $\Sigma MA = 0$ and $\Sigma MB = 0$, where B is any point such that the line AB is not perpendicular to the x-direction, we see that R must be zero, and thus the body is in equilibrium. Therefore, an alternative set of equilibrium equations is

❖ **Case 1**

$$\Sigma F_x = 0 \qquad \Sigma M_A = 0 \qquad \Sigma M_B = 0$$

where the two points A and B must not lie on a line perpendicular to the x-direction.

❖ **Case 2**

$$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0$$

where A, B, and C are any three points not on the same straight line.
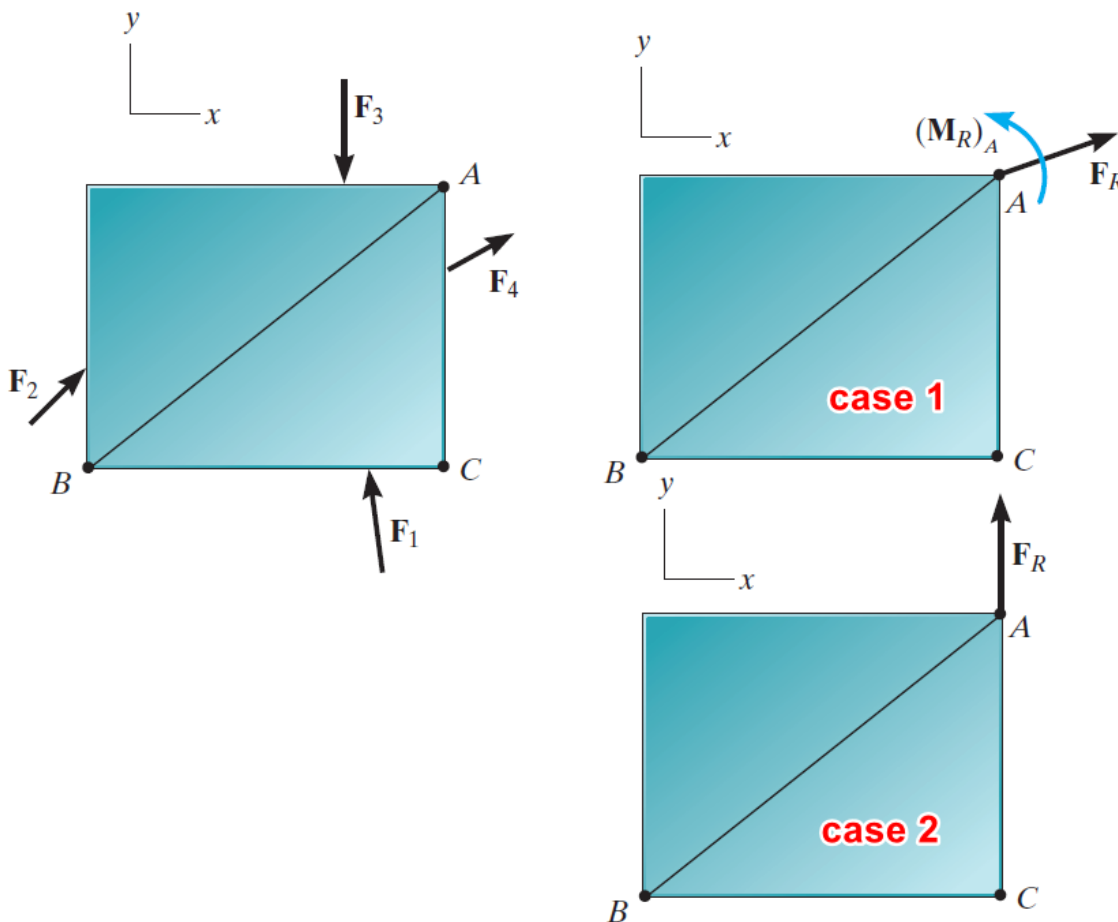
3.4 Constrains and Statical Determinacy

The equilibrium equations developed in this article are both necessary and sufficient conditions to establish the equilibrium of a body However; they do not necessarily provide all the information required to calculate all the unknown forces which may act on a body in equilibrium. Whether the equations are adequate to determine all the unknowns depends on the characteristics of the constraints against possible movement of the body provided by its supports. By *constraint* we mean the restriction of movement.

We must be aware of the nature of the constraints before we attempt to solve an equilibrium roblem. A body can be recognized as *statically indeterminate* when there are more unknown external reactions than there are available independent equilibrium equations for the force system involved. It is always well to count the number of unknown variables on a given body and to be certain that an equal number of independent equations can be written; otherwise, effort might be wasted in attempting an impossible solution with the aid of the equilibrium equations only. The unknown variables may be forces, couples, distances, or angles.
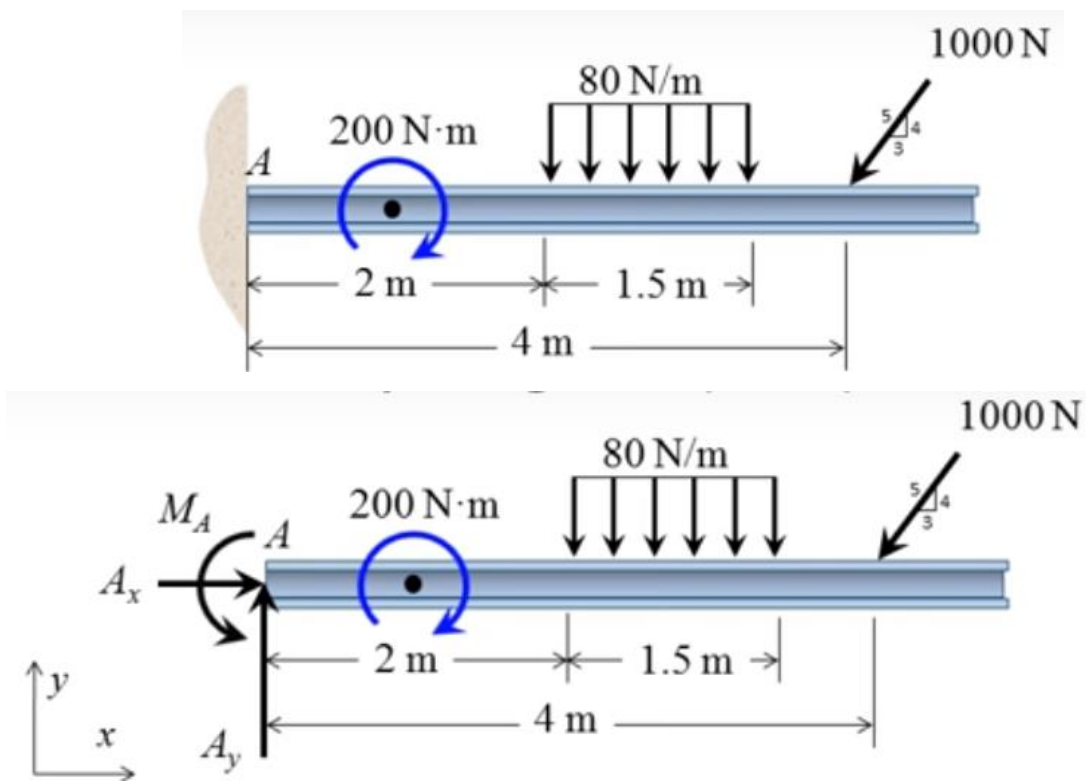
Unknowns	No. of Unknowns	Equilibrium Eqs.	No. of Equilibrium Eqs.
AX AY MA	3	$\Sigma Fx=0$ $\Sigma Fy=0$ $\Sigma Ma=O$	3

APPROACH TO SOLVING PROBLEMS

The sample problems at the end of this article illustrate the application of free-body diagrams and the equations of equilibrium to typical statics problems. These solutions should be studied thoroughly. In the problem work of this chapter and throughout mechanics, it is important to develop a logical and systematic approach which includes the following steps:

1. Identify clearly the quantities which are known and unknown.

2. Make an unambiguous choice of the body (or system of connected bodies treated as a single body) to be isolated and draw its complete free-body diagram, labeling all external known and unknown but identifiable forces and couples which act on it.

3. Choose a convenient set of reference axes, always using right-handed axes when vector cross products are employed. Choose moment centers with a view to simplifying the calculations. Generally the best choice is one through which as many unknown forces pass as possible. Simultaneous solutions of equilibrium equations are frequently necessary, but can be minimized or avoided by a careful choice of reference axes and moment centers.

4. Identify and state the applicable force and moment principles or equations which govern the equilibrium conditions of the problem. In the following sample problems these relations are shown in brackets and precede each major calculation.

5. Match the number of independent equations with the number of unknowns in each problem.

6. Carry out the solution and check the results. In many problems engineering judgment can be developed by first making a reasonable guess or estimate of the result prior to the calculation and then comparing the estimate with the calculated value.

Procedure for Analysis

Coplanar force equilibrium problems for a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y coordinate axes in any suitable orientation.

- Remove all supports and draw an outlined shape of the body.

- Show all the forces and couple moments acting on the body.

- Label all the loadings and specify their directions relative to the x or y axis. The sense of a force or couple moment having an *unknown* magnitude but known line of action can be *assumed*.

- Indicate the dimensions of the body necessary for computing the moments of forces.

Equations of Equilibrium.

- Apply the moment equation of equilibrium, $\Sigma M_O = 0$, about a point (O) that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about O, and a *direct solution* for the third unknown can be determined.

- When applying the force equilibrium equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, orient the x and y axes along lines that will provide the simplest resolution of the forces into their x and y components.

- If the solution of the equilibrium equations yields a negative scalar for a force or couple moment magnitude, this indicates that the sense is opposite to that which was assumed on the free-body diagram.

Example

Determine the support reactions at the roller at A and the pin at B. Neglect the weight and size of the beam.

400 N

60°

A B

|— 4m —|— 4m —|
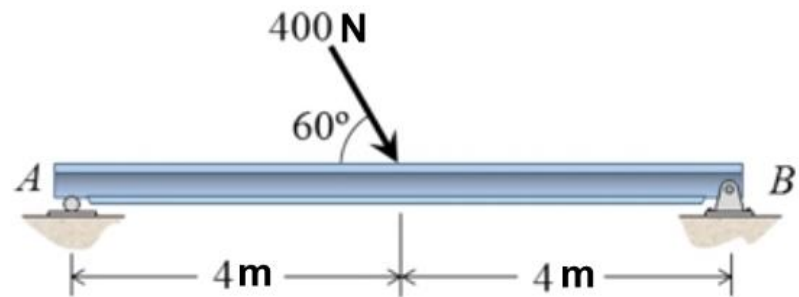
solution

There are 3 unknowns and 3 equilibrium Eqs. Therefore, the problem can be solved

400 N

60°

A B → B_x

$\uparrow y$

x

|— 4 ft —|— 4 ft —|

F_A B_y
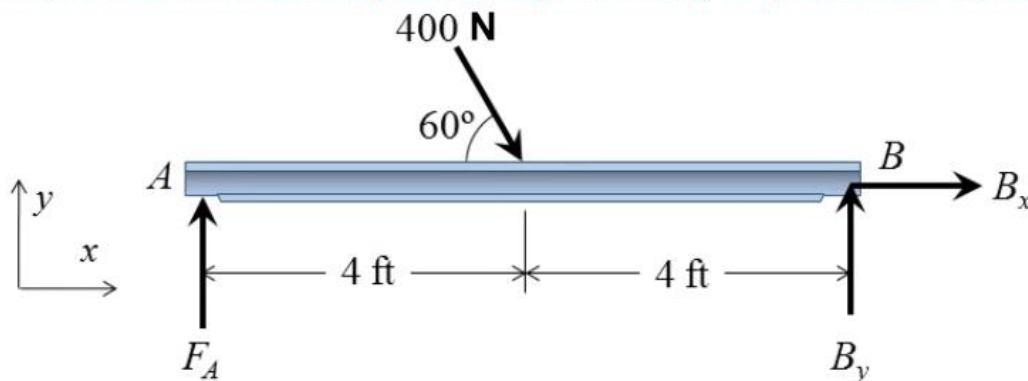
$$\sum F_x = 400 \cdot \cos 60° + B_x = 0$$

$$\sum F_y = F_A - 400 \cdot \sin 60° + B_y = 0$$

$$\sum M_B = -F_A \cdot 8 + 400 \cdot \sin 60° \cdot 4 = 0$$

$\therefore$ $\begin{cases} F_x = 173\,\text{N} \\ B_x = -200\,\text{N} \\ B_y = 173\,\text{N} \end{cases}$

400 N

60°

A B ← 200 N

$\uparrow y$

x

|— 4m —|— 4m —|

173 N   173 N
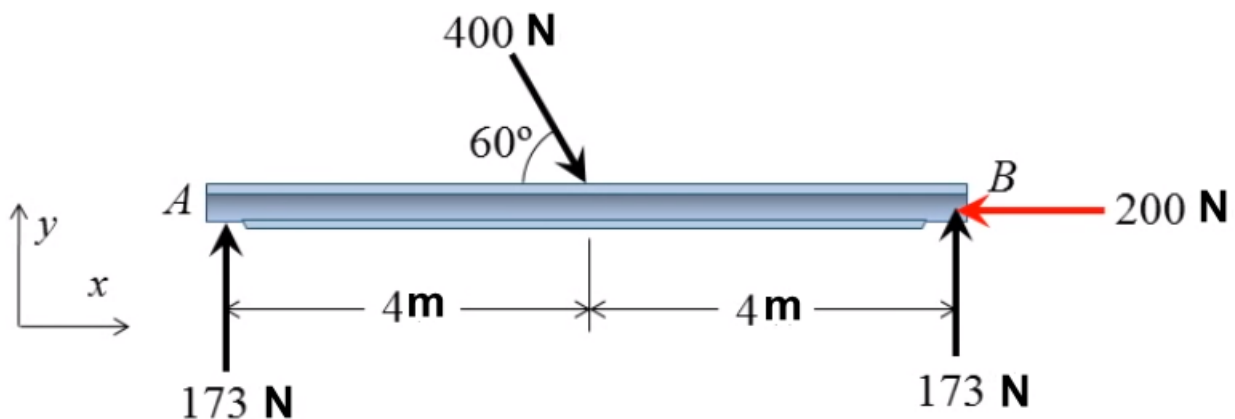
Example

Determine the support reactions at the fixed support, A. Neglect the weight and size of the beam.
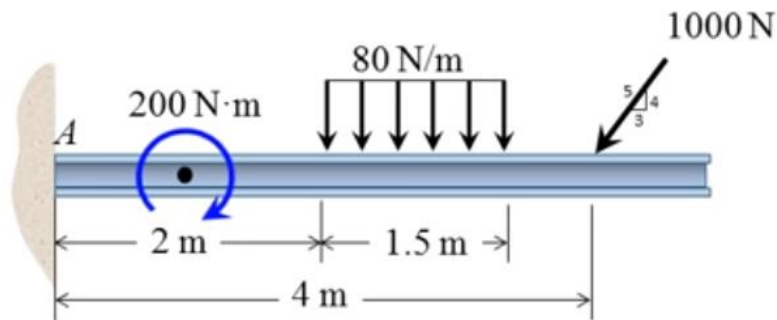
Solution

There are 3 unknowns and 3 equilibrium Eqs. That means the problem can be solved

Free body diagram (FBD)
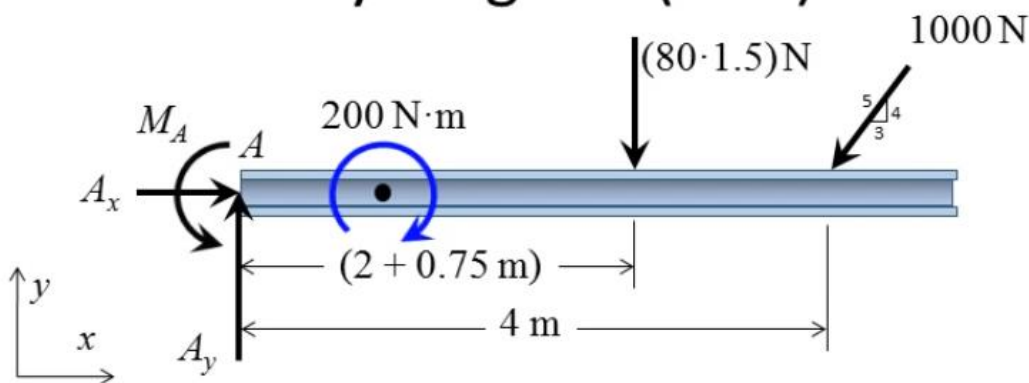
$$\sum F_x = A_x - \frac{3}{5} \cdot 1000\text{N} = 0$$

$$\sum F_y = A_y - 80 \cdot 1.5\text{N} - \frac{4}{5} \cdot 1000\text{N} = 0$$

$$\sum M_A = M_A - 200\text{N} \cdot \text{m} - 80 \cdot 1.5\text{N} \cdot 2.75\text{m}$$

$$- \frac{4}{5} \cdot 1000\text{N} \cdot 4\text{m} = 0$$

$$\therefore \begin{cases} A_x = 600 \text{ N} \\ A_y = 920 \text{ N} \\ M_A = 3730 \text{ N} \cdot \text{m} \end{cases}$$

Example

Determine the support reactions at the smooth collar at A, rocker at B and cable at C. Neglect the weight and size of member ABC.
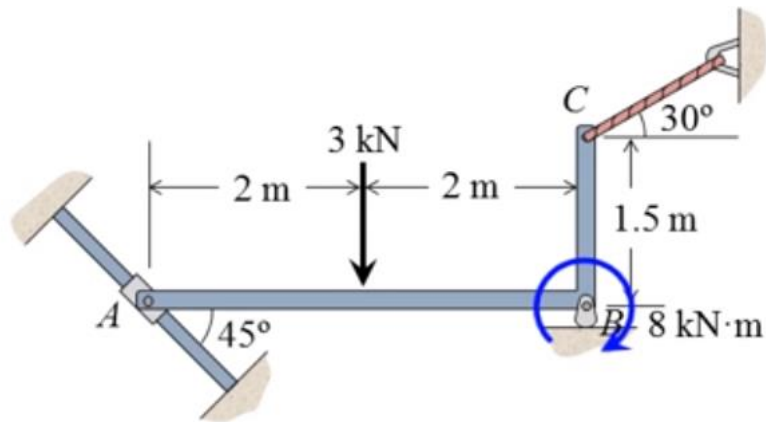
Solution
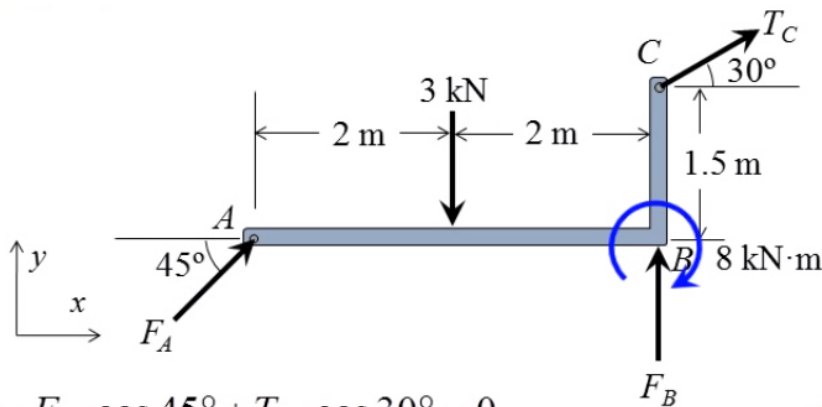
$$\sum F_x = F_A \cdot \cos 45° + T_C \cdot \cos 30° = 0$$

$$\sum F_y = F_A \cdot \sin 45° - 3kN + F_B + T_C \cdot \sin 30° = 0 \qquad \therefore$$

$$\sum M_C = -F_A \cdot \sin 45° \cdot 4m + F_A \cdot \sin 45° \cdot 1.5m$$
$$+ 3kN \cdot 2m - 8kN \cdot m = 0$$

$$\begin{cases} F_A = -1.13 \text{ kN} \\ F_B = 3.34 \text{ kN} \\ T_C = 0.924 \text{ kN} \end{cases}$$
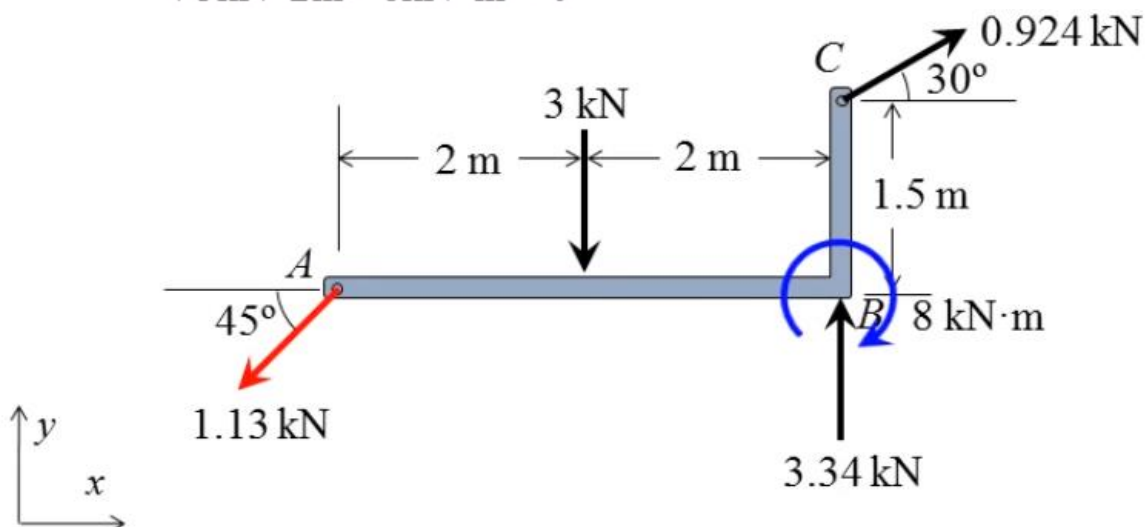
Determine the support reactions on the member in Fig. 5–19a. The collar at A is fixed to the member and can slide vertically along the vertical shaft.

Solution
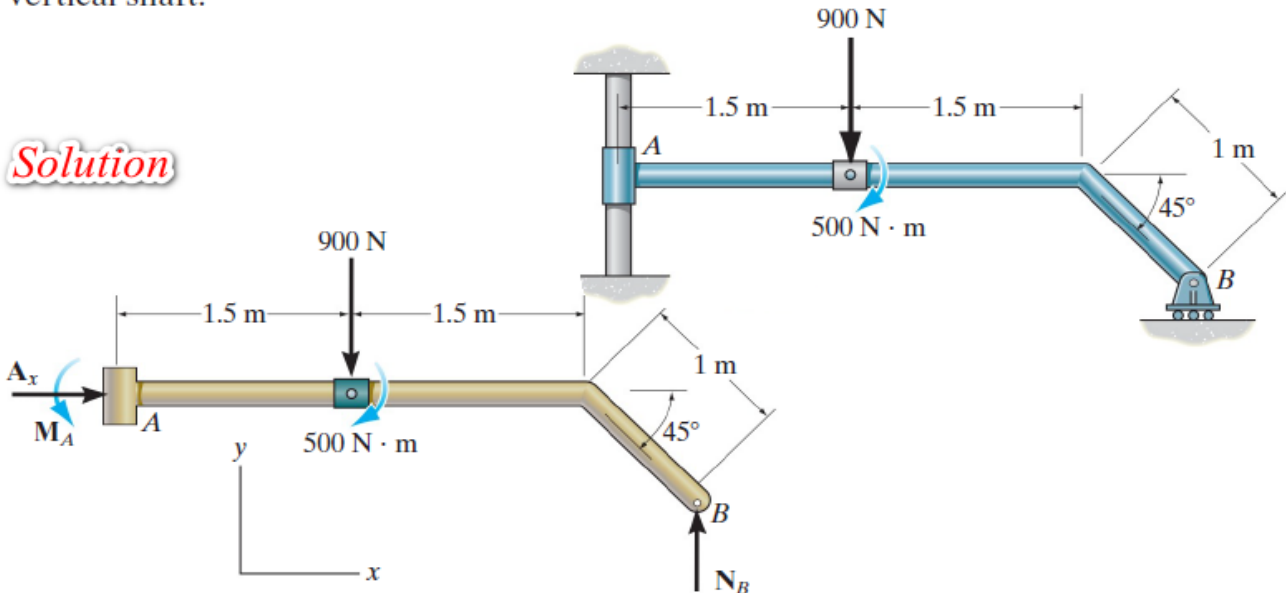
Free-Body Diagram. Removing the supports, the free-body diagram of the member is shown in Fig. 5–19b. The collar exerts a horizontal force A_x and moment M_A on the member. The reaction N_B of the roller on the member is vertical.

Equations of Equilibrium. The forces A_x and N_B can be determined directly from the force equations of equilibrium.

$$\xrightarrow{+} \Sigma F_x = 0; \qquad A_x = 0 \qquad \qquad Ans.$$

$$+\uparrow \Sigma F_y = 0; \qquad N_B - 900 \text{ N} = 0$$

$$N_B = 900 \text{ N} \qquad \qquad Ans.$$

The moment M_A can be determined by summing moments either about point A or point B.

$$\zeta + \Sigma M_A = 0;$$

$$M_A - 900 \text{ N}(1.5 \text{ m}) - 500 \text{ N} \cdot \text{m} + 900 \text{ N} [3 \text{ m} + (1 \text{ m}) \cos 45°] = 0$$

$$M_A = -1486 \text{ N} \cdot \text{m} = 1.49 \text{ kN} \cdot \text{m} \,\zeta \qquad \qquad Ans.$$

or

$$\zeta + \Sigma M_B = 0; \quad M_A + 900 \text{ N} [1.5 \text{ m} + (1 \text{ m}) \cos 45°] - 500 \text{ N} \cdot \text{m} = 0$$

$$M_A = -1486 \text{ N} \cdot \text{m} = 1.49 \text{ kN} \cdot \text{m} \,\zeta \qquad \qquad Ans.$$

The negative sign indicates that M_A has the opposite sense of rotation to that shown on the free-body diagram.

EXAMPLE

The lever ABC is pin supported at A and connected to a short link BD as shown in Fig. 5–22a. If the weight of the members is negligible, determine the force of the pin on the lever at A.

SOLUTION

(a)
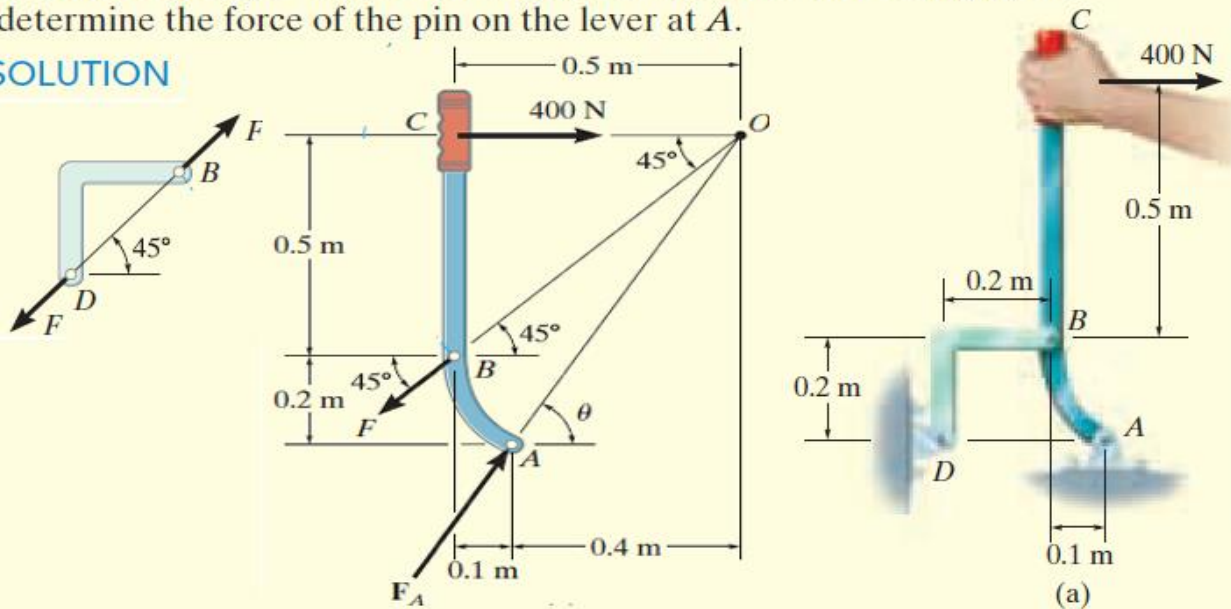
Equations of Equilibrium. By requiring the force system to be concurrent at O, since $\Sigma M_O = 0$, the angle θ which defines the line of action of F_A can be determined from trigonometry,

$$\theta = \tan^{-1}\left(\frac{0.7}{0.4}\right) = 60.3°$$

Using the x, y axes and applying the force equilibrium equations,

$\xrightarrow{+} \Sigma F_x = 0;$ $F_A \cos 60.3° - F \cos 45° + 400 \text{ N} = 0$

$+\uparrow \Sigma F_y = 0;$ $F_A \sin 60.3° - F \sin 45° = 0$

Solving, we get

$$F_A = 1.07 \text{ kN} \qquad \qquad \textit{Ans.}$$

$$F = 1.32 \text{ kN}$$

NOTE: We can also solve this problem by representing the force at A by its two components A_x and A_y and applying $\Sigma M_A = 0$, $\Sigma F_x = 0$, $\Sigma F_y = 0$ to the lever. Once A_x and A_y are determined, we can get F_A and θ.

The 30-N uniform rod has a length of $l = 1$ m. If $s = 1.5$ m, determine the distance h of placement at the end A along the smooth wall for equilibrium.

SOLUTION

Equations of Equilibrium: Referring to the FBD of the rod shown in Fig. a, write the moment equation of equilibrium about point A.

$$\zeta + \Sigma M_A = 0; \qquad T \sin \phi (1) - 3 \sin \theta (0.5) = 0$$

$$T = \frac{1.5 \sin \theta}{\sin \phi}$$

Using this result to write the force equation of equilibrium along y axis,

$$+\uparrow \Sigma F_y = 0; \qquad \left(\frac{15 \sin \theta}{\sin \phi} \right) \cos (\theta - \phi) - 3 = 0$$

$$\boxed{\sin \theta \cos (\theta - \phi) - 2 \sin \phi = 0} \qquad \text{-----(1)}$$

Geometry: Applying the sine law with $\sin (180° - \theta) = \sin \theta$ by referring to Fig. b,

$$\frac{\sin \phi}{h} = \frac{\sin \theta}{1.5}; \qquad \boxed{\sin \theta = \left(\frac{h}{1.5} \right) \sin \theta} \qquad \text{-----(2)}$$

Substituting Eq. (2) into (1) yields

$$\sin \theta \left[\cos (\theta - \phi) - \frac{4}{3} h \right] = 0$$

since $\sin \theta \neq 0$, then

$$\cos (\theta - \phi) - (4/3)h \qquad \boxed{\cos (\theta - \phi) = (4/3)h} \qquad \text{----(3)}$$

Again, applying law of cosine by referring to Fig. b,

$$l^2 = h^2 + 1.5^2 - 2(h)(1.5) \cos (\theta - \phi)$$

$$\boxed{\cos (\theta - \phi) = \frac{h^2 + 1.25}{3h}} \qquad \text{------------(4)}$$

Equating Eqs. (3) and (4) yields

$$\frac{4}{3}h = \frac{h^2 + 1.25}{3h}$$

$$3h^2 = 1.25$$

$$\boxed{h = 0.645 \text{ m}} \qquad \qquad \textbf{Ans.}$$
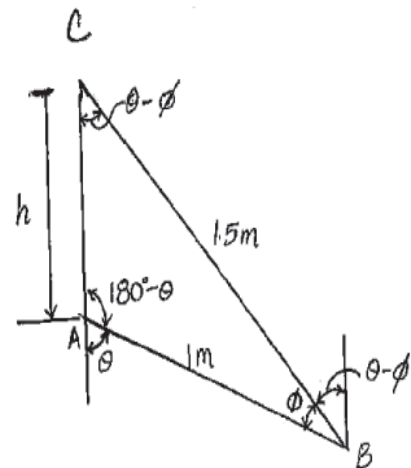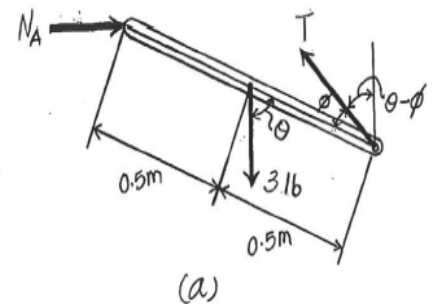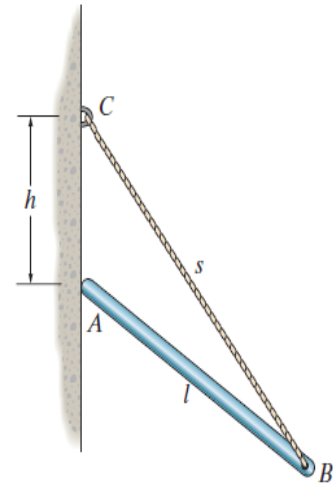
Example The man has a weight W and stands at the center of the plank. If the planes at A and B are smooth, determine the tension in the cord in terms of W and θ.
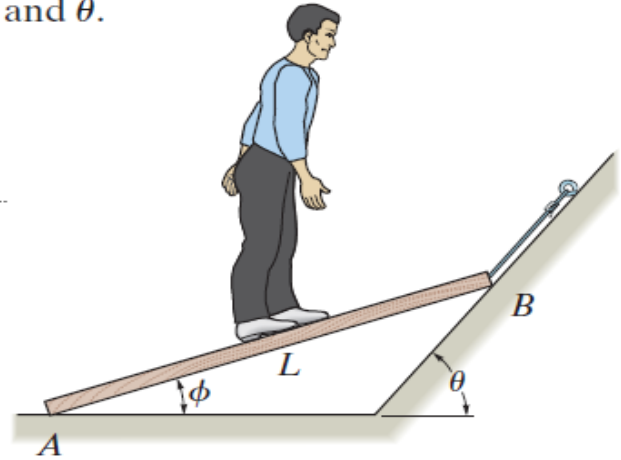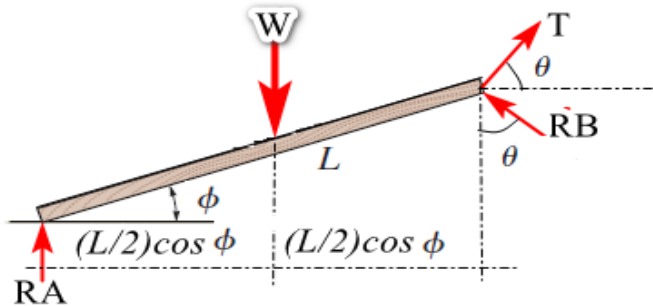
SOLUTION

$\zeta + \Sigma M_B = 0;$ $W\left(\dfrac{L}{2}\cos\phi\right) - N_A(L\cos\phi) = 0$ $N_A = \dfrac{W}{2}$

$\xrightarrow{} \Sigma F_x = 0;$ $T\cos\theta - N_B\sin\theta = 0$ -------------------- (1)

$+\uparrow \Sigma F_y = 0;$ $T\sin\theta + N_B\cos\theta + \dfrac{W}{2} - W = 0$ --------------(2)

Solving Eqs. (1) and (2) yields:

$$T = \dfrac{W}{2}\sin\theta$$ **Ans.**

$$N_B = \dfrac{W}{2}\cos\theta$$

H.W

A uniform glass rod having a length L is placed in the smooth hemispherical bowl having a radius r. Determine the angle of inclination θ for equilibrium.
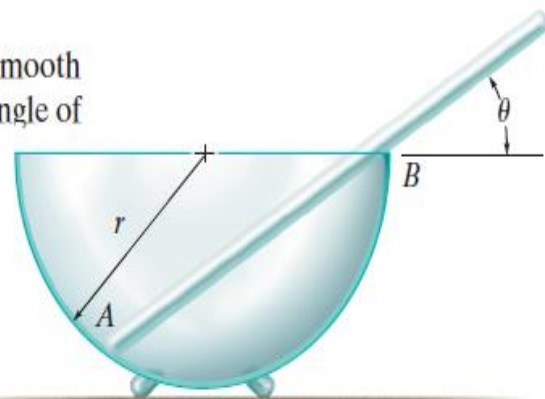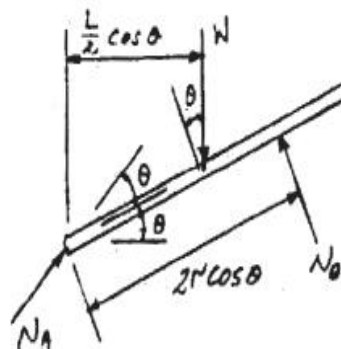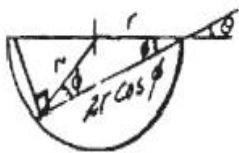
Ans: $\theta = \cos^{-1}\left(\dfrac{L + \sqrt{L^2 + 128r^2}}{16r}\right)$

Note: By observation $\phi = \theta$.

The frame in Fig. 6–33a supports the 50-kg cylinder. Determine the horizontal and vertical components of reaction at A and the force at C.
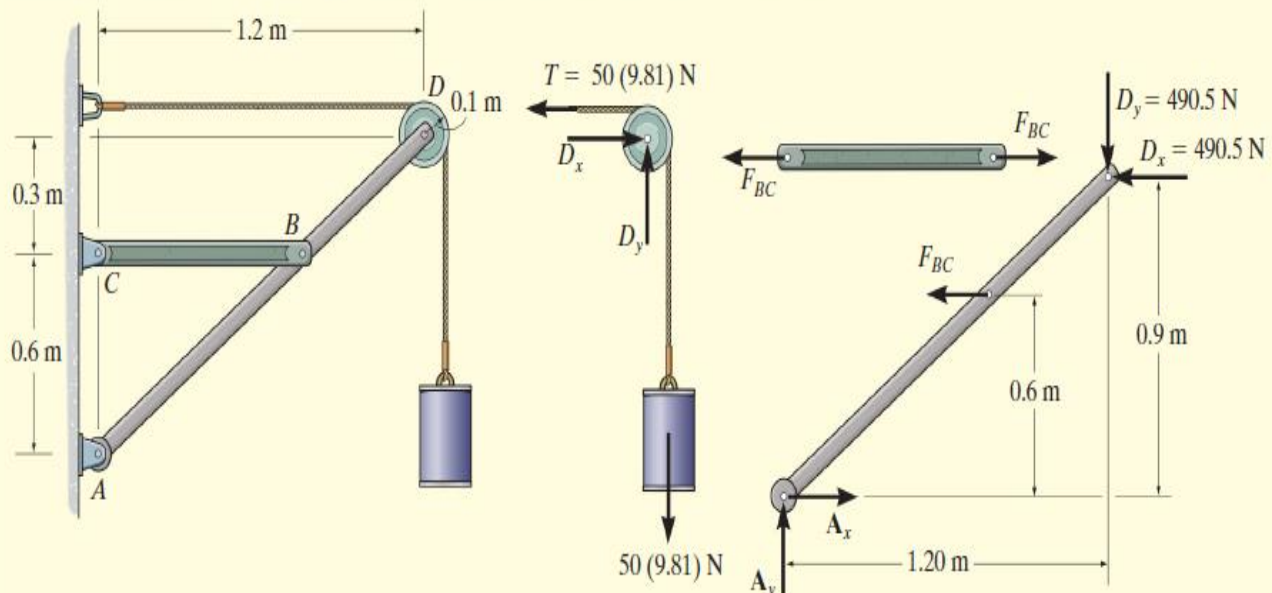
SOLUTION

Free-Body Diagrams. The free-body diagram of pulley D, along with the cylinder and a portion of the cord (a system), is shown in Fig. 6–33b. Member BC is a two-force member as indicated by its free-body diagram. The free-body diagram of member ABD is also shown.

Equations of Equilibrium. We will begin by analyzing the equilibrium of the pulley. The moment equation of equilibrium is automatically satisfied with $T = 50(9.81)$ N, and so

$\xrightarrow{+} \Sigma F_x = 0;$ $D_x - 50(9.81)\text{ N} = 0$ $D_x = 490.5$ N

$+\uparrow \Sigma F_y = 0;$ $D_y - 50(9.81)\text{ N} = 0$ $D_y = 490.5$ N   *Ans.*

Using these results, F_{BC} can be determined by summing moments about point A on member ABD.

$\zeta + \Sigma M_A = 0;$ $F_{BC}(0.6\text{ m}) + 490.5\text{ N}(0.9\text{ m}) - 490.5\text{ N}(1.20\text{ m}) = 0$

$$F_{BC} = 245.25 \text{ N} \qquad \textit{Ans.}$$

Now A_x and A_y can be determined by summing forces.

$\xrightarrow{+} \Sigma F_x = 0;$ $A_x - 245.25\text{ N} - 490.5\text{ N} = 0$ $A_x = 736$ N   *Ans.*

$+\uparrow \Sigma F_y = 0;$ $A_y - 490.5\text{ N} = 0$ $A_y = 490.5$ N   *Ans.*

Example

Determine the horizontal and vertical components of force at pins B and C.
The suspended cylinder has a mass of 75 kg.
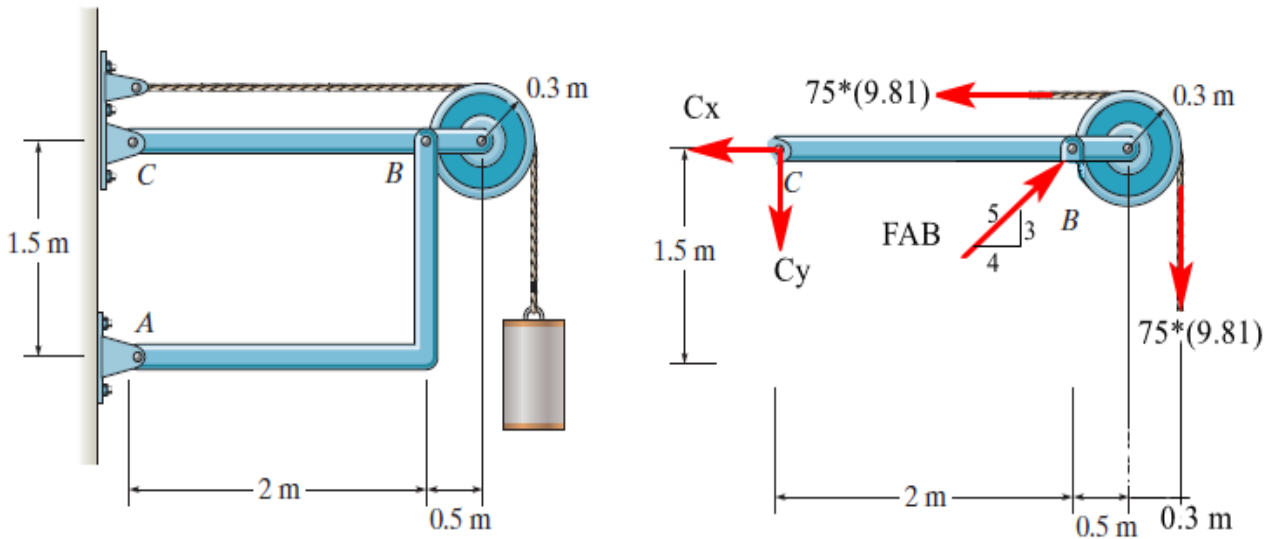
SOLUTION

Free Body Diagram. The solution will be very much simplified if one realizes that member AB is a two force member. Also, the tension in the cable is equal to the weight of the cylinder and is constant throughout the cable.

Equations of Equilibrium. Consider the equilibrium of member BC by referring to its FBD, Fig. a,

$$\zeta + \Sigma M_C = 0; \quad F_{AB}\left(\frac{3}{5}\right)(2) + 75(9.81)(0.3) - 75(9.81)(2.8) = 0$$

$$F_{AB} = 1532.81 \text{ N}$$

$$\zeta + \Sigma M_B = 0; \quad C_y(2) + 75(9.81)(0.3) - 75(9.81)(0.8) = 0$$

$$C_y = 183.94 \text{ N} = 184 \text{ N} \qquad \text{Ans.}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad 1532.81\left(\frac{4}{5}\right) - 75(9.81) - C_x = 0$$

$$C_x = 490.5 \text{ N} \qquad \text{Ans.}$$

Thus,

$$F_B = F_{AB} = 1532.81 \text{ N}$$

$$B_x = \frac{4}{5}(1532.81) = 1226.25 \text{ N} = 1.23 \text{ kN} \qquad \text{Ans.}$$

$$B_y = \frac{3}{5}(1532.81) = 919.69 \text{ N} = 920 \text{ kN} \qquad \text{Ans.}$$

Example

Determine the horizontal and vertical components of force which pin C exerts on member CDE. The 600-N load is applied to the pin.

SOLUTION

Free Body Diagram. The solution will be very much simplified if one determines the support reactions first and then considers the equilibrium of two of its three members after they are dismembered. The *FBDs* of the entire assembly, member *DBF* and member *ABC* are shown in Figs. *a*, *b* and *c*, respectively.

Equations of Equilibrium. Consider the equilibrium of the entire assembly, Fig. *a*,

$$\zeta + \Sigma M_E = 0; \quad N_A(3) - 300(4.5) - 600(4) = 0 \quad N_A = 1250 \text{ N}$$

Next, write the moment equation of equilibrium about point D for member *DBF*, Fig. *b*.

$$\zeta + \Sigma M_D = 0; \quad B_x(1.5) - 300(3) = 0 \quad B_x = 600 \text{ N}$$

Finally, consider the equilibrium of member *ABC*, Fig. *c*

$$\xrightarrow{+} \Sigma F_x = 0; \quad 1250 - 600 - C_x = 0 \quad C_x = 650 \text{ N} \qquad \textbf{Ans.}$$

$$\zeta + \Sigma M_B = 0; \quad C_y = 0 \qquad \textbf{Ans.}$$

(a)

(b)

(c)
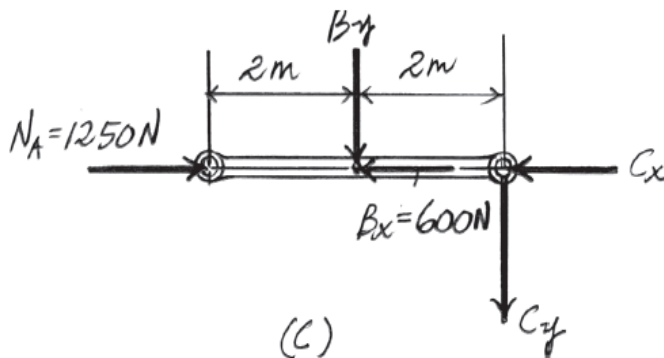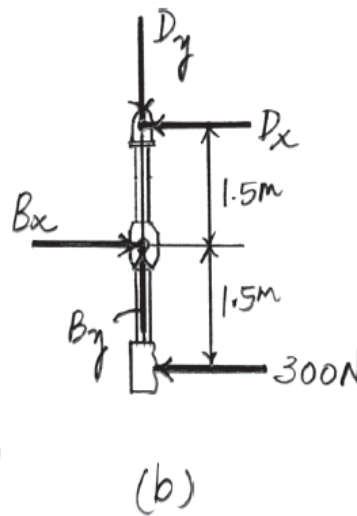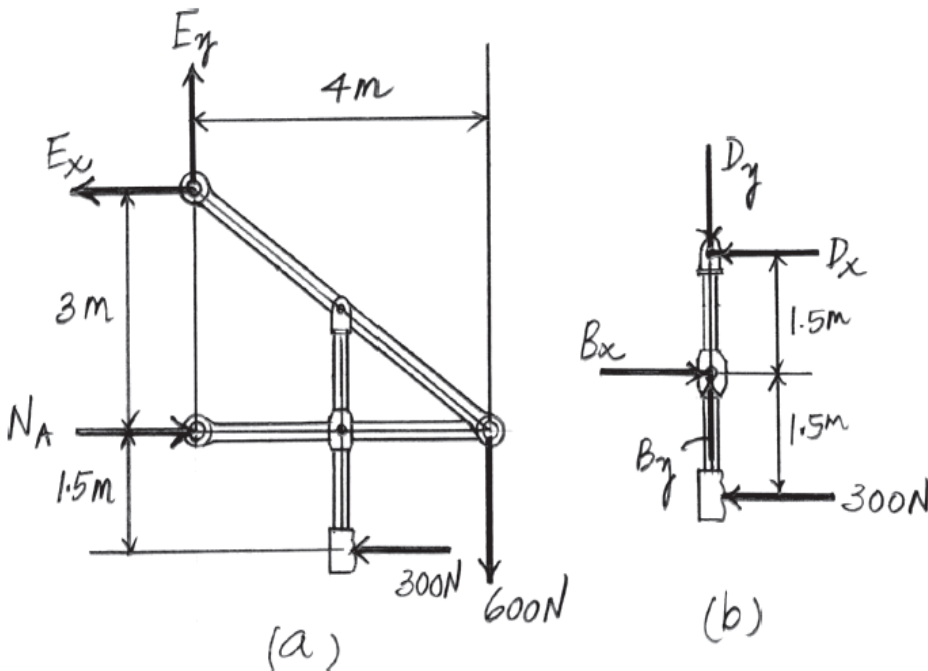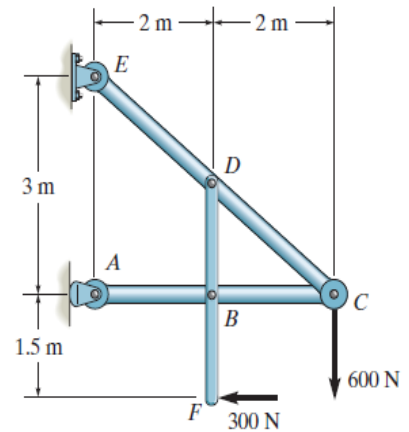
Example

The beam *ABC* is loaded via a 500 N.m couple and a 600 N force as shown. The beam is connected to the rest of the system by a pin joint at *B* and a roller support at *C*. Determine the magnitude of the reaction forces on the beam at the supports.

600 N

B

A   500 N.m

400

300 — 500

Dimensions in mm

20°

C
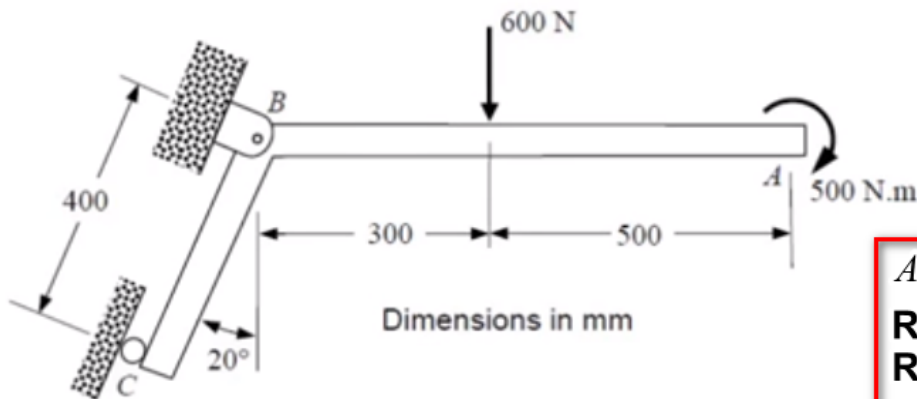
Ans.

RC=1700 N
RBx=1597.5 N
RBy=1181.4 N

Example:Determine the force *P* needed to lift the load. Also, determine the proper placement *x* of the hook for equilibrium. Neglect the weight of the beam.

0.9 m

100 mm   100 mm

100 mm   C

A   B   P

x

6 kN
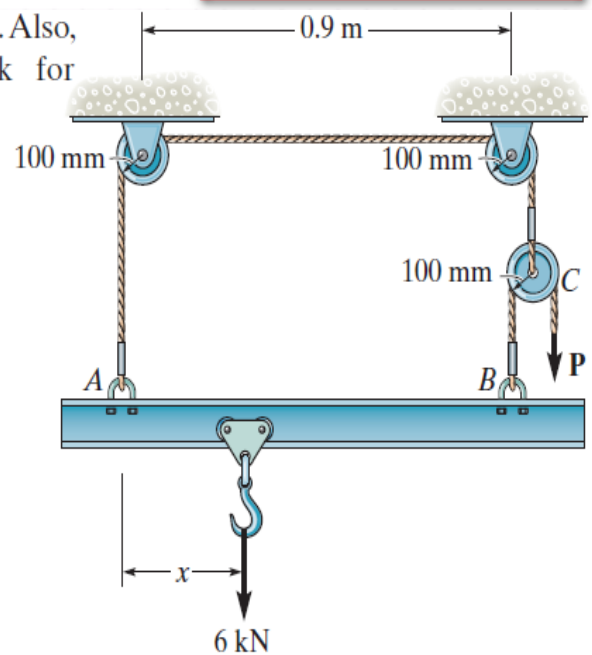
H.W

The uniform rod of length *L* and weight *W* is supported on the smooth planes. Determine its position θ for equilibrium. Neglect the thickness of the rod.

L

θ

ψ ϕ

Ans. $\theta = \tan^{-1}\left(\dfrac{1}{2}\cot\psi - \dfrac{1}{2}\cot\phi\right)$
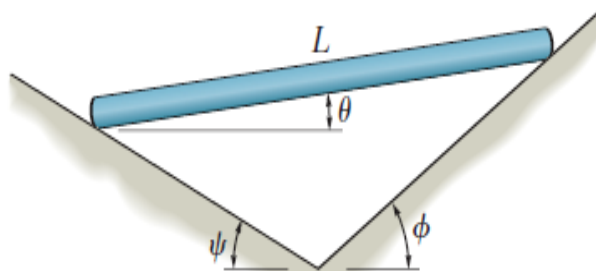
HW

Determine the resultant force at pins A, B, and C on the three-member frame.

Ans:

$A_x = 61.88$ N
$A_y = 569.06$ N
$B_x = 338.12$ N
$B_y = 338.12$ N
$C_x = 461.88$ N
$C_y = -338.12$ N

$F_C = 572$ N
$F_A = 572$ N
$F_B = 478$ N

H.W.
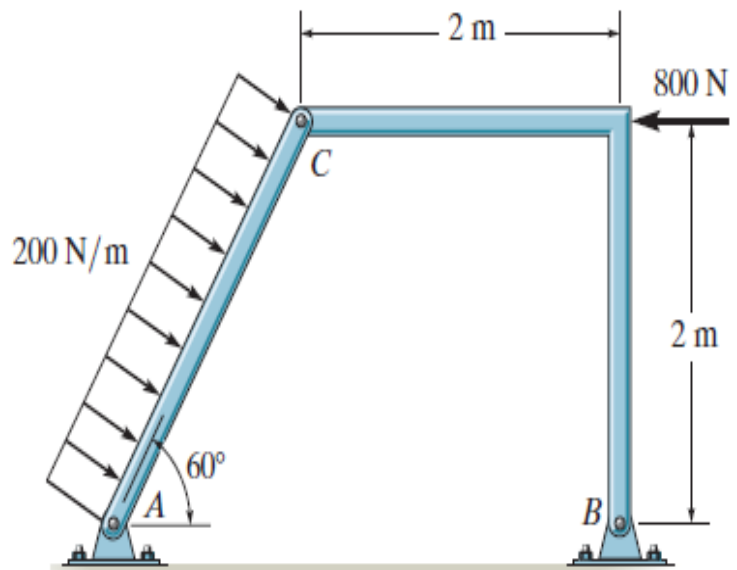
Determine the horizontal and vertical components of force at pins A and D.

Ans:
$A_x = 24.0$ kN
$A_y = 12.0$ kN
$D_x = 18.0$ kN
$D_y = 24.0$ kN
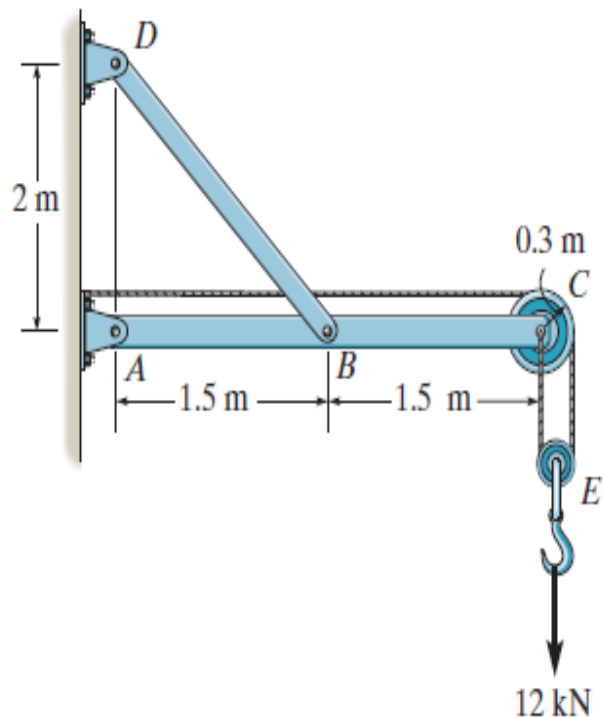
CHAPTER FOUR
FRICTION

4.1 Introduction

In the previous sections, we assumed that surfaces in contact are either frictionless or rough. If they are frictionless, the force each surface exerts on the other is normal to the surfaces, and the two surfaces can move freely with respect to each other. If they are rough, tangential forces can develop that prevent the motion of one surface with respect to the other.

Friction is a force that resists the movement of two contacting surfaces that slide relative to one another. This force always acts tangent to the surface at the points of contact and is directed so as to oppose the possible or existing motion between the surfaces.

There are two types of friction: *dry* friction, sometimes called *Coulomb* friction, and *fluid* friction or viscosity. Fluid friction develops between layers of fluid moving at different velocities.

4.2 Theory of Dry Friction

The theory of dry friction can be explained by considering the effects caused by pulling horizontally on a block of uniform weight **W** which is resting on a rough horizontal surface that is non rigid or deformable as show in fig. below where:

W= block of uniform weight,	ΔNn = Distributed normal force
ΔFn =Distributed frictional force ,	P = Horizontal applied force
N= Σ ΔNn= normal force	$\Sigma y=0$ $N = w$
F= Σ ΔFn= frictional force	$\Sigma x=0$ $F = P$

The effect of the *distributed* normal and frictional loadings is indicated by their *resultants* **N** and **F** on the free-body diagram, Fig. 8–1*d*. Notice that **N** acts a distance *x* to the right of the line of action of **W**, Fig. 8–1*d*. This location, which coincides with the centroid or geometric center of the normal force distribution in Fig. 8–1*b*, is necessary in order to balance the "tipping effect" caused by **P**. For example, if **P** is applied at a height *h* from the surface, Fig. 8–1*d*, then moment equilibrium about point *O* is satisfied if $Wx = Ph$ or $x = Ph > W$.

(a)   (b)   (c)   Resultant normal and frictional forces (d)
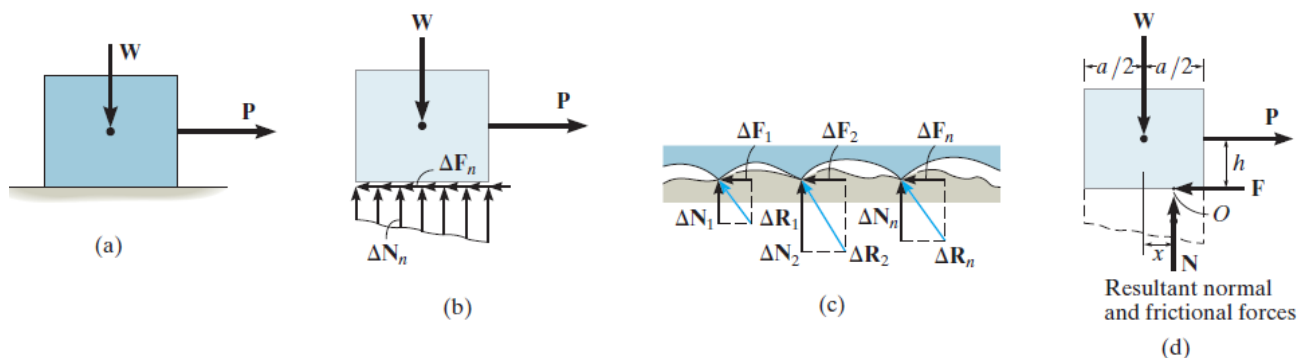
4.3 The static and kinetic friction forces

It was found that, as the magnitude F of the friction force increases from   0 to F_m, the point of application A of the resultant N of the normal forces of contact moves to the right. In this way, the couples formed by P and F and by W and N, respectively, remain balanced. If N reaches B before F reaches its maximum value Fm, the block starts to tip about B before it can start sliding.
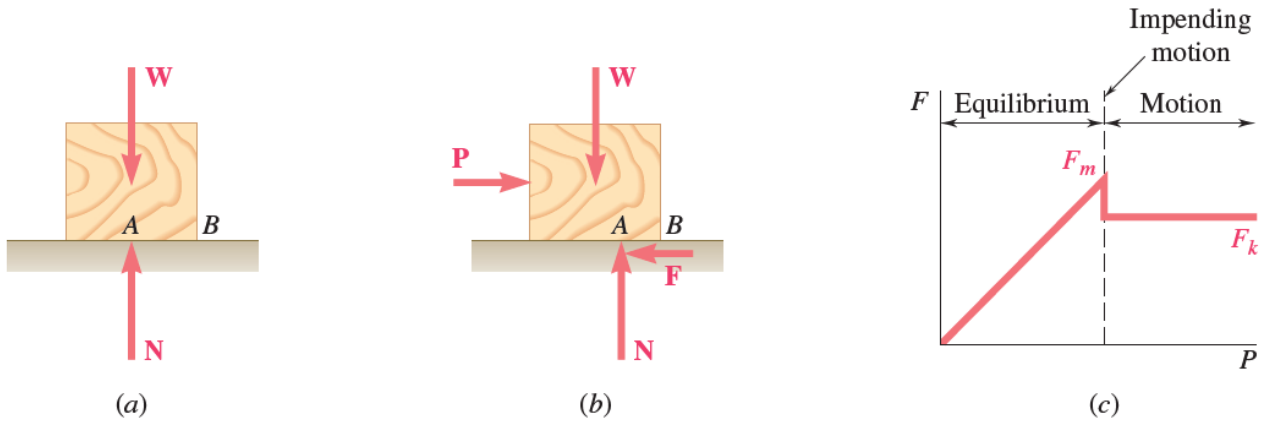
Fig. 4.11 (*a*) Block on a horizontal plane, friction force is zero; (*b*) a horizontally applied force **P** produces an opposing friction force **F**; (*c*) graph of **F** with increasing **P**.

4.5 Coefficients of Friction

Experimental evidence shows that the maximum value Fm of the static friction force is proportional to the normal component N of the reaction of the surface. We have Static friction

$$F_m = \mu_s\, N \tag{4.8}$$

where μs is a constant called the coefficient of static friction. Similarly, we can express the magnitude Fk of the kinetic-friction force in the form Kinetic friction

$$F_k = \mu_k N \tag{4.9}$$

where μ_k is a constant called the coefficient of kinetic friction. The coefficients of friction μ_s and μ_k do not depend upon the area of the surfaces in contact. Both coefficients, however, depend strongly on the nature of the surfaces in contact.

From this discussion, it appears that four different situations can occur when a rigid body is in contact with a horizontal surface:

1. The forces applied to the body do not tend to move it along the surface of contact; there is no friction force (Fig. 4.12a).

2. The applied forces tend to move the body along the surface of contact but are not large enough to set it in motion. We can find the static- friction force F that has developed by solving the equations of equilibrium for the body. Since there is no evidence that F has reached its maximum value, the equation $Fm = \mu sN$ cannot be used to determine the friction force (Fig. 4.12b).

3. The applied forces are such that the body is just about to slide. We say that motion is impending. The friction force F has reached its maximum value Fm and, together with the normal force N, balances the applied forces. Both the equations of equilibrium and the equation $Fm = \mu s\ N$ can be used. Note that the friction force has a sense opposite to the sense of impending motion (Fig. 4.12c).

4. The body is sliding under the action of the applied forces, and the equations of equilibrium no longer apply. However, F is now equal to F_k, and we can use the equation $F_k = \mu_k N$. The sense of F_k is opposite to the sense of motion (Fig. 4.12d).

$F = 0$

$N = P + W$

(a) No friction ($P_x = 0$)

$F = P_x$
$F < \mu_s N$
$N = P_y + W$

(b) No motion ($P_x < F_m$)

$F_m = P_x$
$F_m = \mu_s N$
$N = P_y + W$
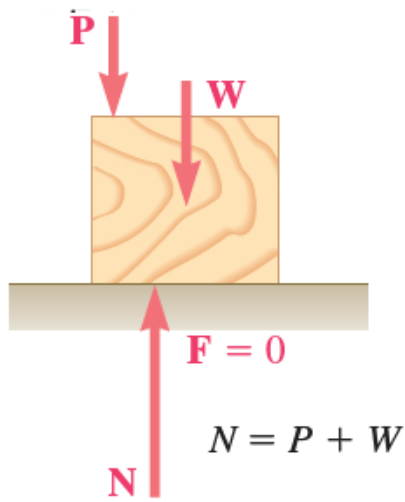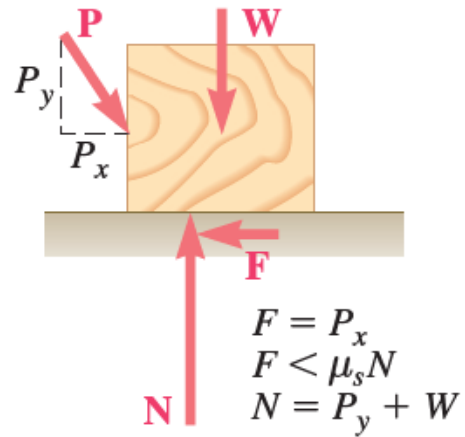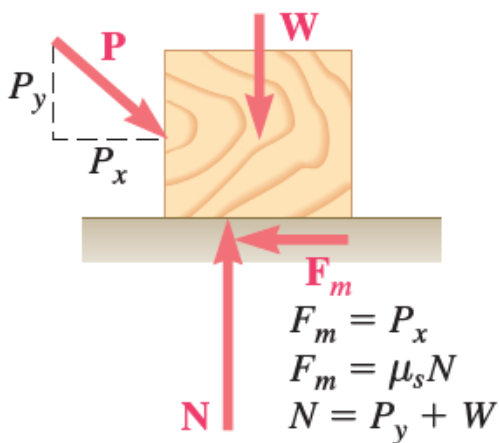
(c) Motion impending $\longrightarrow (P_x = F_m)$

$F_k < P_x$
$F_k = \mu_k N$
$N = P_y + W$
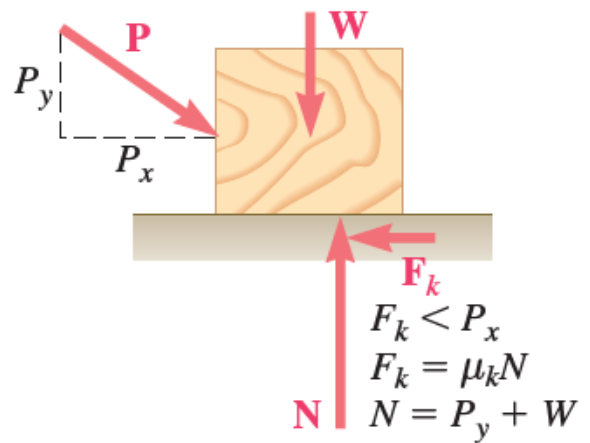
(d) Motion $\longrightarrow (P_x > F_k)$

Fig. 4.12
(a) Applied force is vertical, friction force is zero;
(b) Horizontal component of applied force is less than Fm, no motion occurs;
(c) Horizontal component of applied force equals Fm, motion is impending;
(d) Horizontal component of applied force is greater than Fk, forces are unbalanced and motion continues.

4.4 Angles of Friction

It is sometimes convenient to replace the normal force N and the friction force F by their resultant R. Let's see what happens when we do that.

Angle of static friction

$$\tan\phi_s = \frac{F_m}{N} = \frac{\mu_s N}{N}$$

$$\tan\phi_s = \mu_s$$

Angle of kinetic friction

$$\tan\phi_k = \frac{F_k}{N} = \frac{\mu_k N}{N}$$

$$\tan\phi_k = \mu_k$$

(a) No friction

(b) No motion

(c) Motion impending

(d) Motion

(a) No friction

(b) No motion

(c) Motion impending

(d) Motion
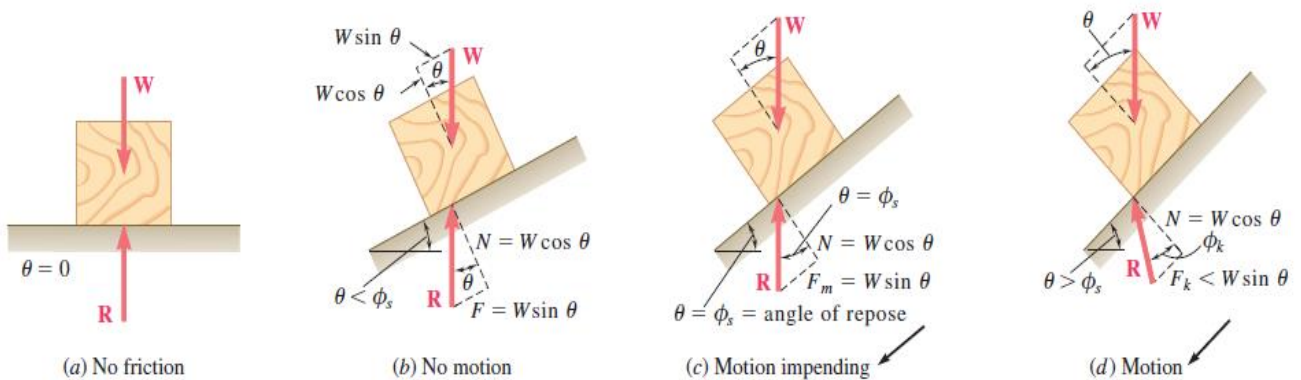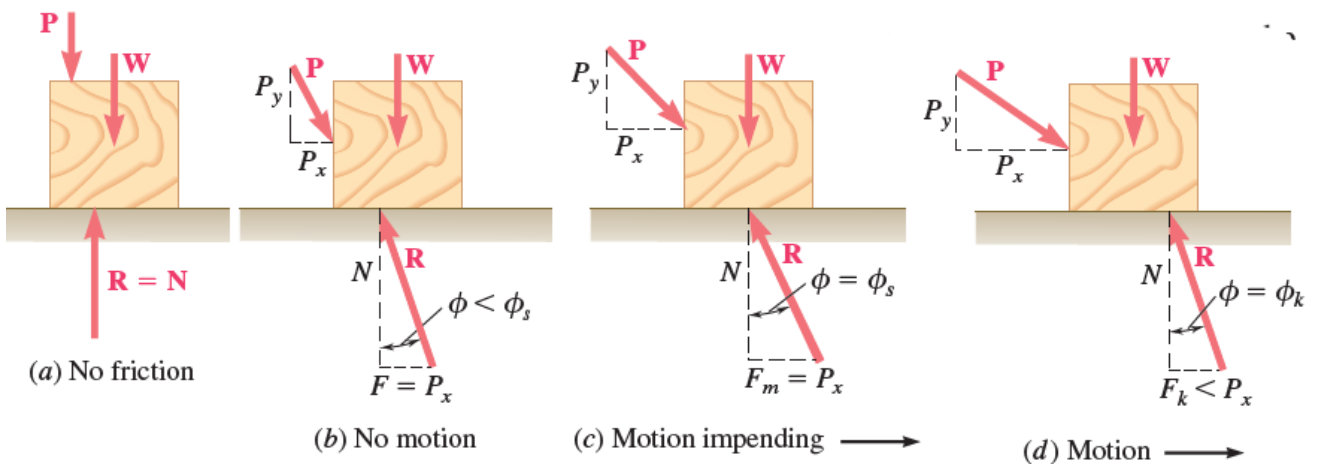
4.5 Problems Involving Dry Friction

Most problems involving friction fall into one of the following three groups.

1. All applied forces are given, and we know the coefficients of friction; we are to determine whether the body being considered remains at rest or slides. The friction force F required to maintain equilibrium is unknown (its magnitude is not equal to μsN) and needs to be determined, together with the normal force N, by drawing a free-body diagram and solving the equations of equilibrium (Fig. 4.15a). We then compare the value found for the magnitude F of the friction force with the maximum value $F_m = \mu_s N$. If F is smaller than or equal to Fm, the body remains at rest. If the value found for F is larger than Fm, equilibrium cannot be maintained and motion takes place; the actual magnitude of the friction force is then $F_k = \mu_k N$.

(a)

2. All applied forces are given, and we know the motion is impending; we are to determine the value of the coefficient of static friction. Here again, we determine the friction force and the normal force by drawing a free body diagram and solving the equations of equilibrium (Fig. 4.15b). Since we know that the value found for F is the maximum value Fm, we determine the coefficient of friction by solving the equation $F_m = \mu_s N$.

(b)

3. The coefficient of static friction is given, and we know that the motion is impending in a given direction; we are to determine the magnitude or the direction of one of the applied forces. The friction force should be shown in the free-body diagram with a sense opposite to that of the impending motion and with a magnitude Fm 5 μsN (Fig. 4.15c). We can then write the equations of equilibrium and determine the desired force. As noted previously, when only three forces are involved, it may be more convenient to represent the reaction of the surface by a single force R and to solve the problem by drawing a force triangle.

(c)
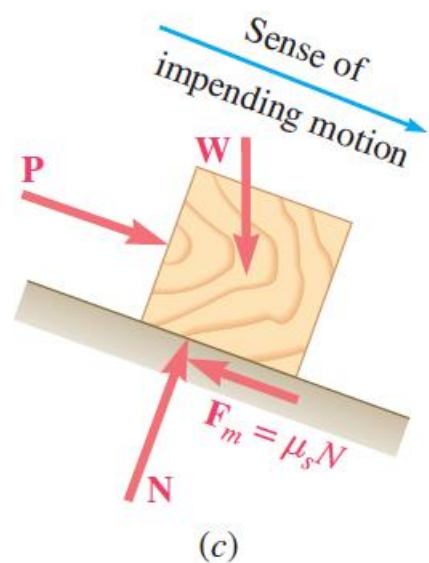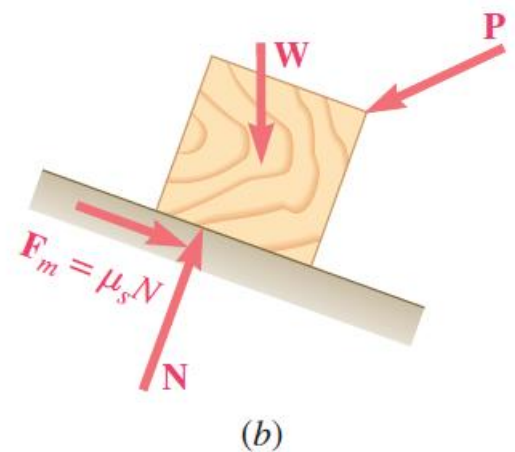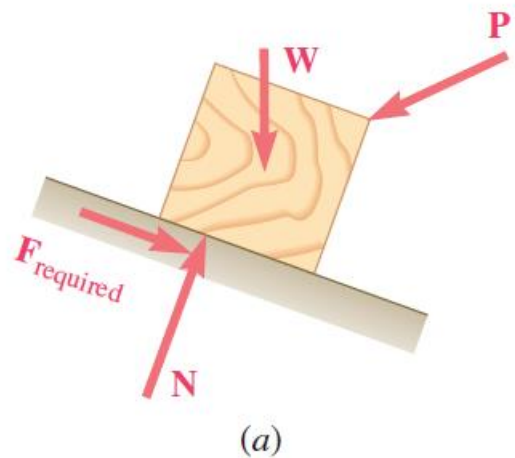
4.6 Procedure for Analysis

Equilibrium problems involving dry friction can be solved using the following procedure:

1-Free-Body Diagrams.

• Draw the necessary free-body diagrams, and unless it is stated in the problem that impending motion or slipping occurs, *always* show the frictional forces as unknowns (i.e., *do not assume $F = \mu N$*).

• Determine the number of unknowns and compare this with the number of available equilibrium equations.

• If there are more unknowns than equations of equilibrium, it will be necessary to apply the frictional equation at some, if not all, points of contact to obtain the extra equations needed for a complete solution.

• If the equation $F = \mu N$ is to be used, it will be necessary to show **F** acting in the correct sense of direction on the free-body diagram.

2-Equations of Equilibrium and Friction.

• Apply the equations of equilibrium and the necessary frictional equations (or conditional equations if tipping is possible) and solve for the unknowns.

Example : A 100 N force acts as shown on a 300-N crate placed on an inclined plane. The coefficients of friction between the crate and the plane are μs 0.25 and μk 0.20. Determine whether the crate is in equilibrium, and find the value of the friction force.

STRATEGY: This is a friction problem of the first type: you know the forces and the friction coefficients and want to determine if the crate moves. You also want to find the friction force.

MODELING and ANALYSIS

Force Required for Equilibrium. First determine the value of the friction force required to maintain equilibrium. Assuming that F is directed down and to the left, draw the free-body diagram of the crate (Fig. 1) and solve the equilibrium equations:

$$\Sigma Fx = 0 \qquad 100 - \frac{3}{5}(300) - F = 0$$

$$F = -80$$
$$F = 80\ N \quad \nearrow$$

$$\Sigma Fy = 0 \qquad N - \frac{4}{5}(300) = 0$$

$$N = 240\ N \quad \nwarrow$$

The force F required to maintain equilibrium is an 80-lb force directed up and to the right; the tendency of the crate is thus to move down the plane. Maximum Friction Force. The magnitude of the maximum friction force that may be developed between the crate and the plane is

$$Fm = \mu_s N = 0.25(240) = 60N$$

Since the value of the force required maintaining equilibrium (80 N) is larger than the maximum value that may be obtained (60 N), equilibrium is not maintained and the crate will slide down the plane.

Actual Value of Friction Force. The magnitude of the actual friction force is

$$Factual = F_k = \mu_k N = 0.20(240) = 48\ N$$

The sense of this force is opposite to the sense of motion; the force is thus directed up and to the right (Fig. 2) $Factual = 48\ N \quad \nearrow$

Note that the forces acting on the crate are not balanced. Their resultant is

$$\frac{3}{5}(300) - 100 - 48 = 32\ N \quad \swarrow$$

REFLECT and THINK: This is a typical friction problem of the first type. Note that you used the coefficient of static friction to determine if the crate moves, but once you found that it does move, you needed the coefficient of kinetic friction to determine the friction force.

Fig. 1 Free-body diagram of crate showing assumed direction of friction force.

Fig. 2 Free-body diagram of crate showing actual friction force.
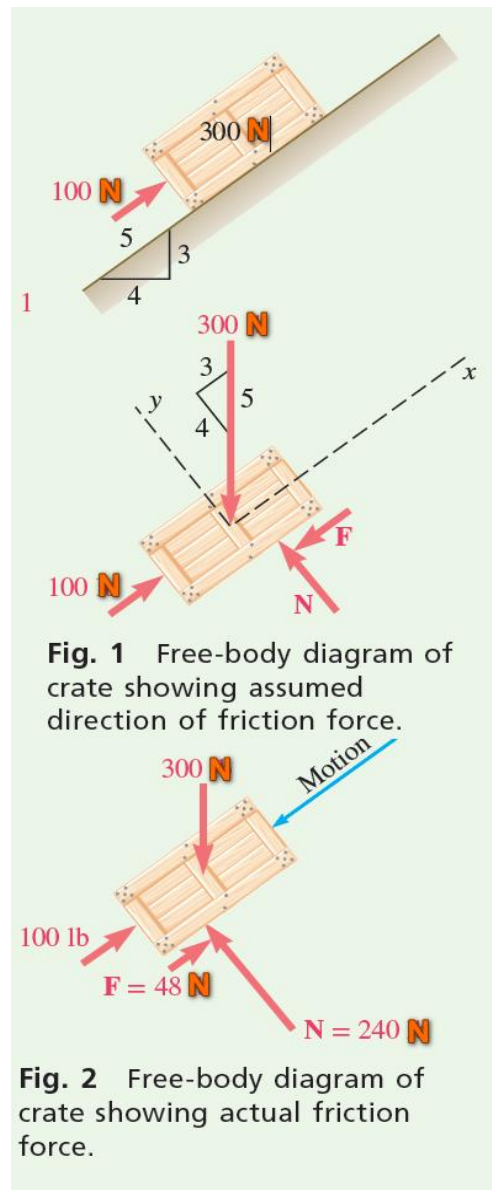
Example: A 100 N force acts as shown on a 100-N crate placed on an inclined plane. The coefficients of friction between the crate and the plane are μs 0.25 and μk 0.20. Determine whether the crate is in equilibrium, and find the value of the friction force.

$\Sigma Fx = 0 \qquad 100 - \dfrac{3}{5}(100) - F = 0$

$\quad F = 40\ N$

$\Sigma Fy = 0 \qquad N - \dfrac{4}{5}(100) = 0$

$\quad N = 80\ N$

$Fm = \mu_s\, N = 0.25(80) = 20N < F{=}40\text{N}$

Equilibrium is not maintained and the crate will slide up the plane.

$F\text{actual} = F_k = \mu_k N = 0.20(80) = 16\ N$

The sense of this force is opposite to the sense of motion; the force is thus directed down and to the left

Note that the forces acting on the crate are not balanced. Their resultant is

$$100 - \dfrac{3}{5}(100) - 16 = 32\ N$$
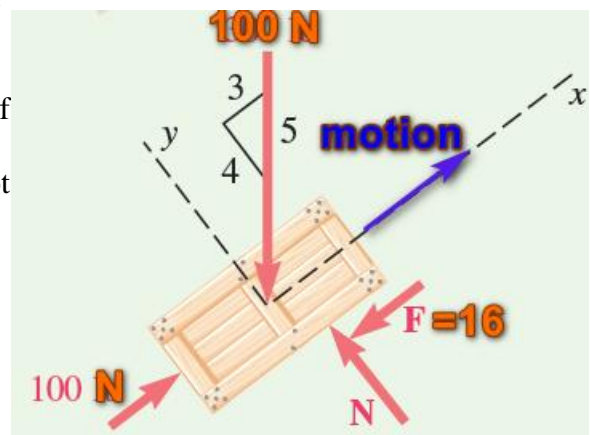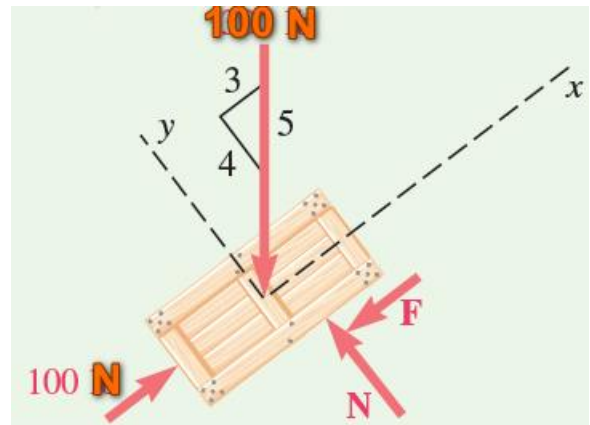
Example: A 100 N force acts as shown on a 200-N crate placed on an inclined plane. The coefficients of friction between the crate and the plane are μs 0.25 and μk 0.20. Determine whether the crate is in equilibrium, and find the value of the friction force.

$\Sigma Fx = 0 \qquad 100 - \dfrac{3}{5}(200) - F = 0$

$\quad F = -20$

$\quad F = 20\ N$

$\Sigma Fy = 0 \qquad N - \dfrac{4}{5}(200) = 0$

$\quad N = 160\ N$

$Fm = \mu_s\, N = 0.25(160) = 40N < \quad F = 20\ N$

Equilibrium is maintained and the crate will be at rest.
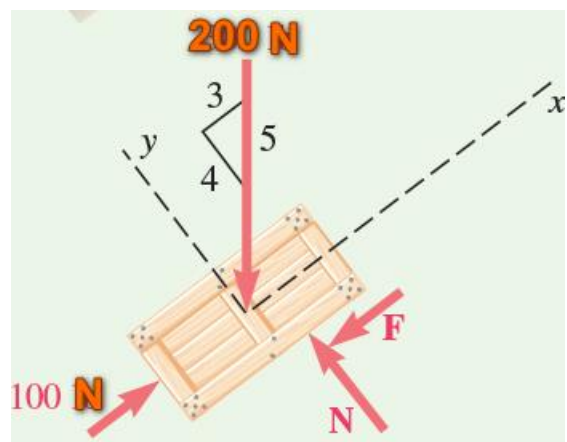
Example A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the force **P** required to (*a*) start the block moving up the incline; (*b*) keep it moving up; (*c*) prevent it from sliding down.

a. Force *P* to Start Block Moving Up.

$\Sigma Fx = 0 \qquad F + (800)\sin(25) - P\cos(25) = 0 \quad ---1$

$\Sigma Fy = 0 \qquad N + (800)\cos(25) - P\sin(25) = 0 \quad ---2$

Due to motion is impending up the incline, thus

$$F = \mu_s N = 0.35\ N \qquad ----3$$

By solving eq.s (1),(2) and (3)

$$P = 780\ N$$
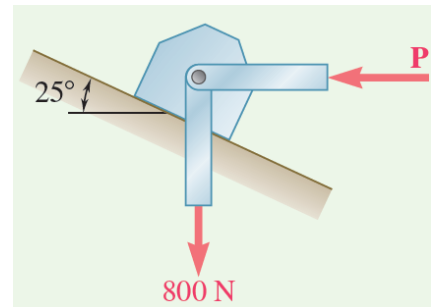
b. Force *P* to Keep Block Moving Up

$\Sigma Fx = 0 \qquad F + (800)\sin(25) - P\cos(25) = 0 \quad ---1$

$\Sigma Fy = 0 \qquad N + (800)\cos(25) - P\sin(25) = 0 \quad ---2$

To keep it moving up the incline

$$F = \mu_k N = 0.25\ N \qquad ------3$$

By solving eq.s (1),(2) and (3)

$$P = 694\ N$$
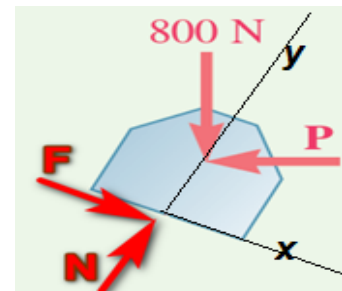
C. Force P to Prevent Block from Sliding Down

$\Sigma Fx = 0 \qquad -F + (800)\sin(25) - P\cos(25) = 0 \quad ---1$

$\Sigma Fy = 0 \qquad N + (800)\cos(25) - P\sin(25) = 0 \quad ---2$

Due to motion is impending up the incline, thus

$$F = \mu_k N = 0.25\ N \qquad ----3$$

By solving eq.s (1),(2) and (3)

$$P = 80.0\ N$$
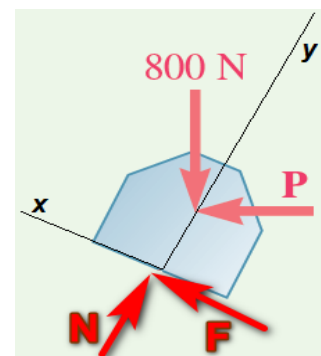
Sample Problem 4.12
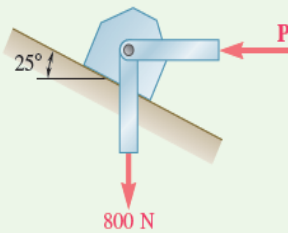
A support block is acted upon by two forces as shown. Knowing that the coefficients of friction between the block and the incline are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine the force **P** required to (a) start the block moving up the incline; (b) keep it moving up; (c) prevent it from sliding down.

STRATEGY: This problem involves practical variations of the third type of friction problem. You can approach the solutions through the concept of the angles of friction.

MODELING:

Free-Body Diagram. For each part of the problem, draw a free-body diagram of the block and a force triangle including the 800-N vertical force, the horizontal force **P**, and the force **R** exerted on the block by the incline. You must determine the direction of **R** in each separate case. Note that, since **P** is perpendicular to the 800-N force, the force triangle is a right triangle, which easily can be solved for **P**. In most other problems, however, the force triangle will be an oblique triangle and should be solved by applying the law of sines.

ANALYSIS:

a. Force *P* to Start Block Moving Up. In this case, motion is impending up the incline, so the resultant is directed at the angle of static friction (Fig. 1). Note that the resultant is oriented to the left of the normal such that its friction component (not shown) is directed opposite the direction of impending motion.

$$P = (800\text{ N}) \tan 44.29° \qquad \mathbf{P} = 780\text{ N} \leftarrow$$

b. Force *P* to Keep Block Moving Up. Motion is continuing, so the resultant is directed at the angle of kinetic friction (Fig. 2). Again, the resultant is oriented to the left of the normal such that its friction component is directed opposite the direction of motion.

$$P = (800\text{ N}) \tan 39.04° \qquad \mathbf{P} = 649\text{ N} \leftarrow$$

c. Force *P* to Prevent Block from Sliding Down. Here, motion is impending down the incline, so the resultant is directed at the angle of static friction (Fig. 3). Note that the resultant is oriented to the right of the normal such that its friction component is directed opposite the direction of impending motion.

$$P = (800\text{ N}) \tan 5.71° \qquad \mathbf{P} = 80.0\text{ N} \leftarrow$$

REFLECT and THINK: As expected, considerably more force is required to begin moving the block up the slope than is necessary to restrain it from sliding down the slope.

$\tan \phi_s = \mu_s$
$= 0.35$
$\phi_s = 19.29°$
$25° + 19.29° = 44.29°$

Fig. 1 Free-body diagram of block and its force triangle—motion impending up incline.

$\tan \phi_k = \mu_k$
$= 0.25$
$\phi_k = 14.04°$
$25° + 14.04° = 39.04°$

Fig. 2 Free-body diagram of block and its force triangle—motion continuing up incline.

$\phi_s = 19.29°$
$25° - 19.29° = 5.71°$

Fig. 3 Free-body diagram of block and its force triangle—motion prevented down the slope.

Example: The uniform thin pole has a weight of 50 N and a length of 6m. If it is placed against the smooth wall and on the rough floor in the position d = 2 m, the coefficient of static friction is 0.3.

1- Will it remain in this position when it is released?

2- Determine the maximum distance d it can be placed from the smooth wall and not slip

Solution

1- No motion

$\Sigma MA = 0 \quad 50(1) - N_2(5.66) = 0$

$\qquad N_2 = 8.81$ N

$\Sigma Fx = 0 \quad N_2 = F_1 = 8.81$ N

$\Sigma Fy = 0 \quad N_1 = 50$ N

to check the motion

$Fm_1 = \mu_s N_1 = 0.3 * 50 = 15$ N $> F_1 = 8.81$ N   O. K.

The pole remains in this position.

2- Impending motion at point A

$F_1 = Fm_1 = \mu_s N_1 = 0.3 N_1$

$\Sigma Fy = 0 \quad N_1 = 50$ N

$F_1 = 0.3 * 50 = 15$ N

$\Sigma Fx = 0 \quad N_2 = F_1 = 15$ N

$\Sigma MA = 0 \quad 50\left(\dfrac{d}{2}\right) - 15(5.66) = 0$

$\boldsymbol{d = 3.4\ m}$

(a)
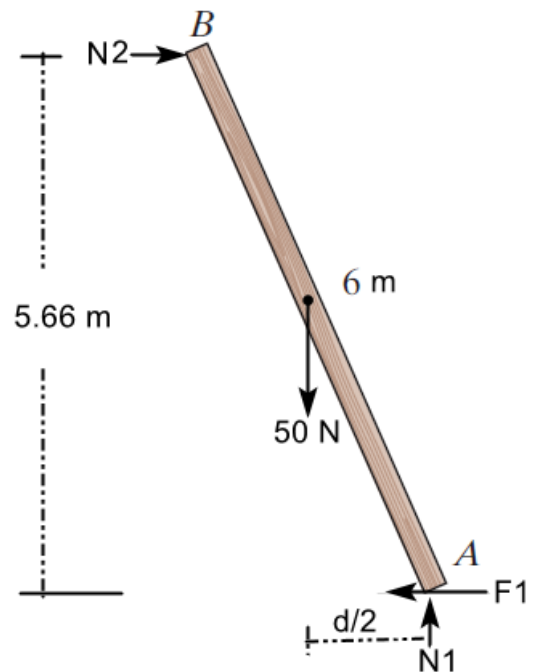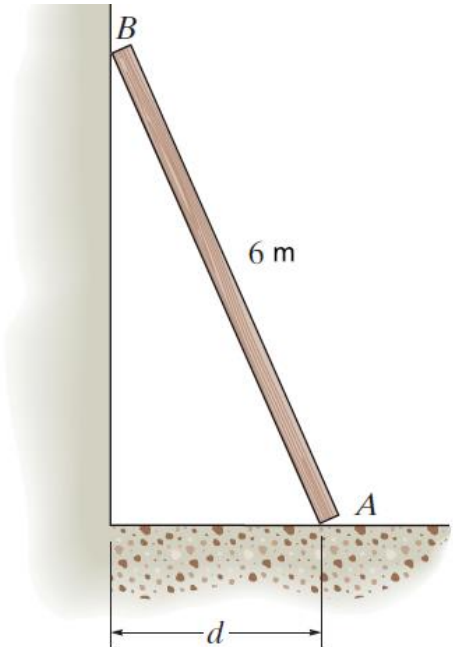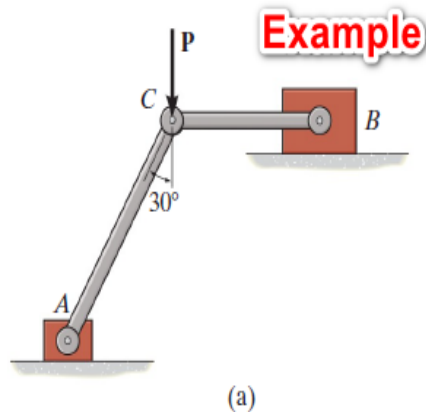
Blocks A and B have a mass of 3 kg and 9 kg, respectively, and are connected to the weightless links shown in Fig. 8–11a. Determine the largest vertical force $\mathbf{P}$ that can be applied at the pin C without causing any movement. The coefficient of static friction between the blocks and the contacting surfaces is $\mu_s = 0.3$.

SOLUTION

Free-Body Diagram. The links are two-force members and so the free-body diagrams of pin C and blocks A and B are shown in Fig. 8–11b. Since the horizontal component of $\mathbf{F}_{AC}$ tends to move block A to the left, $\mathbf{F}_A$ must act to the right. Similarly, $\mathbf{F}_B$ must act to the left to oppose the tendency of motion of block B to the right, caused by $\mathbf{F}_{BC}$. There are seven unknowns and six available force equilibrium equations, two for the pin and two for each block, so that *only one* frictional equation is needed.

Equations of Equilibrium and Friction. The force in links AC and BC can be related to P by considering the equilibrium of pin C.

$$+\uparrow\Sigma F_y = 0; \qquad F_{AC}\cos 30° - P = 0; \qquad F_{AC} = 1.155P$$

$$\xrightarrow{+}\Sigma F_x = 0; \qquad 1.155P\sin 30° - F_{BC} = 0; \qquad F_{BC} = 0.5774P$$

Using the result for F_{AC}, for block A,

$$\xrightarrow{+}\Sigma F_x = 0; \quad F_A - 1.155P\sin 30° = 0; \quad F_A = 0.5774P \qquad (1)$$

$$+\uparrow\Sigma F_y = 0; \quad N_A - 1.155P\cos 30° - 3(9.81\text{ N}) = 0;$$

$$N_A = P + 29.43\text{ N} \qquad (2)$$

Using the result for F_{BC}, for block B,

$$\xrightarrow{+}\Sigma F_x = 0; \qquad (0.5774P) - F_B = 0; \qquad F_B = 0.5774P \qquad (3)$$

$$+\uparrow\Sigma F_y = 0; \qquad N_B - 9(9.81)\text{ N} = 0; \qquad N_B = 88.29\text{ N}$$

Movement of the system may be caused by the initial slipping of *either* block A or block B. If we assume that block A slips first, then

$$F_A = \mu_s N_A = 0.3N_A \qquad (4)$$

Substituting Eqs. 1 and 2 into Eq. 4,

$$0.5774P = 0.3(P + 29.43)$$

$$P = 31.8\text{ N} \qquad \textit{Ans.}$$

Substituting this result into Eq. 3, we obtain $F_B = 18.4$ N. Since the maximum static frictional force at B is $(F_B)_{max} = \mu_s N_B = 0.3(88.29\text{ N}) = 26.5\text{ N} > F_B$, block B will not slip. Thus, the above assumption is correct. Notice that if the inequality were not satisfied, we would have to assume slipping of block B and then solve for P.
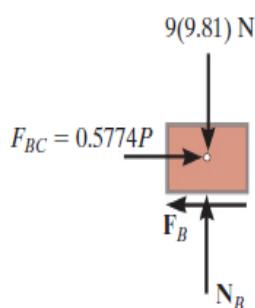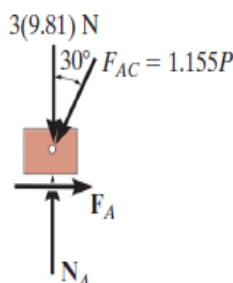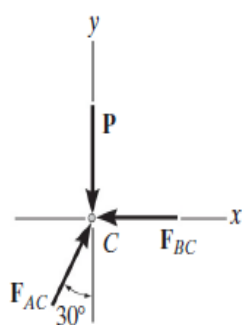
(b)

Fig. 8–11

Example: Determine the max. value which P may have before any sliping takes place

Solution

Case 1: The body (50kg) slips only

$F_1 = Fm_1 = \mu_s N_1 = 0.3 N_1$

$F_2 = Fm_2 = \mu_s N_2 = 0.4 N_2$

F.B.D. of Block (30kg)

$\Sigma Fy = 0 \quad N_1 + (30 * 9.81)\cos(30^o) = 0$

$N_1 = 225 \, N$

$F_1 = 0.30\,(225) = 76.5 \, N$

F.B.D. of Block (50kg)

$\Sigma Fy = 0 \qquad N_2 - (50 * 9.81)\cos(30^o) - 225 = 0$

$N_2 = 680 \, N$

$F_2 = 0.4\,(680) = 272 \, N$

$\Sigma Fx = 0 \qquad P - 76.5 - 272 + (50 * 9.81)\sin(30^o) = 0$

$P = 103.1 \, N$

Case2: body (50kg) and body (40kg) slip as unit

$F_1 = \mu_s N_1 = 0.3 N_1$

$F_3 = \mu_s N_3 = 0.45 N_3$

F.B.D. of Block (30kg)

$\Sigma Fy = 0 \quad N_1 + (30 * 9.81)\cos(30^o) = 0$

$N_1 = 225 \, N$

$F_1 = 0.30\,(225) = 76.5 \, N$

F.B.D. of Block (50kg) and body (40kg)

$\Sigma Fy = 0 \qquad N_3 - (50 * 9.81)\cos(30^o) - (40 * 9.81) - 225 = 0$

$N_3 = 1019 \, N$

$F_3 = 0.45\,(1019) = 495 \, N$

$\Sigma Fx = 0 \qquad P - 76.5 - 495 + ((50 + 40) * 9.81)\sin(30^o) = 0$

$P = 93.8 \, N$

∴ $P = 93.8 \, N$

case 1

case 2
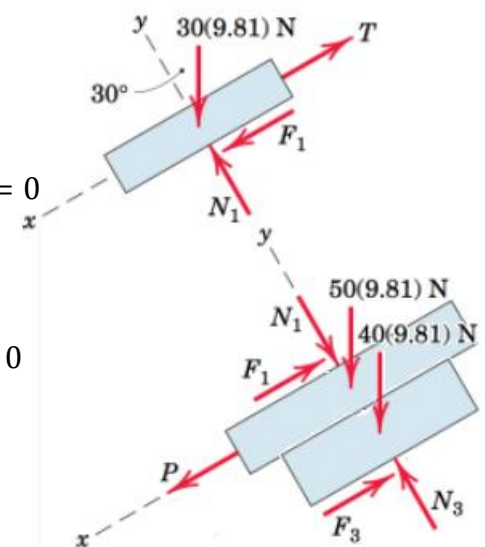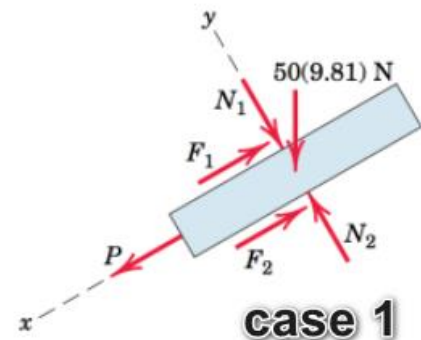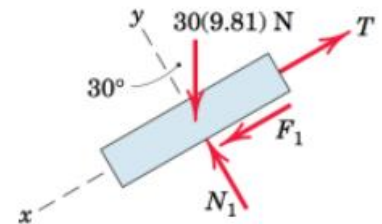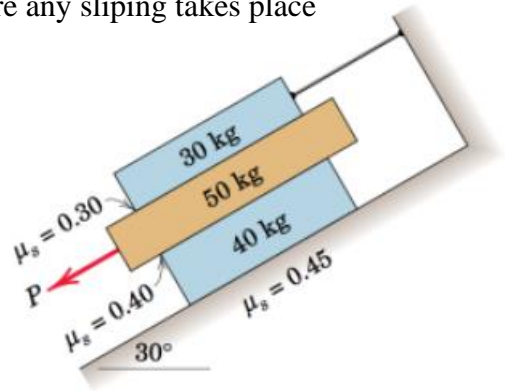
Example: Determine the smallest couple moment that can be applied to the 180N wheel that will cause impending motion. The uniform concrete block has a weight of 232N. The coefficient of static friction is 0.4 at all contacting surfaces.

Solution
There are two possibilities

Case 1: the wheel slips first

$F_1 = Fm_1 = \mu_s N_1 = 0.4\, N_1$
$F_2 = Fm_2 = \mu_s N_2 = 0.4\, N_2$

F.B.D. of the wheel
$\Sigma Fy = 0 \qquad N1 + F_2 = 232$
$\qquad\qquad\qquad N1 + 0.4\, N_2 = 232 \qquad --(1)$
$\Sigma Fx = 0 \qquad N_2 = F_1$
$\qquad\qquad\qquad N_2 = 0.4 * N_1 \qquad ----(2)$
$N_1 = 200\ N \qquad N_2 = 80\ N$
$F_1 = 80\ N \qquad\quad F_2 = 32\ N$
$\Sigma Mo = 0 \qquad F_1 * 0.15 + F_2 * .015 - M = 0$
$\quad 80 * 0.15 + 32 * .015 - M = 0$
$M = 16.8\ N.m$
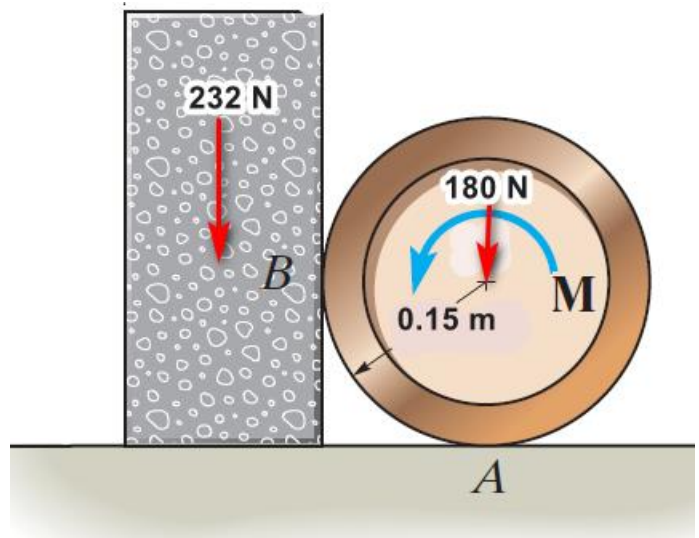
Case 2: The block slips first
$F_2 = Fm_2 = \mu_s N_2 = 0.4\, N_2$
$F_3 = Fm_3 = \mu_s N_3 = 0.4\, N_3$

F.B.D. of the block
$\Sigma Fy = 0 \qquad N3 - F_2 = 180$
$\qquad\qquad\qquad N3 + 0.4\, N_2 = 180 \qquad --(1)$
$\Sigma Fx = 0 \qquad N_2 = F_3$
$\qquad\qquad\qquad N_2 = 0.4 * N_3 \qquad ----(2)$
By solving eq.s (1) and (2)
$N_3 = 214.3\ N \qquad N_2 = 85.7\ N$
$F_3 = 85.7\ N \qquad\quad F_2 = 34.3\ N$
F.B.D. of the wheel
$\Sigma Fx = 0 \qquad F_1 = N_2 = 85.7\ N$

$\Sigma Mo = 0 \qquad F_1 * 0.15 + F_2 * .015 - M = 0$
$\quad 85.72 * 0.15 + 34.29 * .015 - M = 0$
$M = 46.9\ N.m$
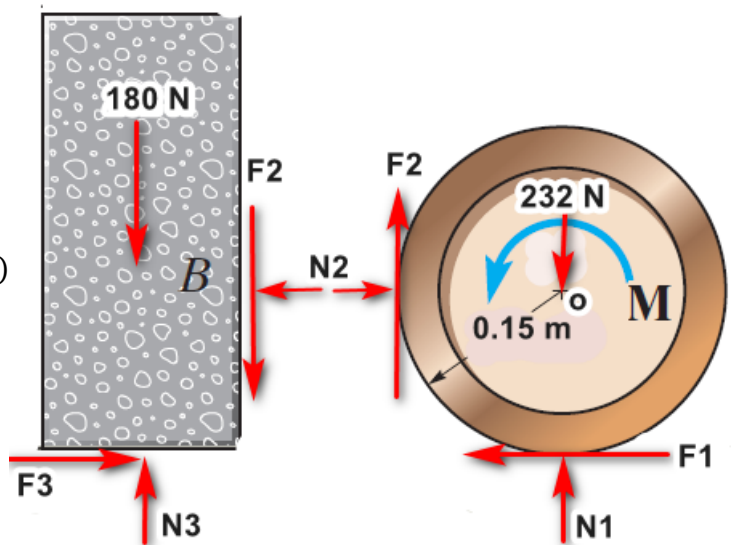
∴ **Use smaller   M= 46.9 N.M**

Example Blocks A and B have a mass of 40 N and 50 N, respectively. Using the coefficients of static friction indicated, determine the largest force P which can be applied to the cord without causing motion. There are pulleys at C and D.
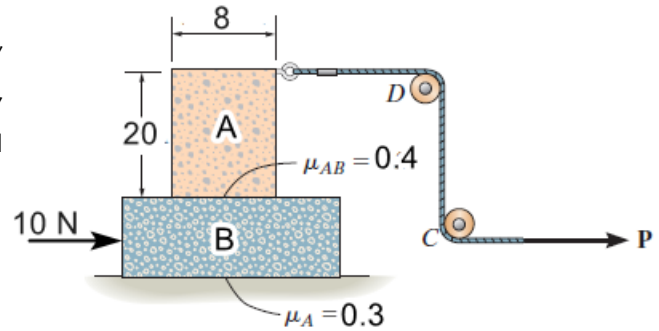
Solution

Case1 : Block A slips first

$$F_1 = Fm_1 = \mu_s N_1 = 0.4 N_1$$
$$\Sigma Fy = 0 \qquad N_2 = 40$$
$$F_1 = 0.4 * 40 = 16$$
$$\Sigma Fx = 0 \qquad F_2 = P$$
$$P = 16 \text{ N}$$
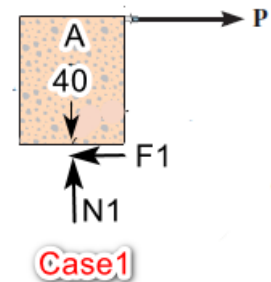
Case 2: Block A and Block B slip as a unit

$$F_2 = Fm_2 = \mu_s N_2 = 0.4 N_2$$
$$\Sigma Fy = 0 \qquad N_2 - 50 - 40 = 0$$
$$N_2 = 90$$
$$F_2 = 0.3 * 90 = 27$$
$$\Sigma Fx = 0 \qquad F_2 - 10 - P = 0$$
$$P = 17 \text{ N}$$
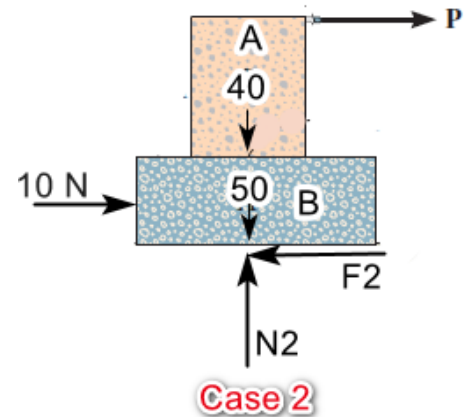
Case 3 Block A tips about Point O

$$\Sigma Mo = 0 \qquad 40 * 4 - 20P = 0$$
$$P = 8 \text{ N}$$
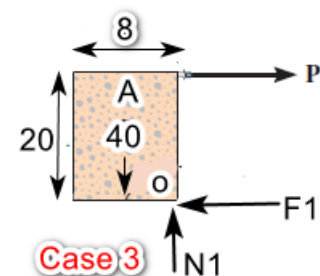
$$\therefore Use\ smallest\ value\ of \quad P = 8 \ N$$

H.W. A cable is attached to the 20-kg plate B, passes over a fixed peg at C, and is attached to the block at A. Using the coefficients of static friction shown, determine the smallest mass of block A so that it will prevent sliding motion of B down the plane.

$\mu_A = 0.2$
$\mu_B = 0.3$
$30°$
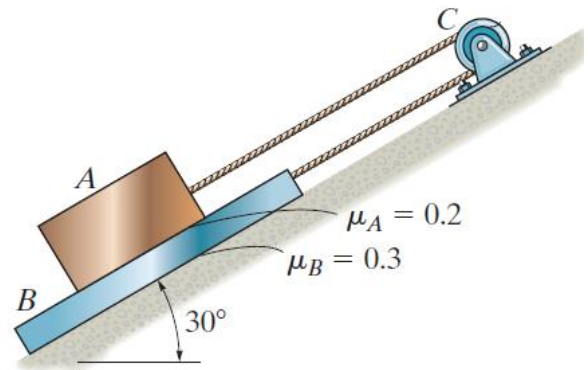
H.W Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest vertical force P which can be applied to the cord without causing motion.

300 mm

$\mu_B = 0.4$

B   400 mm

A

$\mu_A = 0.3$
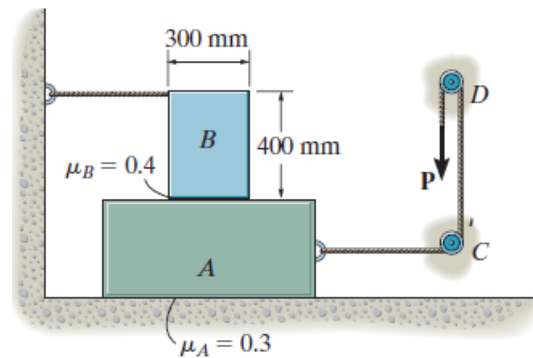
H.W The uniform thin pole has a weight of 50 N and a length of 26m. If it is placed against the rough wall and on the rough box in the position d = 10 m,
μs = 0.3 at points A, B and C
P= 150 N
Box weight = 100 N
Determine the distance x when the motion of the pole impend

X

B

P

26

A

C

2 2

d

H.W.
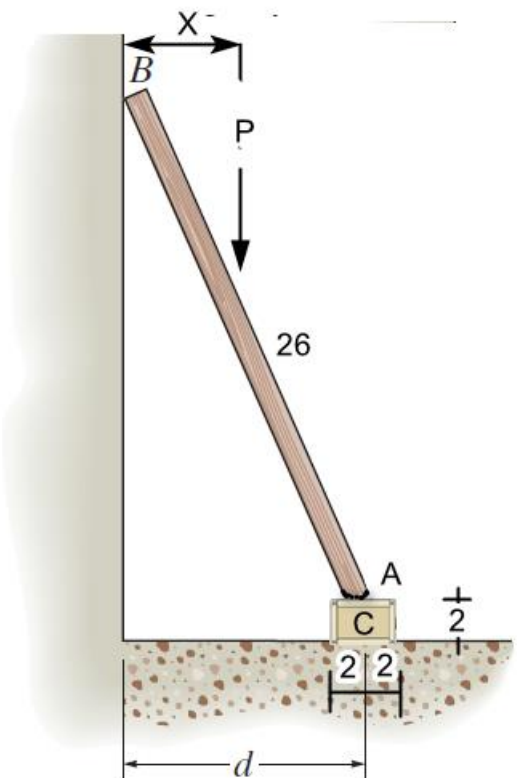
Determine the horizontal force P required to cause slippage to occur. The friction coefficients for the three pairs of mating surfaces are indicated. The top block is free to move vertically.

P

100 kg

50 kg

$\mu = 0.60$

$\mu = 0.40$

20 kg $\mu = 0.30$
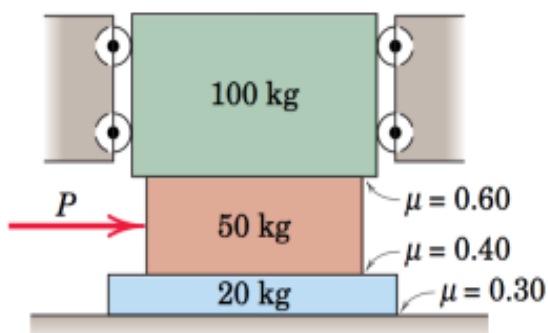
4.7 Wedges

A wedge is a simple machine that is often used to transform an applied force into much larger forces, directed at approximately right angles to the applied force. Wedges also can be used to slightly move or adjust heavy loads.

Consider, for example, the wedge shown in Fig. 8–12a, which is used to lift the block by applying a force to the wedge. Free-body diagrams of the block and wedge are shown in Fig. 8–12b. Here we have excluded the weight of the wedge since it is usually small compared to the weight W of the block. Also, note that the frictional forces F1 and F2 must oppose the motion of the wedge. Likewise, the frictional force F3 of the wall on the block must act downward so as to oppose the block's upward motion.

The locations of the resultant normal forces are not important in the force analysis since neither the block nor wedge will "tip." Hence the moment equilibrium equations will not be considered. There are seven unknowns, consisting of the applied force P, needed to cause motion of the wedge, and six normal and frictional forces. The seven available equations consist of four force equilibrium equations, $\Sigma Fx = 0$, $\Sigma Fy = 0$ applied to the wedge and block, and three frictional equations, $F = \mu N$ applied at each surface of contact.

If the block is to be lowered, then the frictional forces will all act in a sense opposite to that shown in Fig. 8–12b. Provided the coefficient of friction is very small or the wedge angle u is large, then the applied force P must act to the right to hold the block. Otherwise, P may have a reverse sense of direction in order to pull on the wedge to remove it. If P is not applied and friction forces hold the block in place, then the wedge is referred to as self-locking.
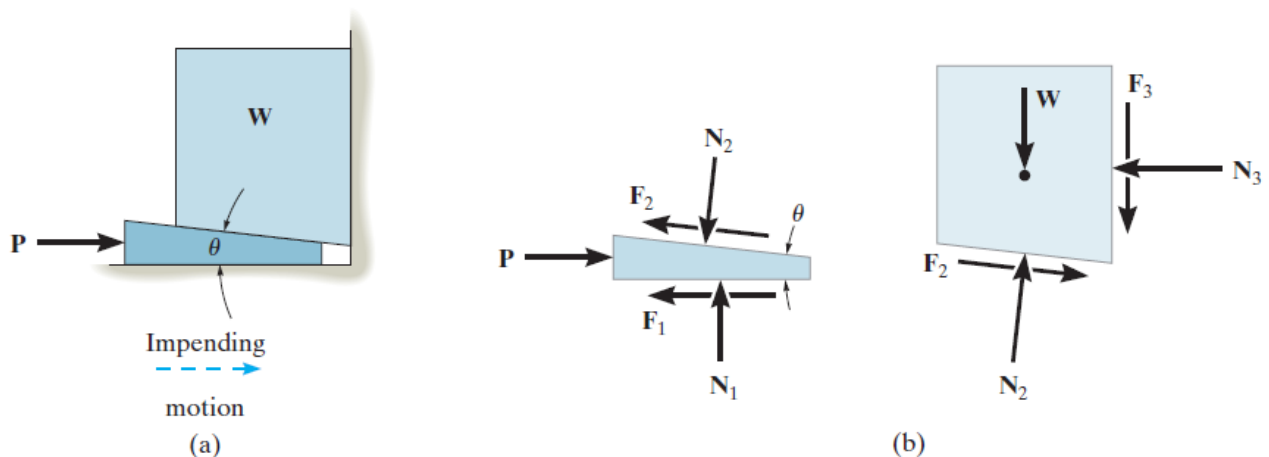
Fig. 8–12

The answer by using the friction resultant and friction angle
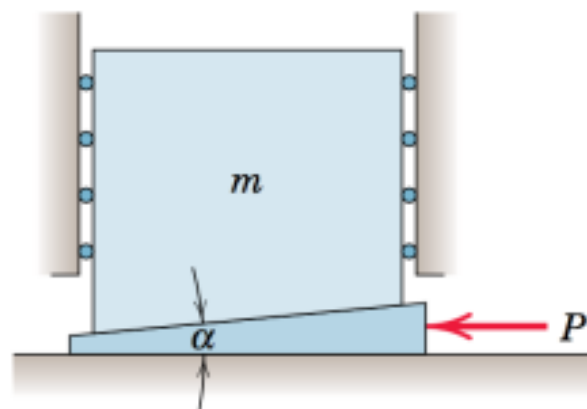
(a)
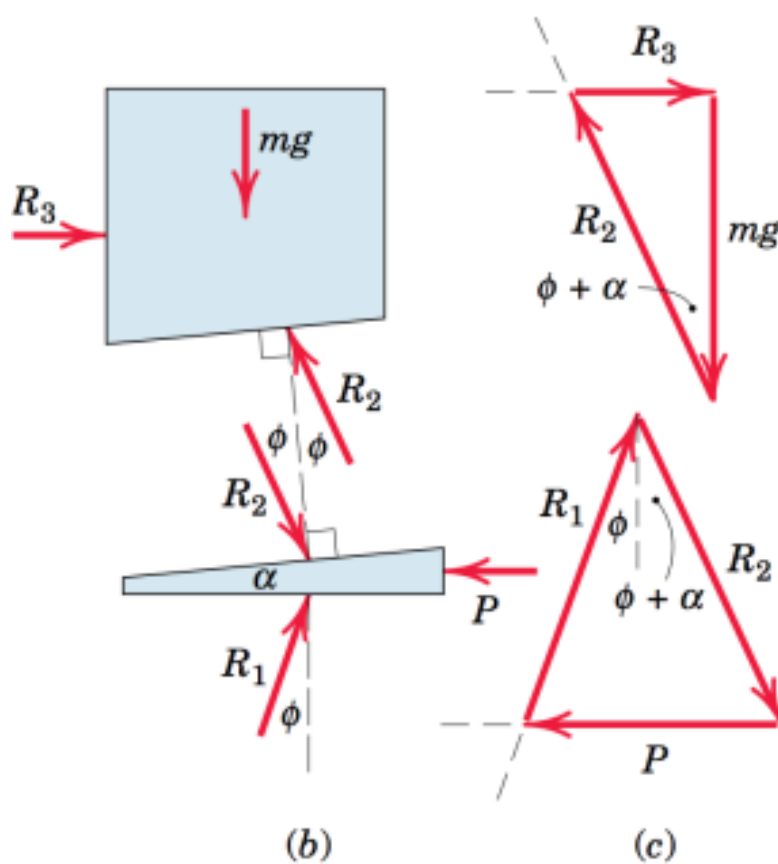
(b)

(c)

Forces to raise load

The uniform stone in Fig. 8–13a has a mass of 500 kg and is held in the horizontal position using a wedge at B. If the coefficient of static friction is $\mu_s = 0.3$ at the surfaces of contact, determine the minimum force P needed to remove the wedge. Assume that the stone does not slip at A.
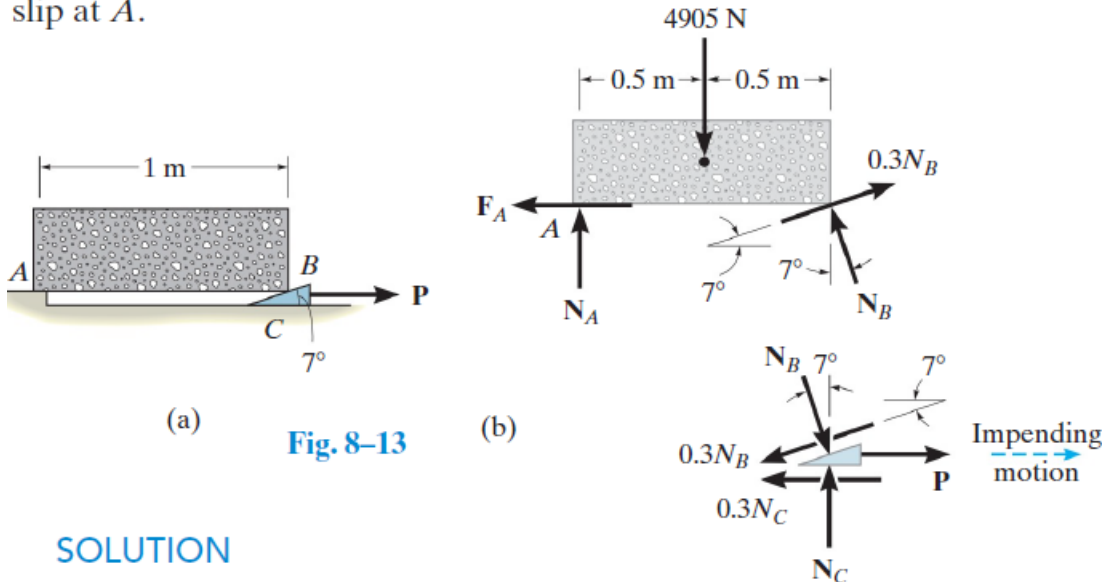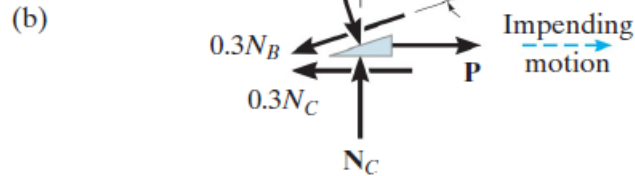
Fig. 8–13

SOLUTION

The minimum force P requires $F = \mu_s N$ at the surfaces of contact with the wedge. The free-body diagrams of the stone and wedge are shown in Fig. 8–13b. On the wedge the friction force opposes the impending motion, and on the stone at $A, F_A \leq \mu_s N_A$, since slipping does not occur there. There are five unknowns. Three equilibrium equations for the stone and two for the wedge are available for solution. From the free-body diagram of the stone,

$\zeta + \Sigma M_A = 0;$ -4905 N$(0.5$ m$) + (N_B \cos 7° $ N$)(1$ m$)$

$$+ (0.3N_B \sin 7° \text{ N})(1 \text{ m}) = 0$$
$$N_B = 2383.1 \text{ N}$$

Using this result for the wedge, we have

$+\uparrow \Sigma F_y = 0;$ $N_C - 2383.1 \cos 7° $ N $- 0.3(2383.1 \sin 7° $ N$) = 0$

$N_C = 2452.5$ N

$\xrightarrow{+} \Sigma F_x = 0;$ $2383.1 \sin 7° $ N $- 0.3(2383.1 \cos 7° $ N$) +$

$$P - 0.3(2452.5 \text{ N}) = 0$$

$P = 1154.9$ N $= 1.15$ kN   *Ans.*

NOTE: Since P is positive, indeed the wedge must be pulled out. If P were zero, the wedge would remain in place (self-locking) and the frictional forces developed at B and C would satisfy $F_B < \mu_s N_B$ and $F_C < \mu_s N_C$.

Example

Determine the smallest values of forces P 1 and P 2 required to rise block A while preventing A from moving horizontally. The coefficient of static friction for all surfaces of contact is 0.3, and the weight of wedges B and C is negligible compared to the weight of block A.

F.B.D. of block A

F.B.D. od block B

The friction force from block C opposes the relative motion of block B.

Impending motion of lower surface of block B relative to block C.
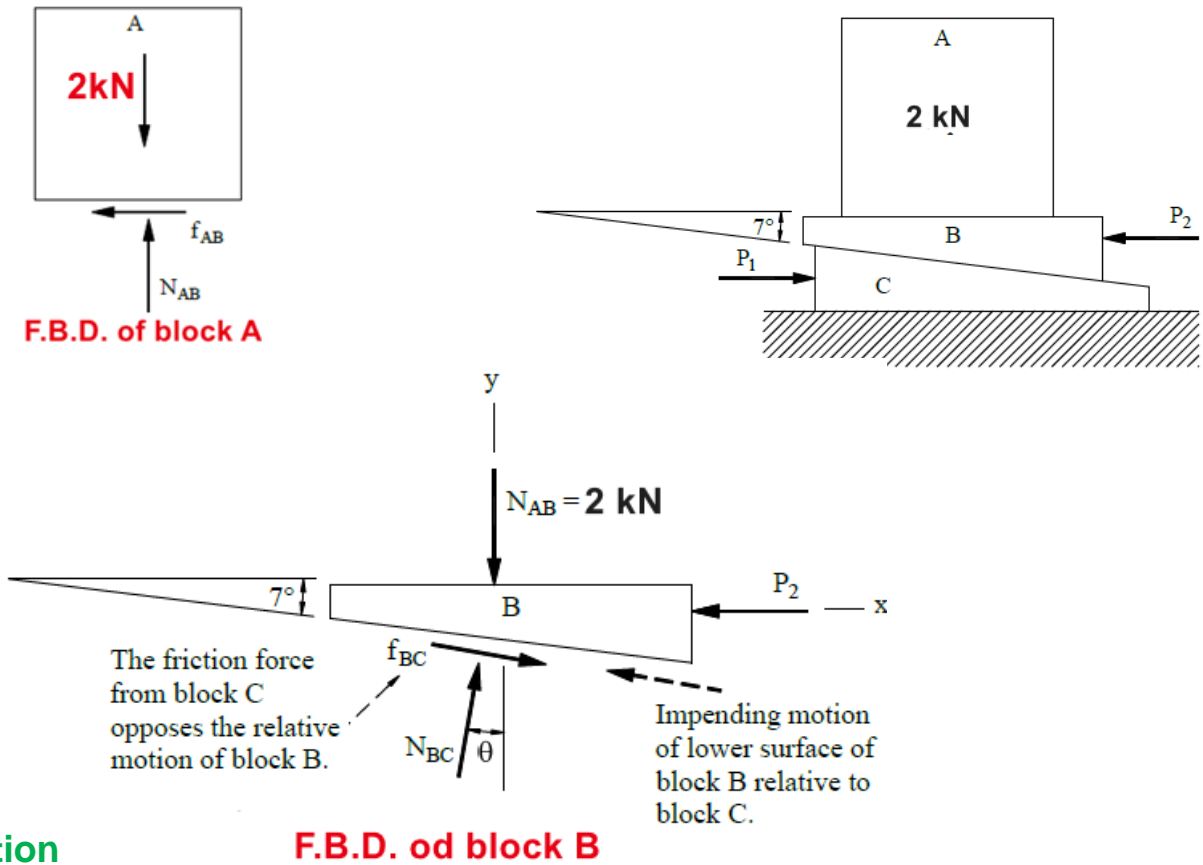
Solution

1 − Equations of equilibrium for block A

$\Sigma Fx = 0 \qquad fAB = 0$

$\Sigma Fy = 0 \qquad NAB = 2 \text{ kN}$

2 − Equations of equilibrium for block B

$\Sigma Fx = 0 \qquad fBC \sin(7^o) + NBC \cos(7^o) - P_2 = 0 \qquad ---1$

$\Sigma Fy = 0 \qquad - fBC \cos(7^o) + NBC \sin(7^o) - 2 = 0 \qquad ---2$

Slip impends between blocks B and C

$fBC = \mu_s NBC = 0.3 NBC \qquad ----3$

By Solving Eqs. 1, 2, and 3 we obtained

$NBC = 2.09 \text{ kN} \qquad fBC = 0.628 \text{ kN} \qquad P_2 = 0.878 \text{ kN}$

2 – Equations of equilibrium for block C

The friction force f_{BC} from block B opposes the relative motion of block C.

$\theta = 7°$ $N_{BC} = 2.092$ kN

Impending motion of upper surface of C relative to B

The friction force f_C from the floor opposes the relative motion of block B.

Impending motion of lower surface of C relative to the floor

F.B.D. of block C
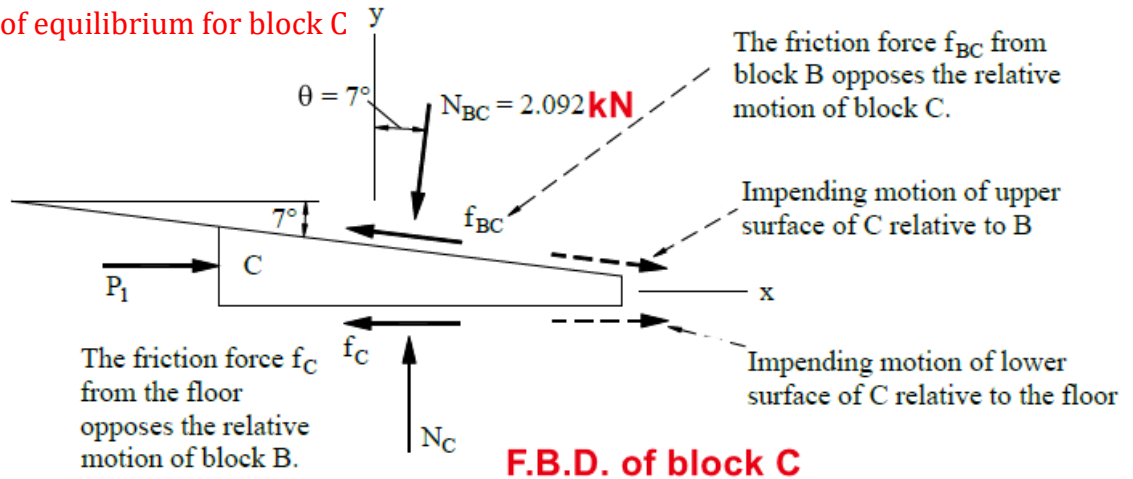
$\Sigma Fx = 0$ $- fBC \, scosin(7°) - 2.09 \sin(7) + P_1 - f_C = 0$ $--- 3$

$\Sigma Fy = 0$ $- fBC \sin(7°) + 2.09 \cos(7°) + N_C = 0$ $--- 4$

Slip impends between blocks C and floor

$f_C = \mu_s N_C = 0.3 N_C$

Solving Eqs. 1, 2, and 3

$N_C = 2.09 \, kN$ $f_C = 0.628 \, kN$ $P_1 = 0.878 \, kN$

Example:

If $P = 250$ N, determine the required minimum Weight for block B so that the wedge will not move to the right. Neglect the weight of A. The coefficient of static friction for all contacting surfaces is $\mu s = 0.35$. Neglect friction at the rollers.

Solution

1 – Equations of equilibrium for block A
Slip impends between blocks A and floor

$F_2 = \mu_s N_2 = 0.35 N_2$

Slip impends between blocks A and B

$F_1 = \mu_s N_1 = 0.35 N_1$

$\Sigma Fy = 0$ $N_2 \cos(10°) - (0.35 N_2) \sin(10°) - N_1 = 0$

$\Sigma Fx = 0$ $P - (0.4 N_2) \cos(10°) + N_2 \sin(10°) - 0.4(N_1) = 0$

  N1 =   kN

2 – Equations of equilibrium for block B
Slip impends between blocks A and B

$\Sigma Fy = 0$ $WB = N1 =$ kN
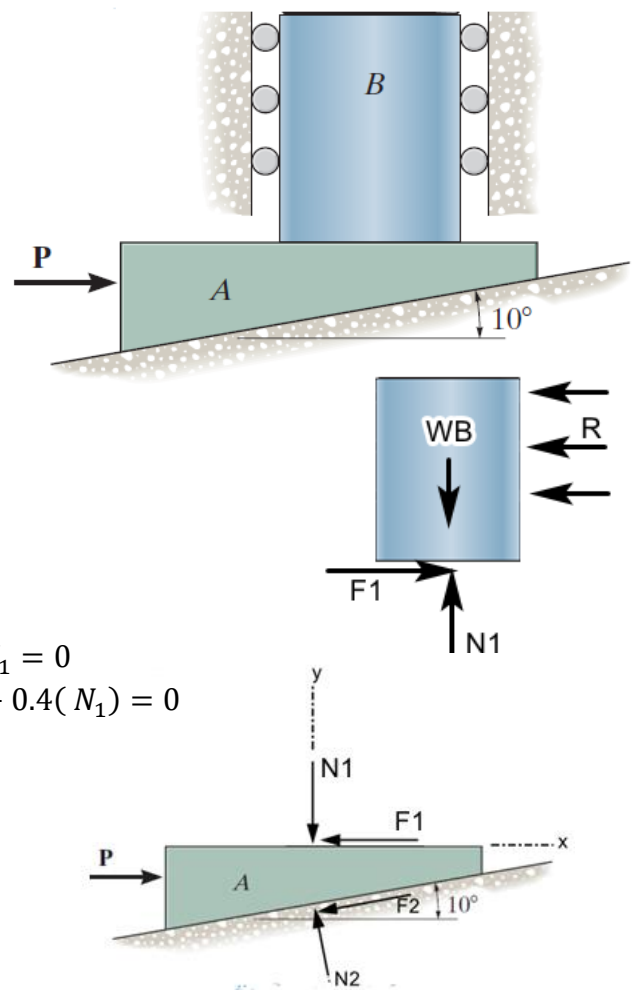
Example : Find P1 need to raise the column A and the minimum P2 to hold the wedge B stationary. The coefficient of static friction for all contacting surfaces is $\mu s = 0.4$. The weight of wedges B and C is negligible compared to the weight of block A

F.B.D. od block B

The friction force from block C opposes the relative motion of block B.

N_{BC}/θ

Impending motion of lower surface of block B relative to block C.
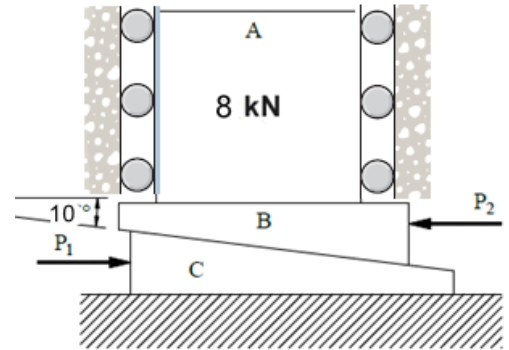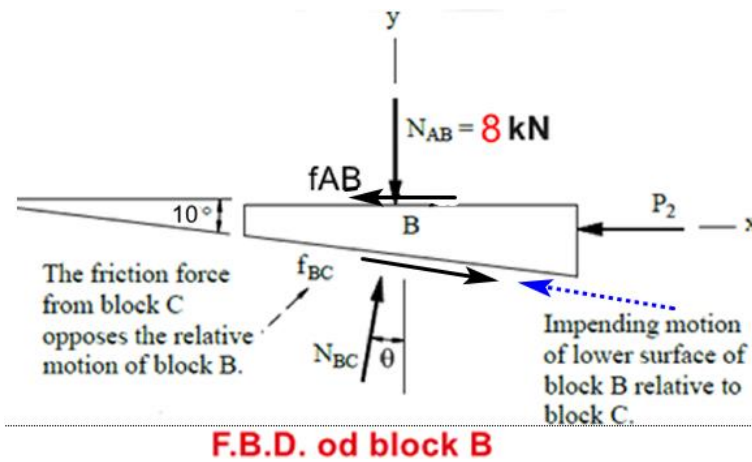
1 – Equations of equilibrium for block A
Slip impends between blocks A and B
$fAB = \mu_s N_{AB} = 0.4 N_{AB}$
$\Sigma Fy = 0 \quad NAB = 8$ kN
$fAB = 0.4 (8) = 3.2 \, kN$
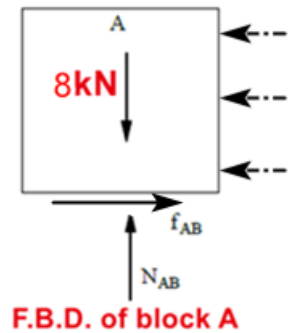
2 – Equations of equilibrium for block B
Slip impends between blocks B and C
$\quad fBC = \mu_s N_{BC} = 0.3 N_{BC}$
$\Sigma Fy = 0 \quad\quad N_{BC} \cos(10^o) - (0.4 N_{BC}) \sin(10^o) - 8 = 0$
$N_{BC} = 8.74 \, kN$
$\Sigma Fx = 0 \quad\quad (0.4 N_{BC}) \cos(10^o) + N_{BC} \sin(10^o) - 0.4(N_{BC}) - P_2 = 0$
$\quad P_2 = 1.76$ kN

2 – Equations of equilibrium for block C
Slip impends between blocks C and floor
$\quad f_C = \mu_s N_C = 0.4 N_C$
$\Sigma Fy = 0 \quad\quad (0.4 N_{BC} \sin(10^o) + N_{BC} \cos(10^o) - (N_C) = 0$
$N_C = \mathbf{8} \, kN$
$\Sigma Fx = 0 \quad\quad P1 - 0.4 \ N_{BC} * \cos(10^o) - N_{BC} \sin(10^o) - 0.4 N_C = 0$
$P_1 = \mathbf{0.878} \, kN$

F.B.D. of block A

The friction force f_{BC} from block B opposes the relative motion of block C.

Impending motion of upper surface of C relative to B

The friction force f_C from the floor opposes the relative motion of block B.

Impending motion of lower surface of C relative to the floor

F.B.D. of block C
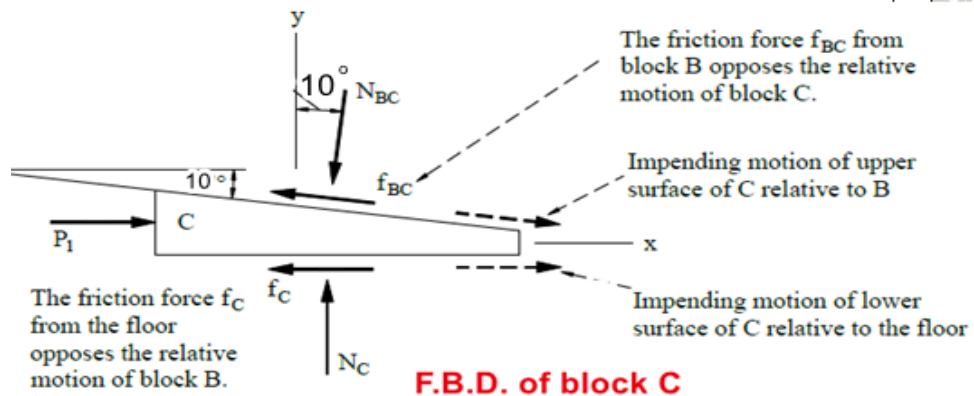
Sample Problem 6/6

The horizontal position of the 500-kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force **P**. If the coefficient of static friction for both wedge surfaces is 0.30 and if the coefficient of static friction between the block and the horizontal surface is 0.60, determine the least force P required to move the block.

Solution. The free-body diagrams of the wedge and the block are drawn with the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ inclined with respect to their normals by the amount of the friction angles for impending motion. The friction angle for limiting static friction is given by $\phi = \tan^{-1} \mu$. Each of the two friction angles is computed and shown on the diagram.

We start our vector diagram expressing the equilibrium of the block at a convenient point A and draw the only known vector, the weight **W** of the block. Next we add $\mathbf{R}_3$, whose 31.0° inclination from the vertical is now known. The vector $-\mathbf{R}_2$, whose 16.70° inclination from the horizontal is also known, must close the polygon for equilibrium. Thus, point B on the lower polygon is determined by the intersection of the known directions of $\mathbf{R}_3$ and $-\mathbf{R}_2$, and their magnitudes become known.

For the wedge we draw $\mathbf{R}_2$, which is now known, and add $\mathbf{R}_1$, whose direction is known. The directions of $\mathbf{R}_1$ and **P** intersect at C, thus giving us the solution for the magnitude of **P**.

Algebraic solution. The simplest choice of reference axes for calculation purposes is, for the block, in the direction a-a normal to $\mathbf{R}_3$ and, for the wedge, in the direction b-b normal to $\mathbf{R}_1$. The angle between $\mathbf{R}_2$ and the a-direction is $16.70° + 31.0° = 47.7°$. Thus, for the block

$$[\Sigma F_a = 0] \qquad 500(9.81) \sin 31.0° - R_2 \cos 47.7° = 0$$

$$R_2 = 3750 \text{ N}$$

For the wedge the angle between $\mathbf{R}_2$ and the b-direction is $90° - (2\phi_1 + 5°) = 51.6°$, and the angle between **P** and the b-direction is $\phi_1 + 5° = 21.7°$. Thus,

$$[\Sigma F_b = 0] \qquad 3750 \cos 51.6° - P \cos 21.7° = 0$$

$$P = 2500 \text{ N} \qquad \qquad Ans.$$

Graphical solution. The accuracy of a graphical solution is well within the uncertainty of the friction coefficients and provides a simple and direct result. By laying off the vectors to a reasonable scale following the sequence described, we obtain the magnitudes of **P** and the **R**'s easily by scaling them directly from the diagrams.
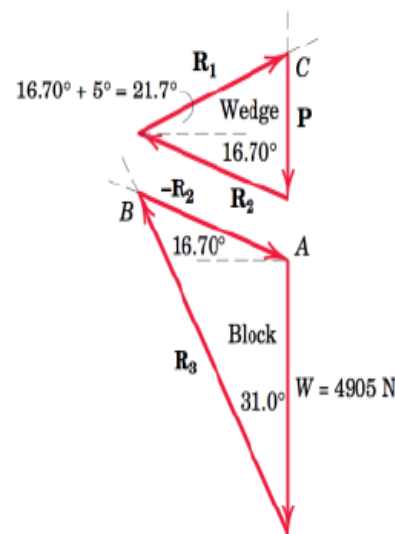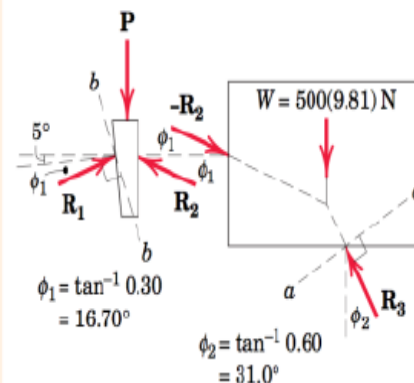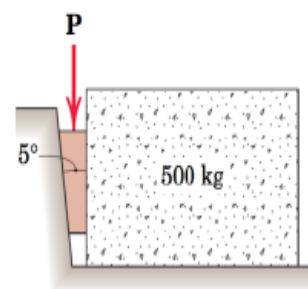
Helpful Hints

① Be certain to note that the reactions are inclined from their normals in the direction to oppose the motion. Also, we note the equal and opposite reactions $\mathbf{R}_2$ and $-\mathbf{R}_2$.

② It should be evident that we avoid simultaneous equations by eliminating reference to $\mathbf{R}_3$ for the block and $\mathbf{R}_1$ for the wedge.

<h1>CHAPTER FIVE</h1>

<h1>TRUSSES</h1>

5.1 Introduction

In the preceding chapters, we studied the equilibrium of a single rigid body, where all forces involved were external to the rigid body. We now consider the equilibrium of structures made of several connected parts. This situation calls for determining not only the external forces acting on the structure, but also the forces that hold together the various parts of the structure. From the point of view of the structure as a whole, these forces are **internal forces**.

The internal forces holding the various parts of the crane together do not appear in the free-body diagram. If, however, we dismember the crane and draw a free-body diagram for each of its component parts, we can see the forces holding the three beams together; since these forces are external forces from the point of view of each component part (*see Fig. below*).

There are many types of structural elements. If an element has pins or hinge supports at both ends and carries no load in-between, it is called a **two-force member**. These elements can only have two forces acting upon them at their hinges. If only two forces act on a body that is in equilibrium, then they must be equal in magnitude, co-linear and opposite in sense. This is known as the two-force principle. These members generally are used in **truss** structures.

A member acted upon by three or more forces that, in general, are not directed along the member, it is called a **multi-force member**. The directions of these forces are unknown; therefore, we need to represent them by two unknown components. **Frames** are designed to support loads and are also usually stationary, fully constrained structures. However, like the crane of Fig. below, frames always contain at least one multi-force member.
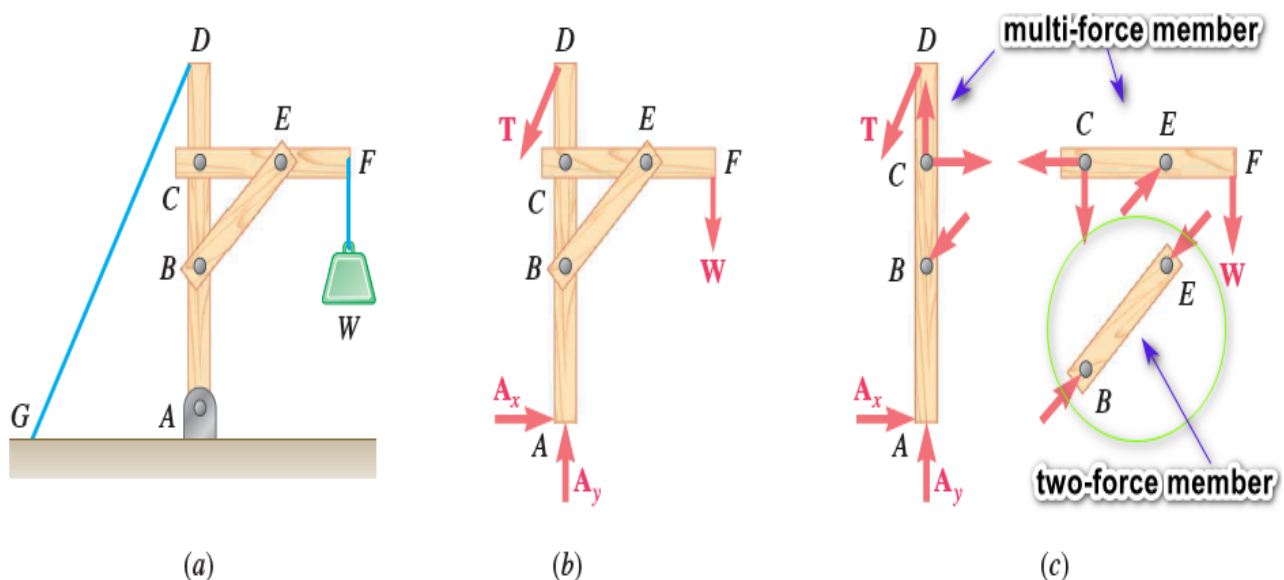
Fig. A structure in equilibrium. (a) Diagram of a crane supporting a load; (b) free-body diagram of the crane; (c) free-body diagrams of the components of the crane.

5.2 Trusses

A *truss* is a structure composed of slender members joined together at their end points. The members commonly used in construction consist of wooden struts or metal bars. In particular, *planar* trusses lie in a single plane and are often used to support roofs and bridges. The truss shown in Fig. 6−1a is an example of a typical roof-supporting truss. In this figure, the roof load is transmitted to the truss *at the joints* by means of a series of *purlins*. Since this loading act in the same plane as the truss, Fig. 6−1b, the analysis of the forces developed in the truss members will be two-dimensional.

Trusses are designed to support loads and are usually stationary, fully constrained structures. Trusses consist exclusively of straight members connected at joints located at the ends of each member. Members of a truss, therefore, are **two-force members**, i.e., members acted upon by two equal and opposite forces directed along the member.

Roof truss

(a)   (b)

Fig. 6–1

Bridge truss

(a)   (b)
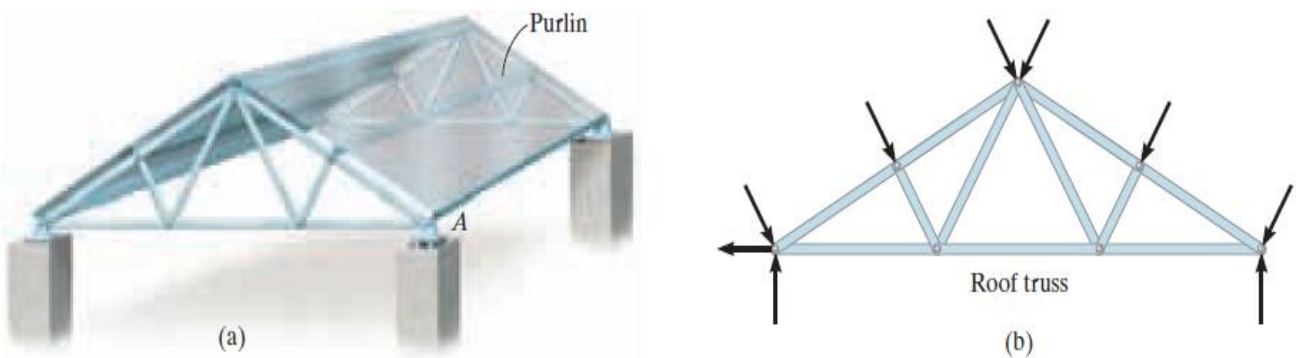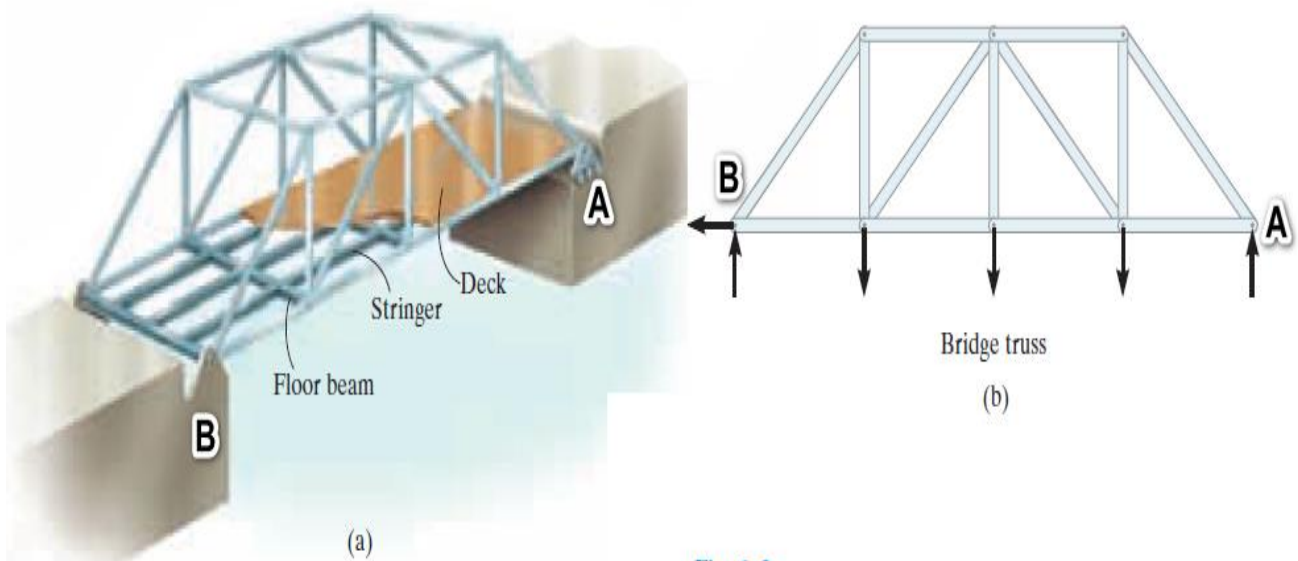
Fig. 6–2

5.2 Assumptions for Design.

To design both the members and the connections of a truss, it is necessary first to determine the *force* developed in each member when the truss is subjected to a given loading. To do this we will make two important assumptions:

1-All loadings are applied at the joints. In most situations, such as for bridge and roof trusses, this assumption is true. Frequently *the weight of the members is neglected* because the force supported by each member is usually much larger than its weight. However, if the weight is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, with half of its magnitude applied at each end of the member.

2- *The members are joined together by smooth pins*. The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called *a gusset plate*, as shown in Fig. 6–3*a*, or by simply passing a *large bolt or pin* through each of the members, Fig. 6–3*b*. We can assume these connections act as pins provided the center lines of the joining members are *concurrent*, as in Fig. 6–3.

Gusset plate

(a)

Fig. 6–3

(b)
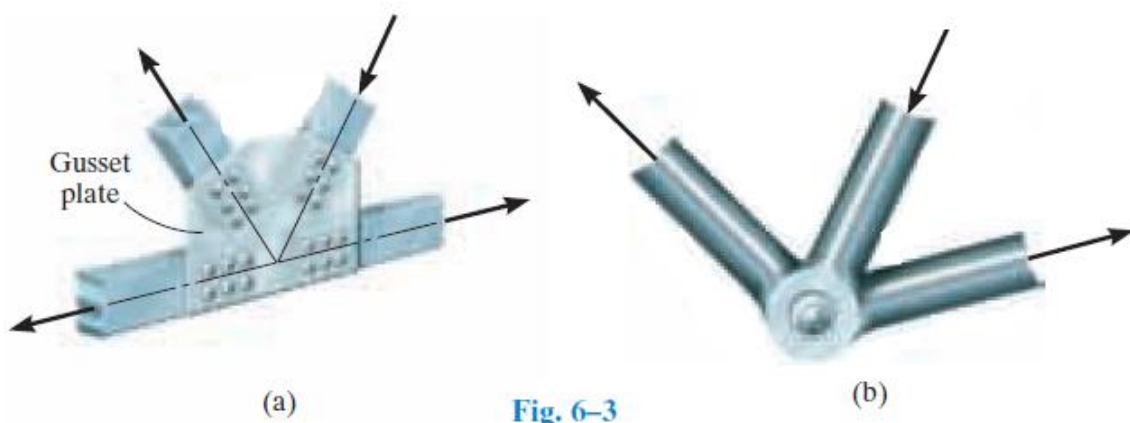
5.4 Member Force

Because of these two assumptions, *each truss member will act as a two force member*, and therefore the force acting at each end of the member will be directed along the axis of the member. If the force tends to *elongate* the member, it is a *tensile force* (T), Fig. 6–4*a*; whereas if it tends to *shorten* the member, it is a *compressive force* (C), Fig. 6–4*b*. In the actual design of a truss it is important to state whether the nature of the force is tensile or compressive. Often, *compression members* must be made *thicker* than tension members because of the **buckling** or column effect that occurs when a member is in compression.
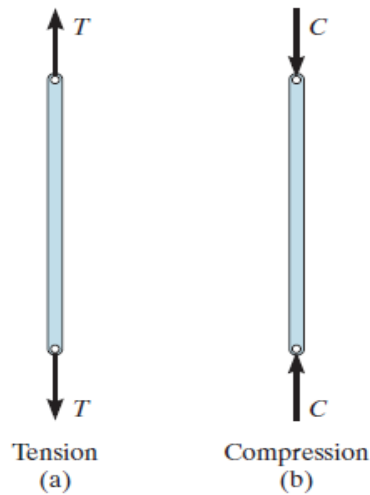
Fig. 6-4 A two-force member of a truss can be in tension or compression

5.5 Simple Truss.

If three members are pin connected at their ends, they form a ***triangular truss*** that will be *rigid*, Fig. 6–5. Attaching two more members and connecting these members to a new joint *D* forms a larger truss, Fig. 6–6. This procedure can be repeated as many times as desired to form an even larger truss. If a truss can be constructed by expanding the basic triangular truss in this way, it is called a ***simple truss***.
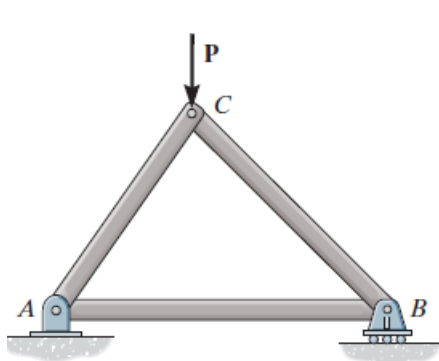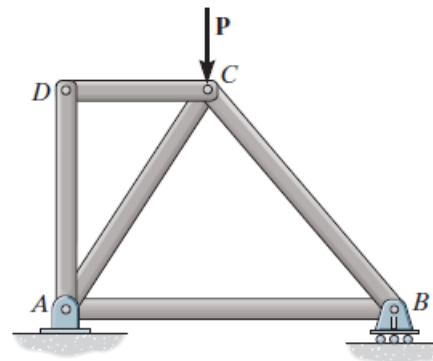
Fig. 6–5   Fig. 6–6

5.6 The Method of Joints

In order to analyze or design a truss, it is necessary to determine the force in each of its members. One way to do this is to use the ***method of joints***. This method is based on the fact that if the entire truss is in equilibrium, then ***each of its joints is also in equilibrium***. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations can then be used to obtain the member forces acting on each joint. Since the members of a *plane truss* are straight two-force members lying in a single plane, each joint is subjected to a force system that is ***coplanar and concurrent***. As a result, ***only ΣFx = 0 and ΣFy = 0 need to be satisfied for equilibrium***.

B

500 N

F_{BC} (compression)

45°

F_{BA}(tension)

(c)

B

500 N

2 m

45°

A

2 m

(a)

B

500 N

45°

F_{BA}(tension)

F_{BC} (compression)

(b)
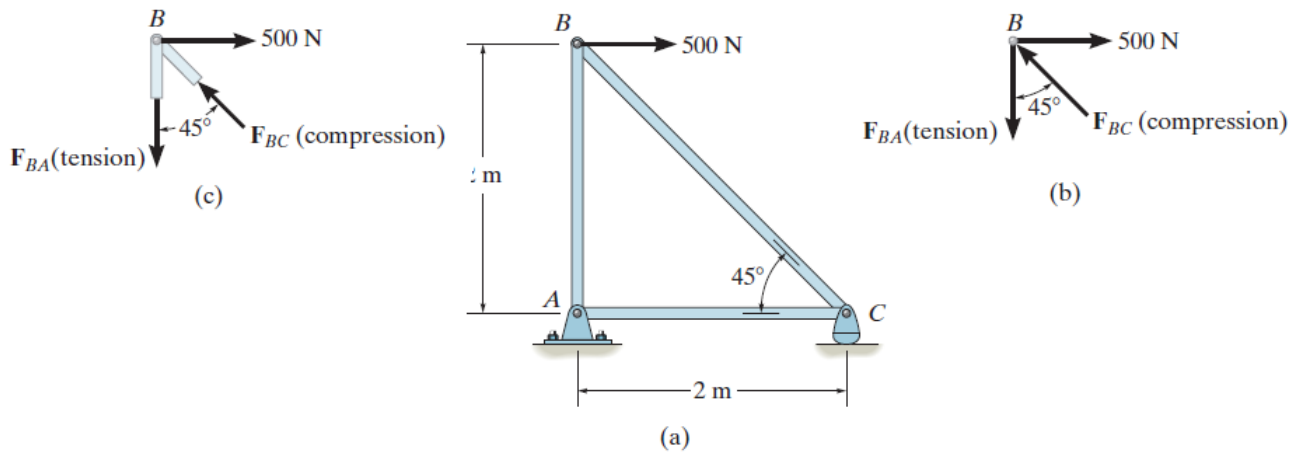
5.7 Procedure for Analysis

The following procedure provides a means for analyzing a truss using the method of joints.

1- Draw the free-body diagram of a joint having at least one known force and at most two unknown forces. (If this joint is at one of the supports, then it may be necessary first to calculate the external reactions at the support.)

2- Use one of the two methods described above for establishing the sense of an unknown force.

3- Orient the x and y axes such that the forces on the free-body diagram can be easily resolved into their x and y components and then apply the two force equilibrium equations $\Sigma Fx = 0$ and $\Sigma Fy = 0.$ Solve for the two unknown member forces and verify their correct sense.

4- Using the calculated results, continue to analyze each of the other joints. Remember that a member in *compression* "pushes" on the joint and a member in *tension* "pulls" on the joint. Also, be sure to choose a joint having at most two unknowns and at least one known force.
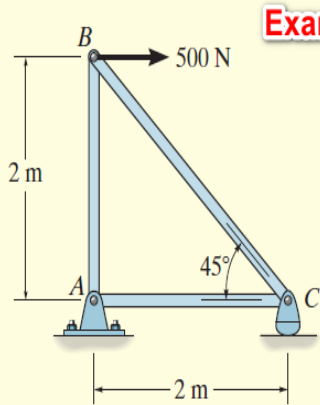
Determine the force in each member of the truss shown in Fig. 6–8a and indicate whether the members are in tension or compression.

(a)

SOLUTION

Since we should have no more than two unknown forces at the joint and at least one known force acting there, we will begin our analysis at joint B.
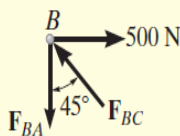
(b)

Joint B. The free-body diagram of the joint at B is shown in Fig. 6–8b. Applying the equations of equilibrium, we have

$$\xrightarrow{+} \Sigma F_x = 0; \quad\quad 500\ \text{N} - F_{BC} \sin 45° = 0 \quad F_{BC} = 707.1\ \text{N (C)} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0; \quad\quad F_{BC} \cos 45° - F_{BA} = 0 \quad F_{BA} = 500\ \text{N (T)} \quad \textit{Ans.}$$
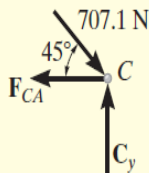
Since the force in member BC has been calculated, we can proceed to analyze joint C to determine the force in member CA and the support reaction at the rocker.

(c)

Joint C. From the free-body diagram of joint C, Fig. 6–8c, we have

$$\xrightarrow{+} \Sigma F_x = 0; \quad -F_{CA} + 707.1 \cos 45°\ \text{N} = 0 \quad F_{CA} = 500\ \text{N (T)} \quad \textit{Ans.}$$
$$+\uparrow \Sigma F_y = 0; \quad\quad C_y - 707.1 \sin 45°\ \text{N} = 0 \quad C_y = 500\ \text{N} \quad \textit{Ans.}$$
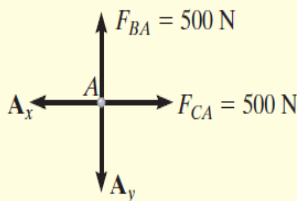
Joint A. Although it is not necessary, we can determine the components of the support reactions at joint A using the results of F_{CA} and F_{BA}. From the free-body diagram, Fig. 6–8d, we have

(d)

$$\xrightarrow{+} \Sigma F_x = 0; \quad\quad 500\ \text{N} - A_x = 0 \quad A_x = 500\ \text{N}$$
$$+\uparrow \Sigma F_y = 0; \quad\quad 500\ \text{N} - A_y = 0 \quad A_y = 500\ \text{N}$$
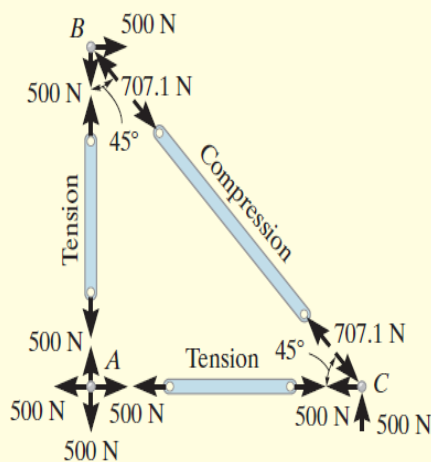
NOTE: The results of the analysis are summarized in Fig. 6–8e. Note that the free-body diagram of each joint (or pin) shows the effects of all the connected members and external forces applied to the joint, whereas the free-body diagram of each member shows only the effects of the end joints on the member.

(e)

Fig. 6–8

Determine the forces acting in all the members of the truss shown in Fig. 6–9a.

SOLUTION

By inspection, there are more than two unknowns at each joint. Consequently, the support reactions on the truss must first be determined. Show that they have been correctly calculated on the free-body diagram in Fig. 6–9b. We can now begin the analysis at joint C. Why?

Joint C. From the free-body diagram, Fig. 6–9c,

$$\xrightarrow{+}\ \Sigma F_x = 0; \qquad\qquad -F_{CD}\cos 30° + F_{CB}\sin 45° = 0$$
$$+\uparrow \Sigma F_y = 0; \qquad 1.5\ \text{kN} + F_{CD}\sin 30° - F_{CB}\cos 45° = 0$$

These two equations must be solved *simultaneously* for each of the two unknowns. Note, however, that a *direct solution* for one of the unknown forces may be obtained by applying a force summation along an axis that is *perpendicular* to the direction of the other unknown force. For example, summing forces along the y' axis, which is perpendicular to the direction of $\mathbf{F}_{CD}$, Fig. 6–9d, yields a *direct solution* for F_{CB}.

$$+\nearrow \Sigma F_{y'} = 0; \qquad 1.5\cos 30°\ \text{kN} - F_{CB}\sin 15° = 0$$
$$F_{CB} = 5.019\ \text{kN} = 5.02\ \text{kN (C)} \qquad\qquad Ans.$$

Then,

$$+\searrow \Sigma F_{x'} = 0;$$
$$-F_{CD} + 5.019\cos 15° - 1.5\sin 30° = 0; \quad F_{CD} = 4.10\ \text{kN (T)} \quad Ans.$$

Joint D. We can now proceed to analyze joint D. The free-body diagram is shown in Fig. 6–9e.

$$\xrightarrow{+}\ \Sigma F_x = 0; \qquad -F_{DA}\cos 30° + 4.10\cos 30°\ \text{kN} = 0$$
$$F_{DA} = 4.10\ \text{kN (T)} \qquad\qquad Ans.$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{DB} - 2(4.10\sin 30°\ \text{kN}) = 0$$
$$F_{DB} = 4.10\ \text{kN (T)} \qquad\qquad Ans.$$

NOTE: The force in the last member, BA, can be obtained from joint B or joint A. As an exercise, draw the free-body diagram of joint B, sum the forces in the horizontal direction, and show that $F_{BA} = 0.776\ \text{kN (C)}$.

(a)

(b)

(c)

(d)

(e)
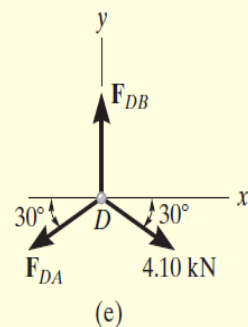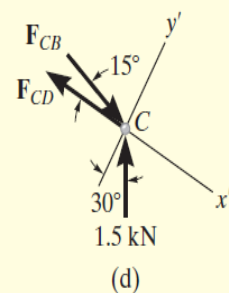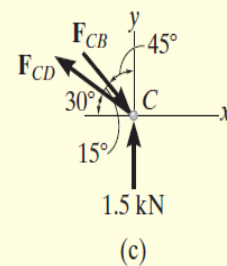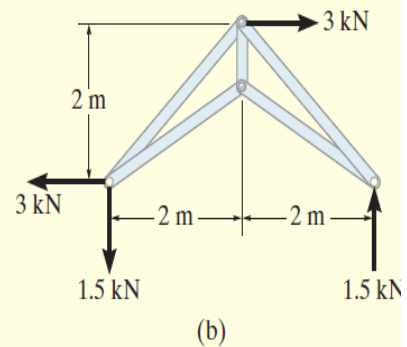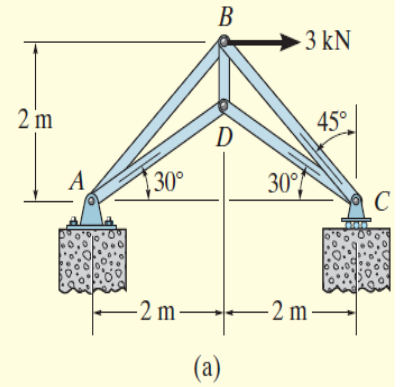
Fig. 6–9

Determine the force in each member of the truss shown in Fig. 6–10a. Indicate whether the members are in tension or compression.
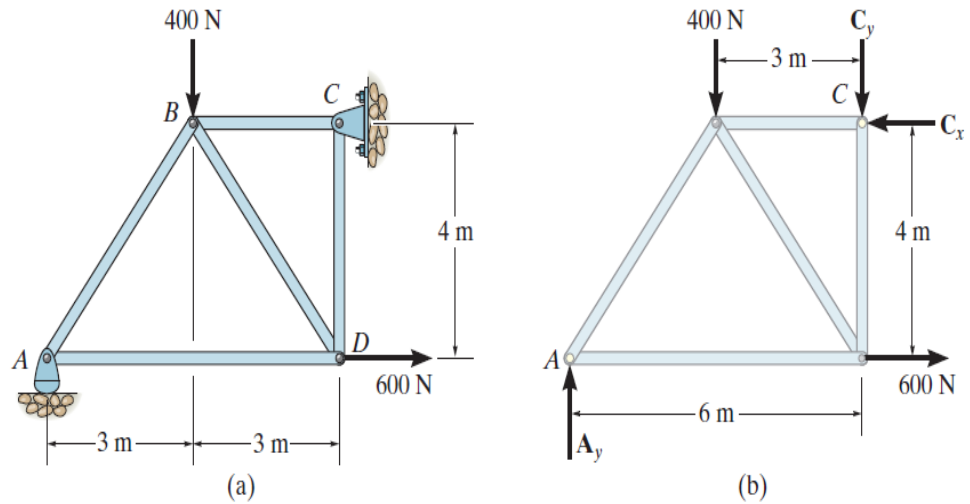
Fig. 6–10

## SOLUTION

Support Reactions. No joint can be analyzed until the support reactions are determined, because each joint has at least three unknown forces acting on it. A free-body diagram of the entire truss is given in Fig. 6–10b. Applying the equations of equilibrium, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 600 \text{ N} - C_x = 0 \qquad\qquad C_x = 600 \text{ N}$$

$$\zeta + \Sigma M_C = 0; \qquad -A_y(6 \text{ m}) + 400 \text{ N}(3 \text{ m}) + 600 \text{ N}(4 \text{ m}) = 0$$

$$A_y = 600 \text{ N}$$

$$+\uparrow \Sigma F_y = 0; \qquad 600 \text{ N} - 400 \text{ N} - C_y = 0 \qquad C_y = 200 \text{ N}$$

The analysis can now start at either joint A or C. The choice is arbitrary since there are one known and two unknown member forces acting on the pin at each of these joints.

Joint A. (Fig. 6–10c). As shown on the free-body diagram, $\mathbf{F}_{AB}$ is assumed to be compressive and $\mathbf{F}_{AD}$ is tensile. Applying the equations of equilibrium, we have

$$+\uparrow \Sigma F_y = 0; \qquad 600 \text{ N} - \tfrac{4}{5} F_{AB} = 0 \qquad F_{AB} = 750 \text{ N} \quad \text{(C)} \qquad \textit{Ans.}$$

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{AD} - \tfrac{3}{5}(750 \text{ N}) = 0 \qquad F_{AD} = 450 \text{ N} \quad \text{(T)} \qquad \textit{Ans.}$$
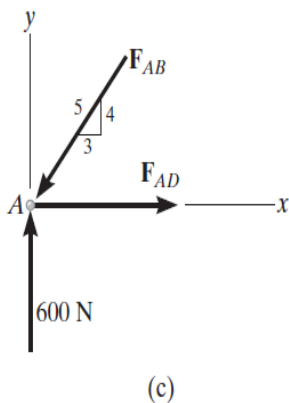
Joint D. (Fig. 6–10d). Using the result for F_{AD} and summing forces in the horizontal direction, Fig. 6–10d, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -450 \text{ N} + \tfrac{3}{5}F_{DB} + 600 \text{ N} = 0 \qquad F_{DB} = -250 \text{ N}$$

The negative sign indicates that $\mathbf{F}_{DB}$ acts in the *opposite sense* to that shown in Fig. 6–10d.* Hence,

$$F_{DB} = 250 \text{ N (T)} \qquad\qquad\qquad Ans.$$

To determine $\mathbf{F}_{DC}$, we can either correct the sense of $\mathbf{F}_{DB}$ on the free-body diagram, and then apply $\Sigma F_y = 0$, or apply this equation and retain the negative sign for F_{DB}, i.e.,

$$+\uparrow \Sigma F_y = 0; \qquad -F_{DC} - \tfrac{4}{5}(-250 \text{ N}) = 0 \qquad F_{DC} = 200 \text{ N (C)} \qquad Ans.$$

Joint C. (Fig. 6–10e).

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{CB} - 600 \text{ N} = 0 \qquad\qquad F_{CB} = 600 \text{ N (C)} \qquad Ans.$$
$$+\uparrow \Sigma F_y = 0; \qquad\qquad 200 \text{ N} - 200 \text{ N} \equiv 0 \quad \text{(check)}$$

NOTE: The analysis is summarized in Fig. 6–10f, which shows the free-body diagram for each joint and member.

(d)

(e)

(f)
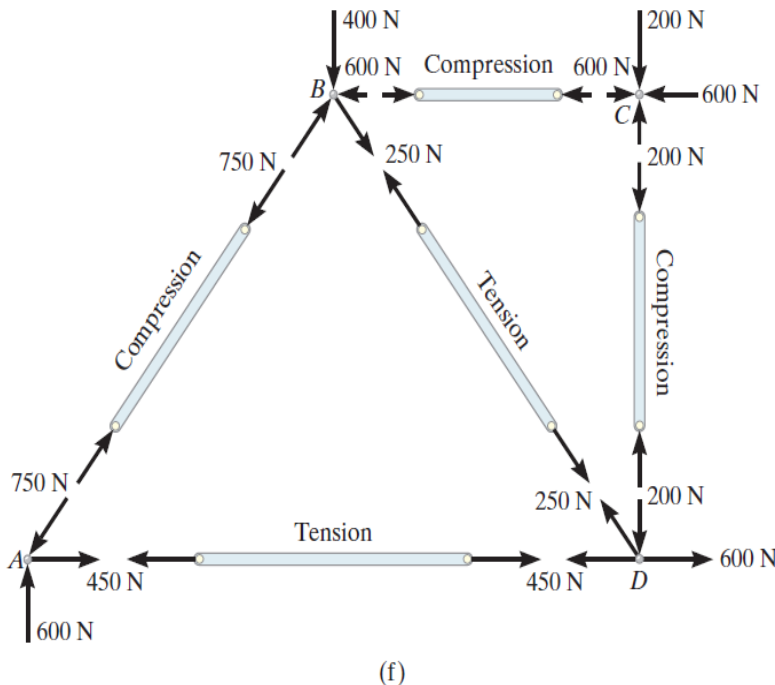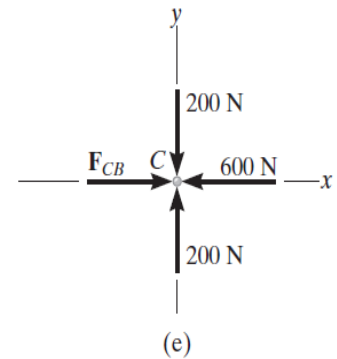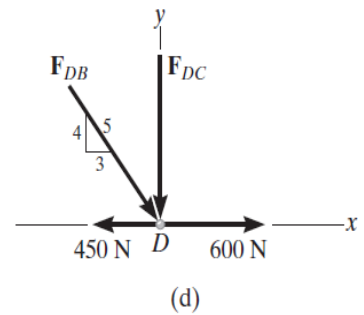
Fig. 6–10 (cont.)

*The proper sense could have been determined by inspection, prior to applying $\Sigma F_x = 0$.

5.8 Zero-Force Members

Zero-force members support no load; however, they are necessary for stability, and are available when additional loadings are applied to the joints of the truss. These members can usually be identified by inspection. They occur at joints where:

1-If only two non-collinear members form a truss joint and no external load or support reaction is applied to the joint, the two members must be zero-force members.(see Case No.1)

(a)

$$\xrightarrow{+} \Sigma F_x = 0;\ F_{AB} = 0$$
$$+\uparrow \Sigma F_y = 0;\ F_{AF} = 0$$

(b)
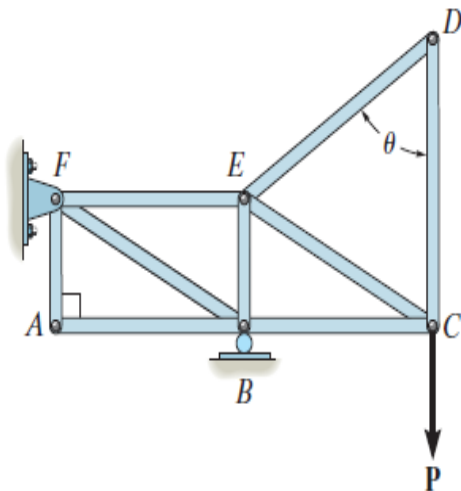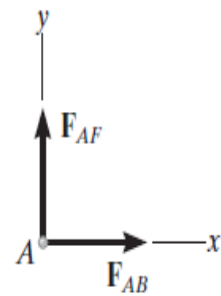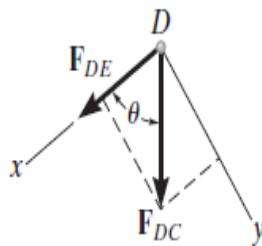
$$+\searrow \Sigma F_y = 0;\ F_{DC} \sin\theta = 0;\ F_{DC} = 0\ \text{since}\ \sin\theta \neq 0$$
$$+\swarrow \Sigma F_x = 0;\ F_{DE} + 0 = 0;\ F_{DE} = 0$$

(c)
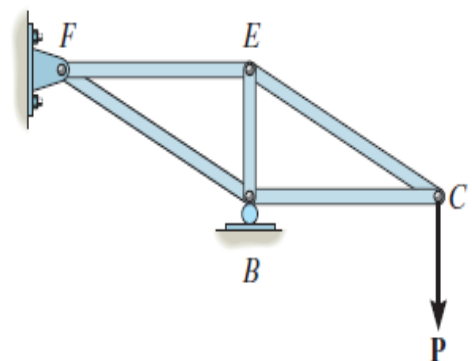
(d)

Case No. 1

2-If three members form a truss joint for which two of the members are collinear, the third member is a zero-force member provided no external force or support reaction has a component that acts along this member. (see Case No.2)

(a)
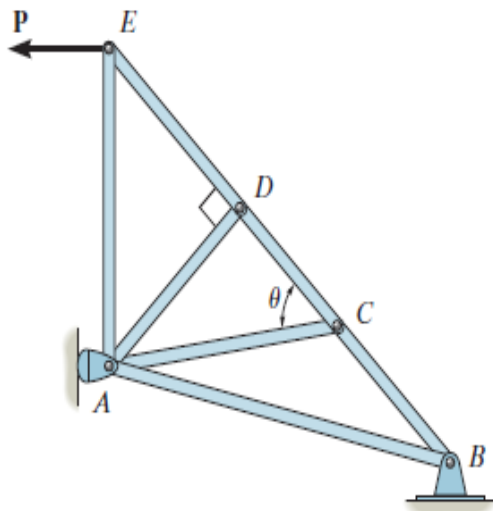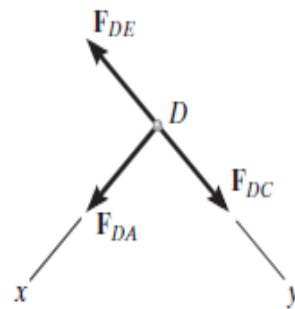
$+\swarrow \Sigma F_x = 0; \quad F_{DA} = 0$
$+\searrow \Sigma F_y = 0; \quad F_{DC} = F_{DE}$

(b)
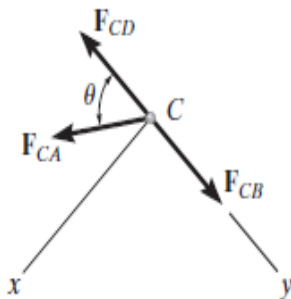
$+\swarrow \Sigma F_x = 0; \quad F_{CA}\sin\theta = 0; \quad F_{CA} = 0 \text{ since } \sin\theta \neq 0;$
$+\searrow \Sigma F_y = 0; \quad F_{CB} = F_{CD}$

(c)
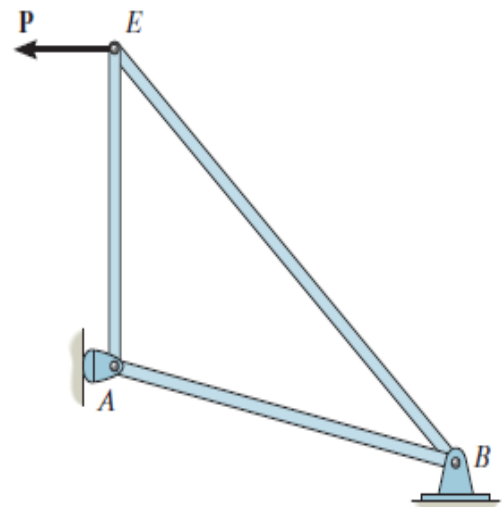
(d)

Case No.2

Example

Using the method of joints, determine all the zero-force members of the *Fink roof truss* shown in Fig   . Assume all joints are pin connected.
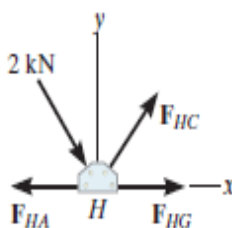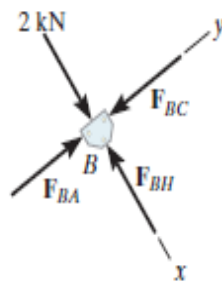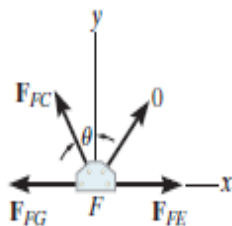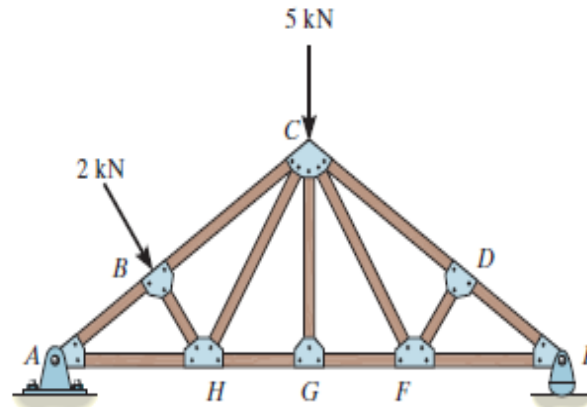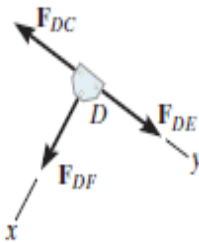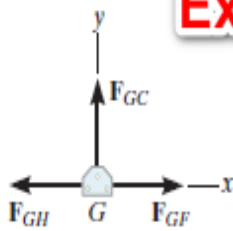
SOLUTION

Look for joint geometries that have three members for which two are collinear. We have

Joint G.

$$+\uparrow \Sigma F_y = 0; \qquad\qquad F_{GC} = 0 \qquad\qquad Ans.$$

Realize that we could not conclude that GC is a zero-force member by considering joint C, where there are five unknowns. The fact that GC is a zero-force member means that the 5-kN load at C must be supported by members CB, CH, CF, and CD.

Joint D.

$$+\nearrow \Sigma F_x = 0; \qquad\qquad F_{DF} = 0 \qquad\qquad Ans.$$

Joint F.

$$+\uparrow \Sigma F_y = 0; \qquad F_{FC} \cos \theta = 0 \quad \text{Since } \theta \neq 90°, \quad F_{FC} = 0 \qquad Ans.$$

NOTE: If joint B is analyzed, Fig. 6–13e,

$$+\searrow \Sigma F_x = 0; \qquad\qquad 2 \text{ kN} - F_{BH} = 0 \quad F_{BH} = 2 \text{ kN} \quad (C)$$

Also, F_{HC} must satisfy $\Sigma F_y = 0$, Fig. 6–13f, and therefore HC is *not* a zero-force member.

5.9 The Method of Sections

When we need to find the force in only a few members of a truss, we can analyze the truss using the *method of sections*. It is based on the principle that if the truss is in equilibrium then any segment of the truss is also in equilibrium. For example, consider the two truss members shown on the left in Fig. 6–14. If the forces within the members are to be determined, then an imaginary section, indicated by the blue line, can be used to cut each member into two parts and thereby "expose" each internal force as "external" to the free-body diagrams shown on the right. Clearly, it can be seen that equilibrium requires that the member in tension (T) be subjected to a "pull," whereas the member in compression (C) is subjected to a "push."

The method of sections can also be used to "cut" or section the members of an entire truss. If the section passes through the truss and the free-body diagram of either of its two parts is drawn, we can then apply the equations of equilibrium to that part to determine the member forces at the "cut section." Since only *three* independent equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_O = 0$) can be applied to the free-body diagram of any segment, then we should try to select a section that, in general, passes through not more than *three* members in which the forces are unknown.
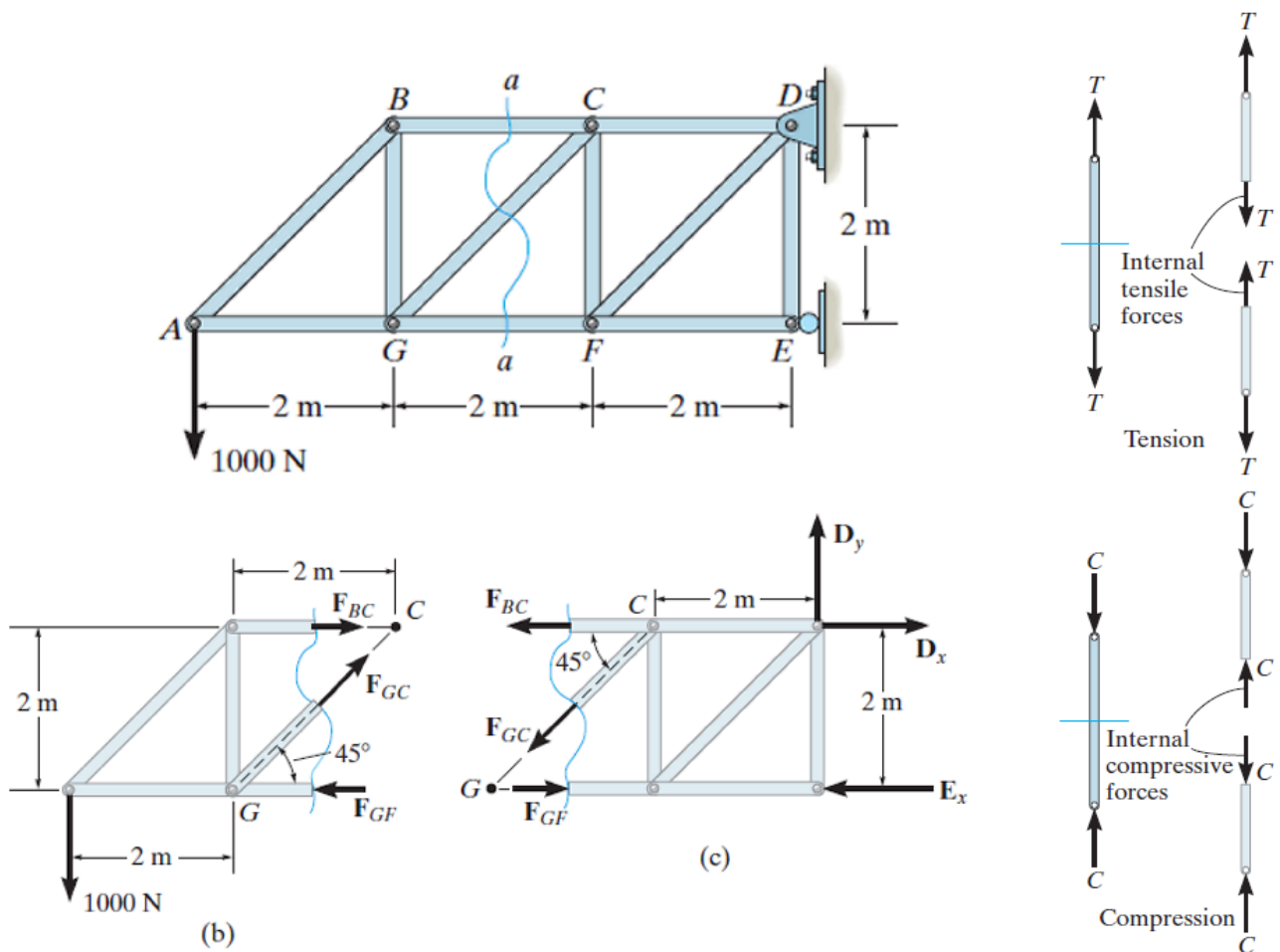
When applying the equilibrium equations, we should carefully consider ways of writing the equations so as to yield a *direct solution* for each of the unknowns, rather than having to solve simultaneous equations. For example, using the truss segment in Fig. *b* and summing moments about *C* would yield a direct solution for $\mathbf{F}_{GF}$ since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GC}$ create zero moment about *C*. Likewise, $\mathbf{F}_{BC}$ can be directly obtained by summing moments about *G*. Finally, $\mathbf{F}_{GC}$ can be found directly from a force summation in the vertical direction since $\mathbf{F}_{GF}$ and $\mathbf{F}_{BC}$ have no vertical components. This ability to *determine directly* the force in a particular truss member is one of the main advantages of using the method of sections.

NOTE2:

   As in the method of joints, there are two ways in which we can determine the correct sense of an unknown member force:

• The correct sense of an unknown member force can in many cases be determined "by inspection." For example, $\mathbf{F}_{BC}$ is a tensile force as represented in Fig. 6–15b since moment equilibrium about G requires that $\mathbf{F}_{BC}$ create a moment opposite to that of the 1000-N force. Also, $\mathbf{F}_{GC}$ is tensile since its vertical component must balance the 1000-N force which acts downward. In more complicated cases, the sense of an unknown member force may be assumed. If the solution yields a negative scalar, it indicates that the force's sense is opposite to that shown on the free-body diagram.

• *Always assume that the unknown member forces at the cut section are tensile forces, i.e., "pulling" on the member. By doing this, the numerical solution of the equilibrium equations will yield positive scalars for members in tension and negative scalars for members in compression.*

5.10 Procedure for Analysis

The forces in the members of a truss may be determined by the method of sections using the following procedure.

1-Free-Body Diagram.

o Make a decision on how to "cut" or section the truss through the members where forces are to be determined.

o Before isolating the appropriate section, it may first be necessary to determine the truss's support reactions. If this is done then the three equilibrium equations will be available to solve for member forces at the section.

o Draw the free-body diagram of that segment of the sectioned truss which has the least number of forces acting on it.

o Use one of the two methods described above for establishing the sense of the unknown member forces.

2- Equations of Equilibrium.

o Moments should be summed about a point that lies at the intersection of the lines of action of two unknown forces, so that the third unknown force can be determined directly from the moment equation.

o **If two of the unknown forces are *parallel*,** forces may be *summed perpendicular* to the direction of these unknowns to determine *directly* the third unknown force.

(a)

(b)

(c)
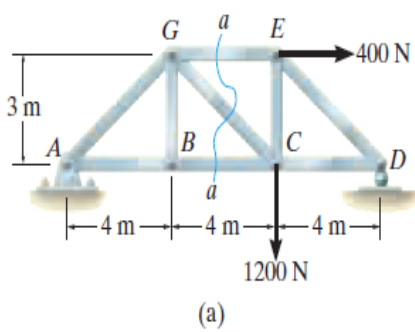
Fig. 6–16

Determine the force in members GE, GC, and BC of the truss shown in Fig. 6–16a. Indicate whether the members are in tension or compression.

SOLUTION

Section aa in Fig. 6–16a has been chosen since it cuts through the *three* members whose forces are to be determined. In order to use the method of sections, however, it is *first* necessary to determine the external reactions at A or D. Why? A free-body diagram of the entire truss is shown in Fig. 6–16b. Applying the equations of equilibrium, we have

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 400 \text{ N} - A_x = 0 \qquad A_x = 400 \text{ N}$$

$$\zeta + \Sigma M_A = 0; \qquad -1200 \text{ N}(8 \text{ m}) - 400 \text{ N}(3 \text{ m}) + D_y(12 \text{ m}) = 0$$

$$D_y = 900 \text{ N}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 1200 \text{ N} + 900 \text{ N} = 0 \qquad A_y = 300 \text{ N}$$
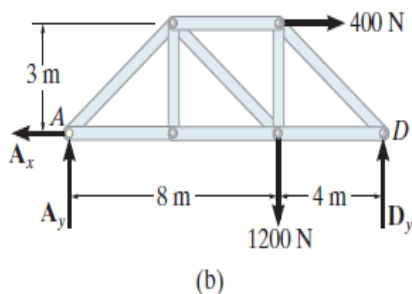
Free-Body Diagram. For the analysis the free-body diagram of the left portion of the sectioned truss will be used, since it involves the least number of forces, Fig. 6–16c.

Equations of Equilibrium. Summing moments about point G eliminates $\mathbf{F}_{GE}$ and $\mathbf{F}_{GC}$ and yields a direct solution for F_{BC}.

$$\zeta + \Sigma M_G = 0; \quad -300 \text{ N}(4 \text{ m}) - 400 \text{ N}(3 \text{ m}) + F_{BC}(3 \text{ m}) = 0$$

$$F_{BC} = 800 \text{ N} \quad (T) \qquad \qquad Ans.$$

In the same manner, by summing moments about point C we obtain a direct solution for F_{GE}.

$$\zeta + \Sigma M_C = 0; \quad -300 \text{ N}(8 \text{ m}) + F_{GE}(3 \text{ m}) = 0$$

$$F_{GE} = 800 \text{ N} \quad (C) \qquad \qquad Ans.$$

Since $\mathbf{F}_{BC}$ and $\mathbf{F}_{GE}$ have no vertical components, summing forces in the y direction directly yields F_{GC}, i.e.,

$$+\uparrow \Sigma F_y = 0; \qquad 300 \text{ N} - \tfrac{3}{5}F_{GC} = 0$$

$$F_{GC} = 500 \text{ N} \quad (T) \qquad \qquad Ans.$$
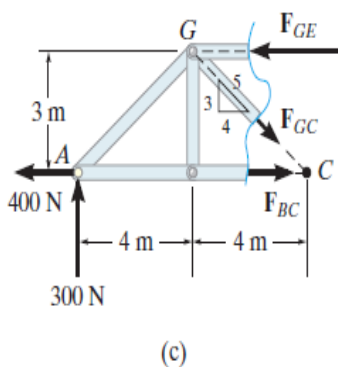
NOTE: Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example, $\Sigma M_C = 0$ requires $\mathbf{F}_{GE}$ to be *compressive* because it must balance the moment of the 300-N force about C.

Determine the force in member *CF* of the truss shown in Fig. 6–17a. Indicate whether the member is in tension or compression. Assume each member is pin connected.

(a)

(b)
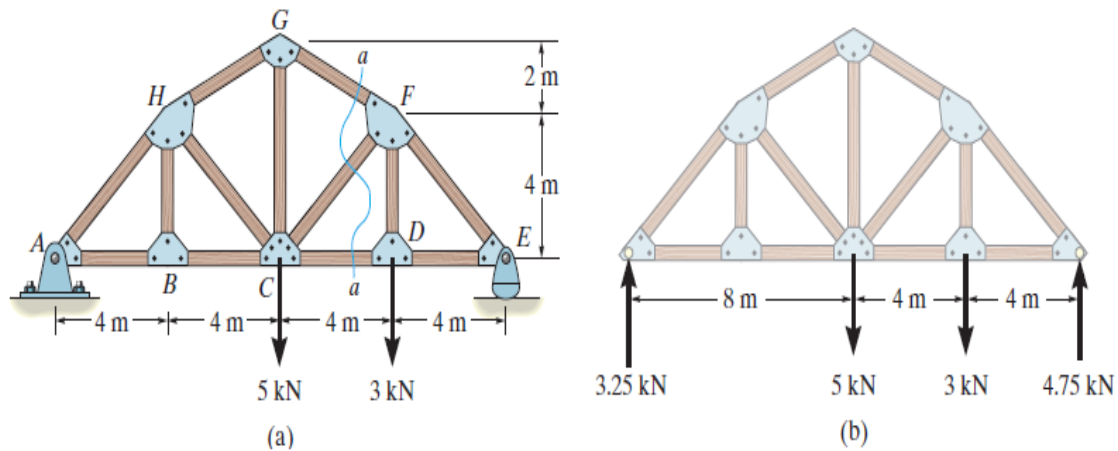
Fig. 6–17

SOLUTION

Free-Body Diagram. Section *aa* in Fig. 6–17a will be used since this section will "expose" the internal force in member *CF* as "external" on the free-body diagram of either the right or left portion of the truss. It is first necessary, however, to determine the support reactions on either the left or right side. Verify the results shown on the free-body diagram in Fig. 6–17b.

The free-body diagram of the right portion of the truss, which is the easiest to analyze, is shown in Fig. 6–17c. There are three unknowns, F_{FG}, F_{CF}, and F_{CD}.

Equations of Equilibrium. We will apply the moment equation about point O in order to eliminate the two unknowns F_{FG} and F_{CD}. The location of point O measured from E can be determined from proportional triangles, i.e., $4/(4 + x) = 6/(8 + x)$, $x = 4$ m. Or, stated in another manner, the slope of member GF has a drop of 2 m to a horizontal distance of 4 m. Since FD is 4 m, Fig. 6–17c, then from D to O the distance must be 8 m.

(c)

An easy way to determine the moment of $\mathbf{F}_{CF}$ about point O is to use the principle of transmissibility and slide $\mathbf{F}_{CF}$ to point C, and then resolve $\mathbf{F}_{CF}$ into its two rectangular components. We have

$$\zeta + \Sigma M_O = 0;$$

$$-F_{CF} \sin 45°(12 \text{ m}) + (3 \text{ kN})(8 \text{ m}) - (4.75 \text{ kN})(4 \text{ m}) = 0$$

$$F_{CF} = 0.589 \text{ kN} \quad (C) \qquad\qquad Ans.$$
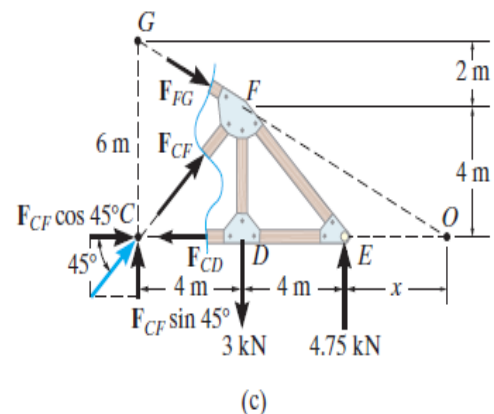
Determine the force in member EB of the roof truss shown in Fig. 6–18a. Indicate whether the member is in tension or compression.
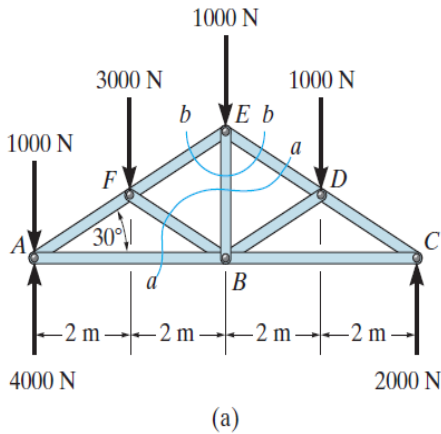
SOLUTION

**Free-Body Diagrams.** By the method of sections, any imaginary section that cuts through EB, Fig. 6–18a, will also have to cut through three other members for which the forces are unknown. For example, section aa cuts through ED, EB, FB, and AB. If a free-body diagram of the left side of this section is considered, Fig. 6–18b, it is possible to obtain $\mathbf{F}_{ED}$ by summing moments about B to eliminate the other three unknowns; however, $\mathbf{F}_{EB}$ cannot be determined from the remaining two equilibrium equations. One possible way of obtaining $\mathbf{F}_{EB}$ is first to determine $\mathbf{F}_{ED}$ from section aa, then use this result on section bb, Fig. 6–18a, which is shown in Fig. 6–18c. Here the force system is concurrent and our sectioned free-body diagram is the same as the free-body diagram for the joint at E.

(a)

(b)

(c)
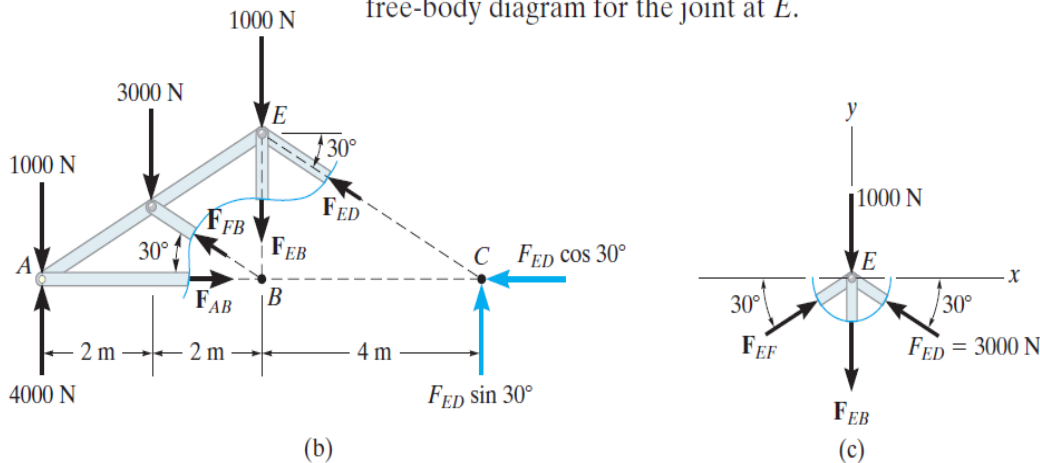
Fig. 6–18

Equations of Equilibrium. In order to determine the moment of $\mathbf{F}_{ED}$ about point B, Fig. 6–18b, we will use the principle of transmissibility and slide the force to point C and then resolve it into its rectangular components as shown. Therefore,

$$\zeta + \Sigma M_B = 0; \quad 1000\ \text{N}(4\ \text{m}) + 3000\ \text{N}(2\ \text{m}) - 4000\ \text{N}(4\ \text{m})$$
$$+ F_{ED} \sin 30°(4\ \text{m}) = 0$$
$$F_{ED} = 3000\ \text{N} \quad (\text{C})$$

Considering now the free-body diagram of section bb, Fig. 6–18c, we have

$$\xrightarrow{+} \Sigma F_x = 0; \quad F_{EF} \cos 30° - 3000 \cos 30°\ \text{N} = 0$$
$$F_{EF} = 3000\ \text{N} \quad (\text{C})$$
$$+ \uparrow \Sigma F_y = 0; \quad 2(3000 \sin 30°\ \text{N}) - 1000\ \text{N} - F_{EB} = 0$$
$$F_{EB} = 2000\ \text{N} \quad (\text{T}) \qquad \qquad \textit{Ans.}$$

CHAPTER SIX

CENTER OF GRAVITY AND CENTROID

6.1 Introduction

Knowing the resultant or total weight of a body and its location is important when considering the effect this force produces on the body. The point of location is called the center of gravity, and in this section we will show how to find it for an irregularly shaped body. We will then extend this method to show how to find the body's center of mass, and its geometric center or centroid.

Center of Gravity.
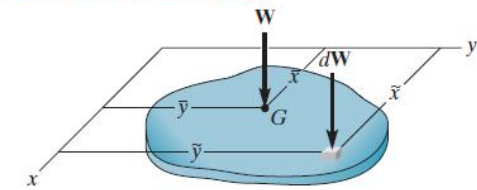
$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}\, dW$$

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW}$$
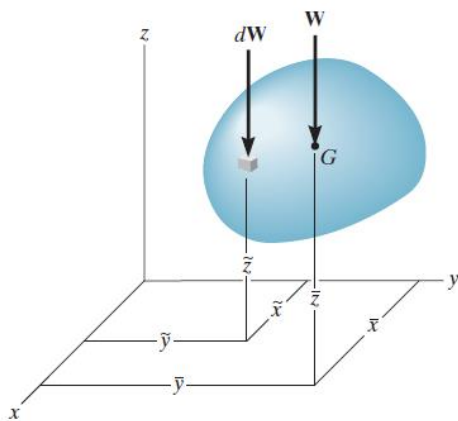
$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\, dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's **center of mass** C_m, Fig. 9–2. This location can be determined by substituting $dW = g\, dm$ into Eqs. 9–1. Provided g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\, dm}{\int dm} \qquad (9\text{–}2)$$
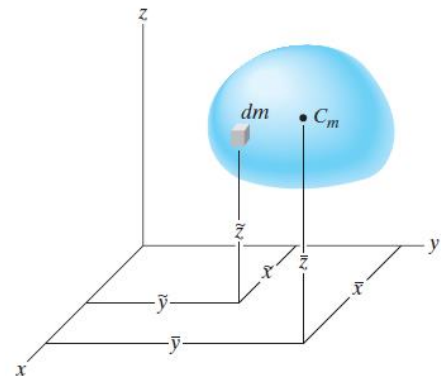
Centroid of a Volume. If the body in Fig. 9–3 is made from a *homogeneous material*, then its density ρ (rho) will be *constant*. Therefore, a differential element of volume dV has a mass $dm = \rho\, dV$. Substituting this into Eqs. 9–2 and canceling out ρ, we obtain formulas that locate the **centroid** C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\, dV}{\int_V dV} \qquad (9\text{–}3)$$
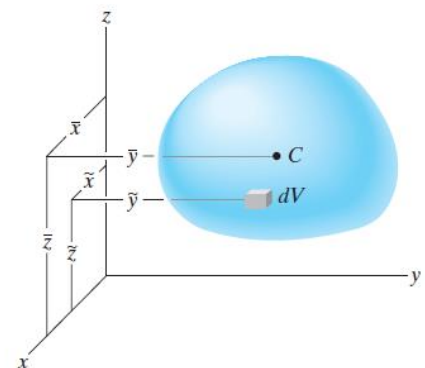
Centroid of an Area. If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 9–5a, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 9–3, namely,

$$\bar{x} = \frac{\displaystyle\int_A \tilde{x}\, dA}{\displaystyle\int_A dA} \qquad \bar{y} = \frac{\displaystyle\int_A \tilde{y}\, dA}{\displaystyle\int_A dA} \qquad\qquad (9\text{–}4)$$

(a) (b) (c)
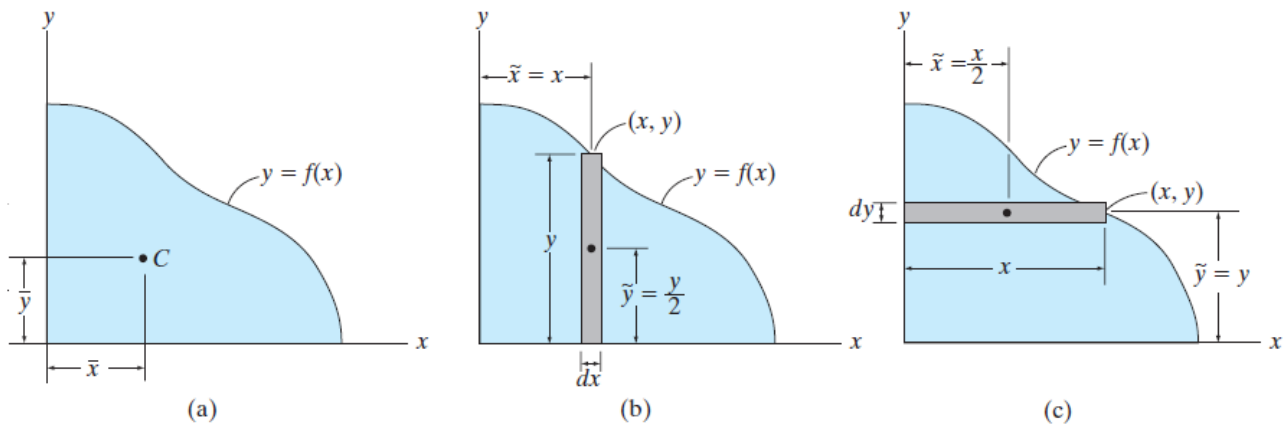
Centroid for Areas Bounded by 2 Curves
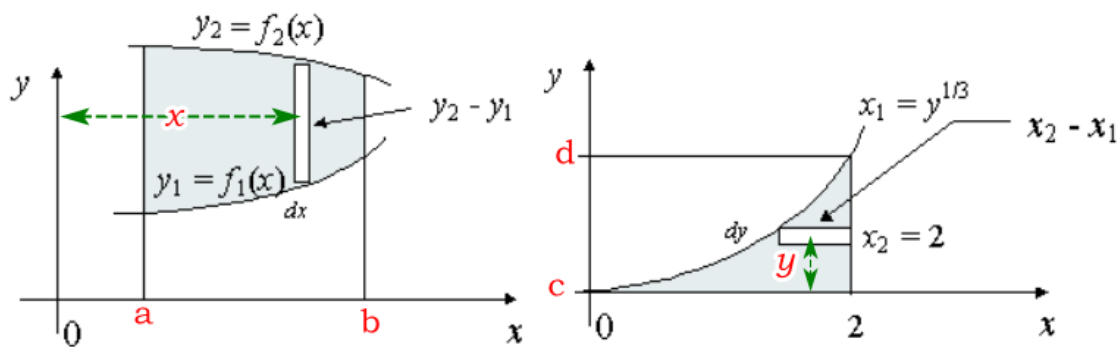
We extend the simple case given above. The "typical" rectangle indicated has width Δx and height $y_2 - y_1$, so the total moments in the x-direction over the total area is given by:

$$\bar{x} = \frac{\text{total moments}}{\text{total area}} = \frac{\displaystyle\int_a^b x\,(y_2 - y_1)\,dx}{\displaystyle\int_a^b (y_2 - y_1)\,dx}$$

For the y coordinate, we have 2 different ways we can go about it.

$$\bar{y} = \frac{\text{total moments}}{\text{total area}} = \frac{\displaystyle\int_c^d y\,(x_2 - x_1)\,dy}{\displaystyle\int_c^d (x_2 - x_1)\,dy}$$

EXAMPLE 9.3

Determine the distance $\bar{y}$ measured from the x axis to the centroid of the area of the triangle shown in Fig. 9–10.
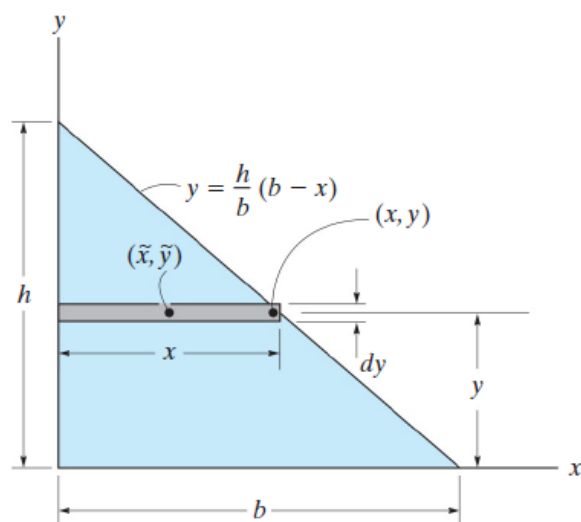
Fig. 9–10

SOLUTION

Differential Element. Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), Fig. 9–10.

Area and Moment Arms. The area of the element is $dA = x\,dy$ $= \dfrac{b}{h}(h - y)\,dy$, and its centroid is located a distance $\tilde{y} = y$ from the x axis.

Integration. Applying the second of Eqs. 9–4 and integrating with respect to y yields

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\,dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^h y\left[\frac{b}{h}(h - y)\,dy\right]}{\displaystyle\int_0^h \frac{b}{h}(h - y)\,dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}$$

$$= \frac{h}{3} \qquad\qquad\qquad Ans.$$

NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

EXAMPLE 9.5

Locate the centroid of the area shown in Fig. 9–12a.

SOLUTION I

Differential Element. A differential element of thickness dx is shown in Fig. 9–12a. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\, dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 9–4 and integrating with respect to x yields

$$\bar{x} = \frac{\displaystyle\int_A \tilde{x}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{1\,m} xy\, dx}{\displaystyle\int_0^{1\,m} y\, dx} = \frac{\displaystyle\int_0^{1\,m} x^3\, dx}{\displaystyle\int_0^{1\,m} x^2\, dx} = \frac{0.250}{0.333} = 0.75\ m \qquad Ans.$$

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{1\,m} (y/2)y\, dx}{\displaystyle\int_0^{1\,m} y\, dx} = \frac{\displaystyle\int_0^{1\,m} (x^2/2)x^2\, dx}{\displaystyle\int_0^{1\,m} x^2\, dx} = \frac{0.100}{0.333} = 0.3\ m\ Ans.$$

(a)

SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 9–12b. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\, dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1 - x}{2}\right) = \frac{1 + x}{2}, \quad \tilde{y} = y$$

Integrations. Applying Eqs. 9–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\displaystyle\int_A \tilde{x}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{1\,m} [(1 + x)/2](1 - x)\, dy}{\displaystyle\int_0^{1\,m} (1 - x)\, dy} = \frac{\dfrac{1}{2}\displaystyle\int_0^{1\,m} (1 - y)\, dy}{\displaystyle\int_0^{1\,m} (1 - \sqrt{y})\, dy} = \frac{0.250}{0.333} = 0.75\ m \quad Ans.$$

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{1\,m} y(1 - x)\, dy}{\displaystyle\int_0^{1\,m} (1 - x)\, dy} = \frac{\displaystyle\int_0^{1\,m} (y - y^{3/2})\, dy}{\displaystyle\int_0^{1\,m} (1 - \sqrt{y})\, dy} = \frac{0.100}{0.333} = 0.3\ m \qquad Ans.$$

(b)

Fig. 9–12

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.
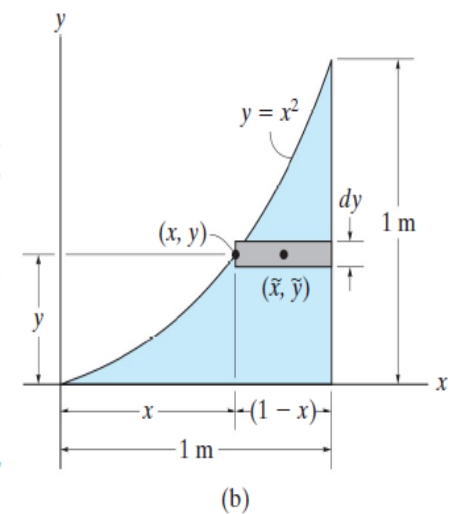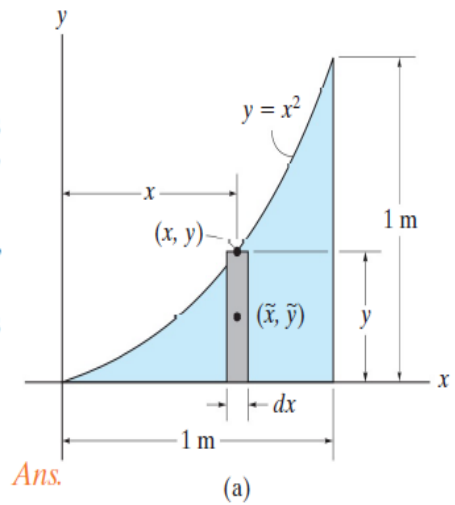
Example

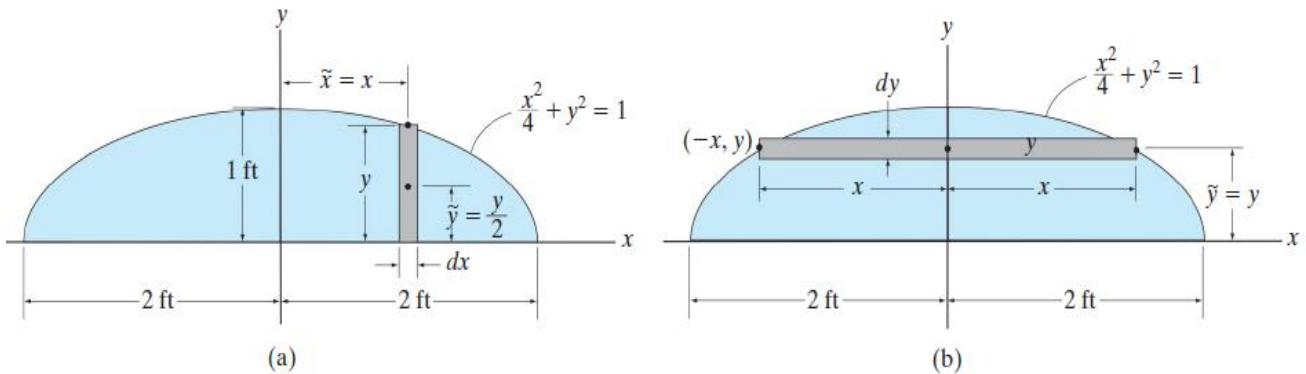Locate the centroid of the semi-elliptical area shown in Fig. 9–13a.

Fig. 9–13

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 9–13a will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\, dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad\qquad Ans.$$

Applying the second of Eqs. 9–4 with $y = \sqrt{1 - \dfrac{x^2}{4}}$, we have

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_{-2\,\text{ft}}^{2\,\text{ft}} \frac{y}{2}\,(y\, dx)}{\displaystyle\int_{-2\,\text{ft}}^{2\,\text{ft}} y\, dx} = \frac{\dfrac{1}{2}\displaystyle\int_{-2\,\text{ft}}^{2\,\text{ft}}\left(1 - \frac{x^2}{4}\right)dx}{\displaystyle\int_{-2\,\text{ft}}^{2\,\text{ft}}\sqrt{1 - \frac{x^2}{4}}\, dx} = \frac{4/3}{\pi} = 0.424\ \text{ft} \qquad Ans.$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 9–13b.

Area and Moment Arms. The area is $dA = 2x\, dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{1\,\text{ft}} y(2x\, dy)}{\displaystyle\int_0^{1\,\text{ft}} 2x\, dy} = \frac{\displaystyle\int_0^{1\,\text{ft}} 4y\sqrt{1 - y^2}\, dy}{\displaystyle\int_0^{1\,\text{ft}} 4\sqrt{1 - y^2}\, dy} = \frac{4/3}{\pi}\ \text{ft} = 0.424\ \text{ft} \quad Ans.$$

Locate the centroid $\bar{x}$ *and* $\bar{y}$ of the shaded area.
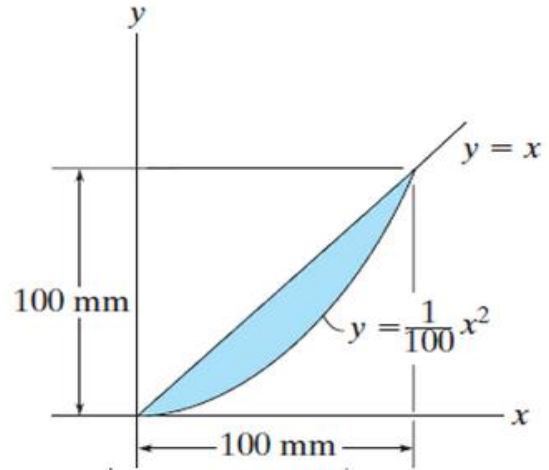
SOLUTION

Area And Moment Arm. Here, $y_2 = x$ and $y_1 = \dfrac{1}{100}x^2$. Thus the area of the differential element shown shaded in Fig. *a* is $dA = (y_2 - y_1)\,dx = (x - \dfrac{1}{100}x^2)dx$ and its centroid is at $\tilde{x} = x$.

Centroid. Perform the integration

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\displaystyle\int_0^{100\text{ mm}} x\left(x - \frac{1}{100}x^2\right)dx}{\displaystyle\int_0^{100\text{ mm}}\left(x - \frac{1}{100}x^2\right)dx}$$

$$= \frac{\left(\dfrac{x^3}{3} - \dfrac{1}{400}x^4\right)\Big|_0^{100\text{ mm}}}{\left(\dfrac{x^2}{2} - \dfrac{1}{300}x^3\right)\Big|_0^{100\text{ mm}}}$$

$$= 50.0 \text{ mm}$$

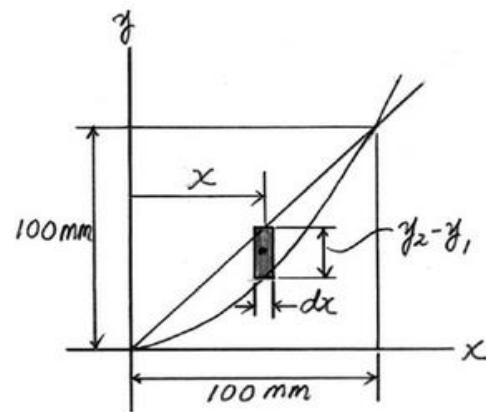***Area And Moment Arm.*** Here, $x_2 = 10y^{1/2}$ and $x_1 = y$. Thus, the area of the differential element shown shaded in Fig. *a* is $dA = (x_2 - x_1)\,dy = (10y^{1/2} - y)dy$ and its centroid is at $\tilde{y} = y$.

Centroid. Perform the integration,

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\displaystyle\int_0^{100\text{ mm}} y\left(10y^{1/2} - y\right)dy}{\displaystyle\int_0^{100\text{ mm}}\left(10y^{1/2} - y\right)dy}$$

$$= \frac{\left(4y^{5/2} - \dfrac{y^3}{3}\right)\Big|_0^{100\text{ mm}}}{\left(\dfrac{20}{3}y^{3/2} - \dfrac{y^2}{2}\right)\Big|_0^{100\text{ mm}}}$$

$$= 40.0 \text{ mm}$$
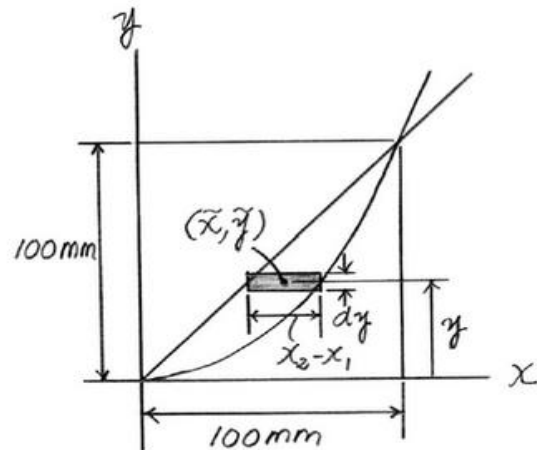
EXAMPLE 9.4

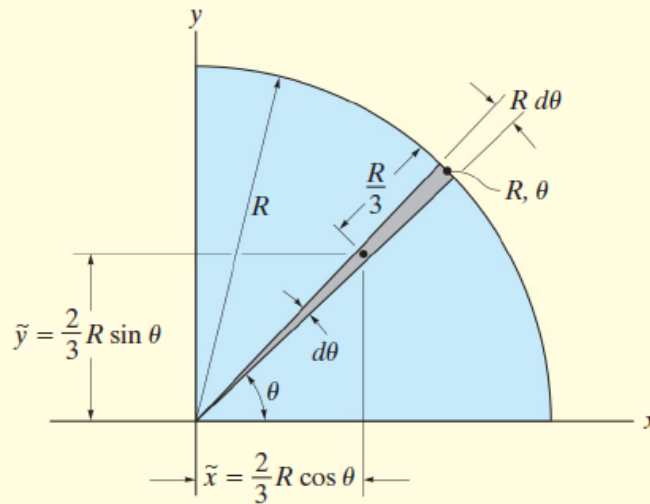Locate the centroid for the area of a quarter circle shown in Fig. 9–11.

Fig. 9–11

SOLUTION

Differential Element. Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9–11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point (R, θ).

Area and Moment Arms. The area of the element is

$$dA = \tfrac{1}{2}(R)(R \, d\theta) = \frac{R^2}{2} d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at $\tilde{x} = \tfrac{2}{3}R \cos \theta$, $\tilde{y} = \tfrac{2}{3}R \sin \theta$.

Integrations. Applying Eqs. 9–4 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\displaystyle\int_A \tilde{x} \, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{\pi/2} \left(\frac{2}{3}R \cos \theta\right)\frac{R^2}{2} d\theta}{\displaystyle\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right)\displaystyle\int_0^{\pi/2} \cos \theta \, d\theta}{\displaystyle\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi} \qquad Ans.$$

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y} \, dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^{\pi/2} \left(\frac{2}{3}R \sin \theta\right)\frac{R^2}{2} d\theta}{\displaystyle\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right)\displaystyle\int_0^{\pi/2} \sin \theta \, d\theta}{\displaystyle\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi} \qquad Ans.$$

Determine the coordinates of the centroid of the shaded area.
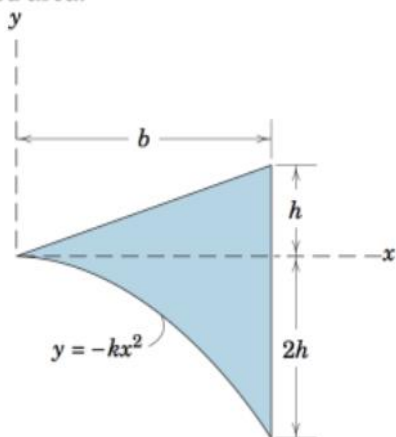
$y = -kx^2$

b, h, $2h$

Determine the y-coordinate of the centroid of the shaded area.

$x = y^2/b$, b, $b/2$
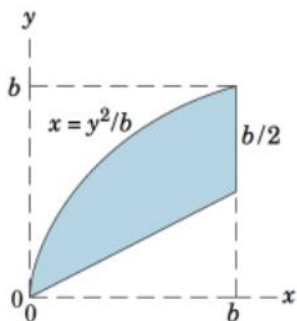
Determine the y-coordinate of the centroid of the shaded area.

$x = ky^2$, b, a
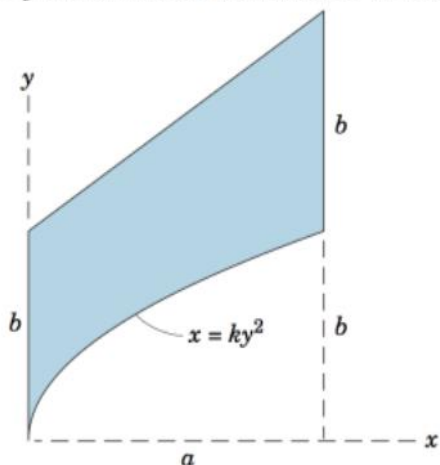
Locate the centroid of the shaded area between the two curves.

$y = \dfrac{x^3}{4}$

$x = \dfrac{y^2}{2}$

Determine the x- and y-coordinates of the centroid of the shaded area.

$y = 1 + \dfrac{x^3}{6}$
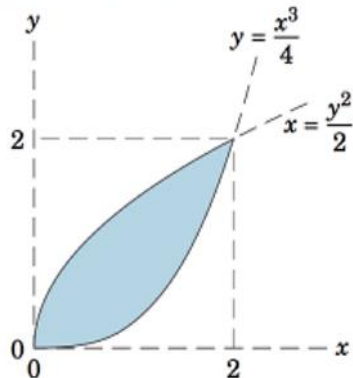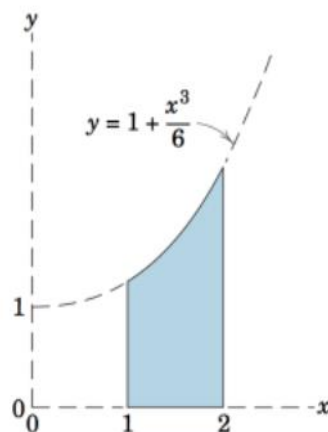
If the shaded area is revolved $360°$ about the y-axis, determine the y-coordinate of the centroid of the resulting volume.

$\dfrac{b}{2}$, $\dfrac{h}{4}$, $\dfrac{3h}{4}$, Parabolic

Determine the x- and y-coordinates of the centroid of the shaded area.

$y = y_0 - kx^3$

300 mm, 3, 4, 150 mm
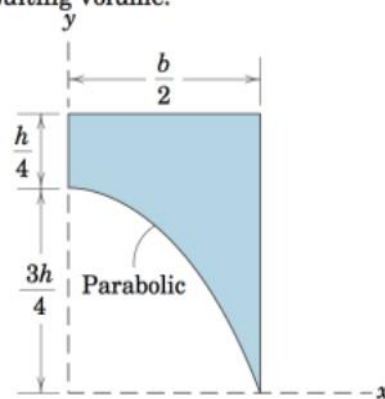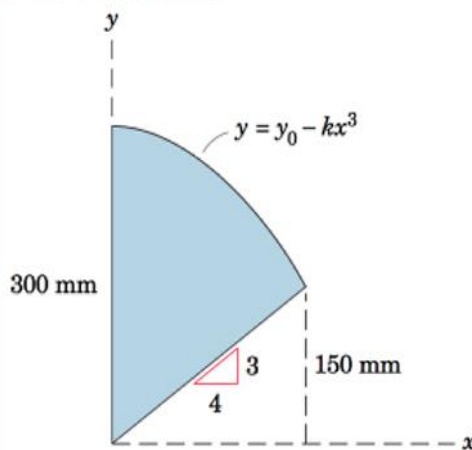
6.2 Centroid for Composite Areas

For composite areas, that can be decomposed to a finite number n of simpler subareas, and provided that the centroids of these subareas are available or easy to find, then the centroid coordinates of the entire area x_c , y_c can be calculated through the following formulas:

$$x_c = \frac{\sum_i^n A_i y_{c,i}}{\sum_i^n A_i}$$

$$y_c = \frac{\sum_i^n A_i x_{c,i}}{\sum_i^n A_i}$$

where A_i is the surface area of subarea i, and $x_{c,i}$, $y_{c,i}$ the centroid coordinates of subarea i. The sum $\sum_i^n A_i$ is equal to the total area A. The sums that appear in the two nominators are the respective first moments of the total area: $S_x = \sum_i^n A_i y_{c,i}$ and $S_y = \sum_i^n A_i x_{c,i}$.

> The above formulas impose the concept that the static moment (first moment of area), around a given axis, for the composite area (considered as a whole), is equivalent to the sum of the static moments of its subareas.

Steps to find the centroid of composite areas

The steps for the calculation of the centroid coordinates, x_c and y_c, of a composite area, are summarized to the following:
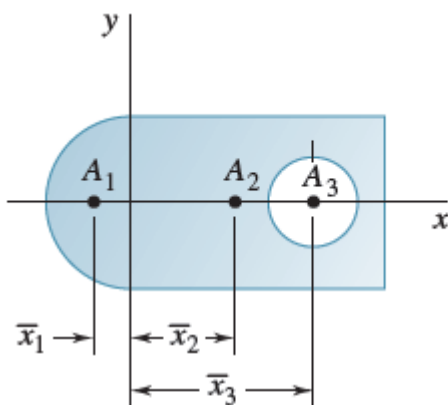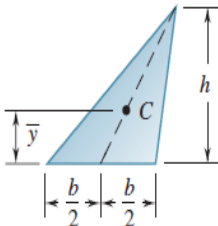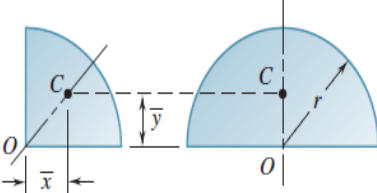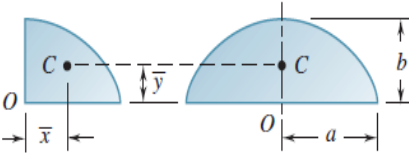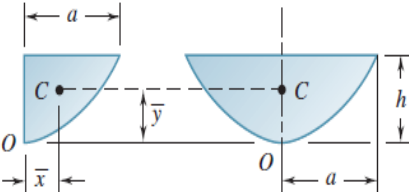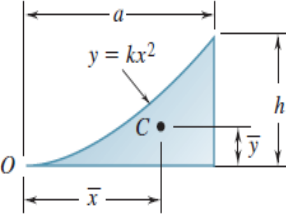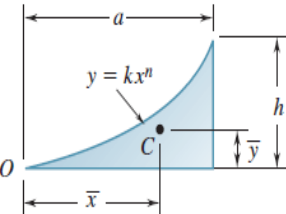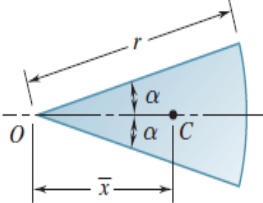
Centroid of a composite area

$$Q_y = \overline{X}\Sigma A = \Sigma \overline{x}A \qquad Q_x = \overline{Y}\Sigma A = \Sigma \overline{y}A$$

	$\overline{x}$	A	$\overline{x}A$
A_1 Semicircle	−	+	−
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	−	−

Fig. When calculating the centroid of a composite area, note that if the centroid of a component area has a negative coordinate distance relative to the origin, or if the area represents a hole, then the first moment is negative.

Centroids of common shapes of areas.

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\dfrac{h}{3}$	$\dfrac{bh}{2}$
Quarter-circular area		$\dfrac{4r}{3\pi}$	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{4}$
Semicircular area		0	$\dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{2}$
Quarter-elliptical area		$\dfrac{4a}{3\pi}$	$\dfrac{4b}{3\pi}$	$\dfrac{\pi ab}{4}$
Semielliptical area		0	$\dfrac{4b}{3\pi}$	$\dfrac{\pi ab}{2}$
Semiparabolic area		$\dfrac{3a}{8}$	$\dfrac{3h}{5}$	$\dfrac{2ah}{3}$
Parabolic area		0	$\dfrac{3h}{5}$	$\dfrac{4ah}{3}$
Parabolic spandrel		$\dfrac{3a}{4}$	$\dfrac{3h}{10}$	$\dfrac{ah}{3}$
General spandrel		$\dfrac{n+1}{n+2}a$	$\dfrac{n+1}{4n+2}h$	$\dfrac{ah}{n+1}$
Circular sector		$\dfrac{2r\sin\alpha}{3\alpha}$	0	αr^2

EXAMPLE

Locate the centroid of the plate area shown in Fig.
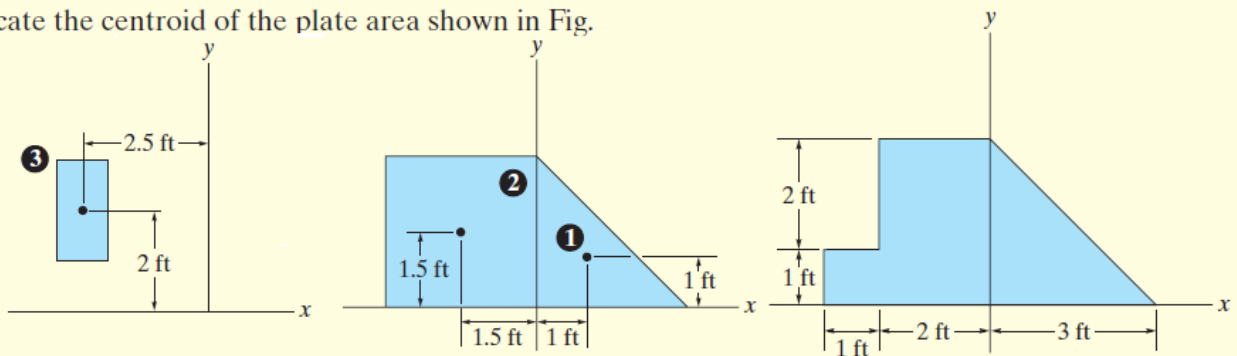
SOLUTION

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	-1.5	1.5	-13.5	13.5
3	$-(2)(1) = -2$	-2.5	2	5	-4
	$\Sigma A = 11.5$			$\Sigma\tilde{x}A = -4$	$\Sigma\tilde{y}A = 14$

Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad Ans.$$

$$\bar{y} = \frac{\Sigma\tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad Ans.$$

Sample Problem 5/6

Locate the centroid of the shaded area.

Solution. The composite area is divided into the four elementary shapes shown in the lower figure. The centroid locations of all these shapes may be obtained from Table D/3. Note that the areas of the "holes" (parts 3 and 4) are taken as negative in the following table:

PART	A mm^2	$\bar{x}$ mm	$\bar{y}$ mm	$\bar{x}A$ mm^3	$\bar{y}A$ mm^3
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	-1414	60	12.73	$-84\,800$	$-18\,000$
4	-800	120	40	$-96\,000$	$-32\,000$
TOTALS	12 790			959 000	650 000

The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959\,000}{12\,790} = 75.0 \text{ mm} \qquad Ans.$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650\,000}{12\,790} = 50.8 \text{ mm} \qquad Ans.$$
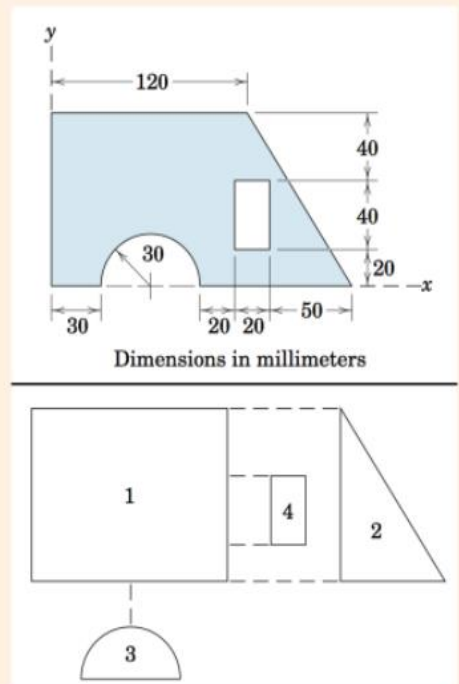
Sample Problem 5.1
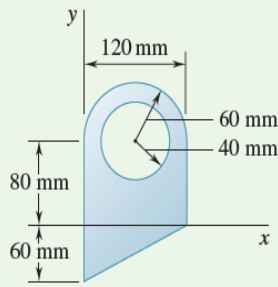
For the plane area shown, determine (*a*) the first moments with respect to the *x* and *y* axes; (*b*) the location of the centroid.

STRATEGY: Break up the given area into simple components, find the centroid of each component, and then find the overall first moments and centroid.

MODELING: As shown in Fig. 1, you obtain the given area by adding a rectangle, a triangle, and a semicircle and then subtracting a circle. Using the coordinate axes shown, find the area and the coordinates of the centroid of each of the component areas. To keep track of the data, enter them in a table. The area of the circle is indicated as negative because it is subtracted from the other areas. The coordinate $\bar{y}$ of the centroid of the triangle is negative for the axes shown. Compute the first moments of the component areas with respect to the coordinate axes and enter them in your table.
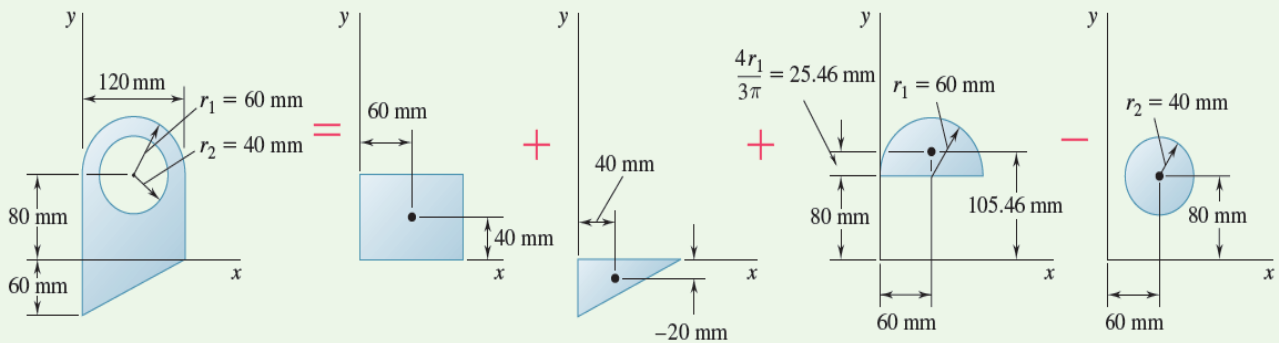
Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \bar{y}A = +506.2 \times 10^3$

Fig. 1 Given area modeled as the combination of simple geometric shapes.

ANALYSIS:

a. First Moments of the Area. Using Eqs. (5.8), you obtain

$$Q_x = \Sigma \bar{y}A = 506.2 \times 10^3 \text{ mm}^3 \qquad Q_x = 506 \times 10^3 \text{ mm}^3 \blacktriangleleft$$
$$Q_y = \Sigma \bar{x}A = 757.7 \times 10^3 \text{ mm}^3 \qquad Q_y = 758 \times 10^3 \text{ mm}^3 \blacktriangleleft$$

b. Location of Centroid. Substituting the values given in the table into the equations defining the centroid of a composite area yields (Fig. 2)

$$\bar{X}\Sigma A = \Sigma \bar{x}A: \qquad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$
$$\bar{X} = 54.8 \text{ mm} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A: \qquad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$
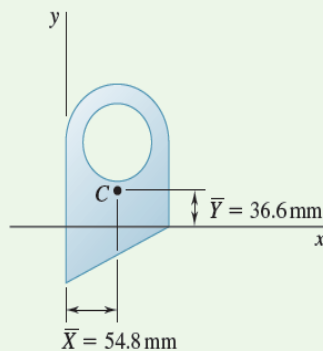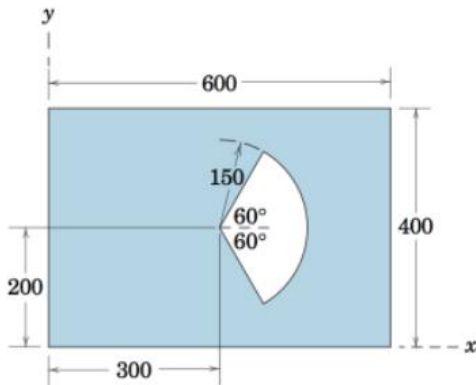$$\bar{Y} = 36.6 \text{ mm} \blacktriangleleft$$

Fig. 2 Centroid of composite area.

5/57 Determine the *x*- and *y*-coordinates of the centroid of the shaded area.

5/58 Determine the coordinates of the centroid of the shaded area.
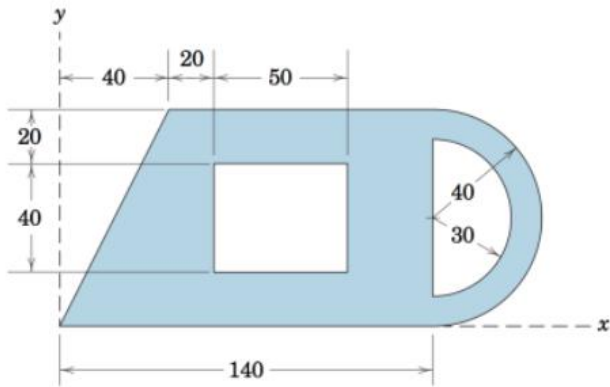
5/56 Determine the *y*-coordinate of the centroid of the shaded area. The triangle is equilateral.
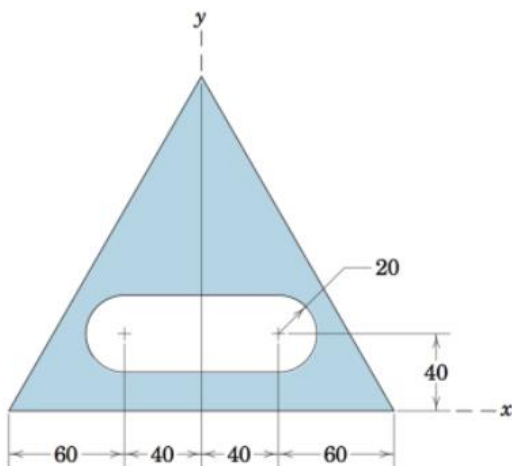

5/59 Determine the *x*- and *y*-coordinates of the centroid of the shaded area.
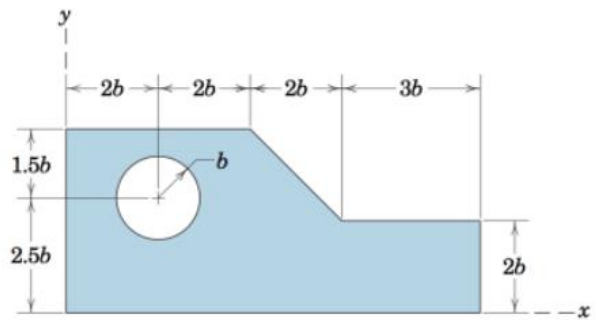
5/53 Calculate the *y*-coordinate of the centroid of the shaded area.
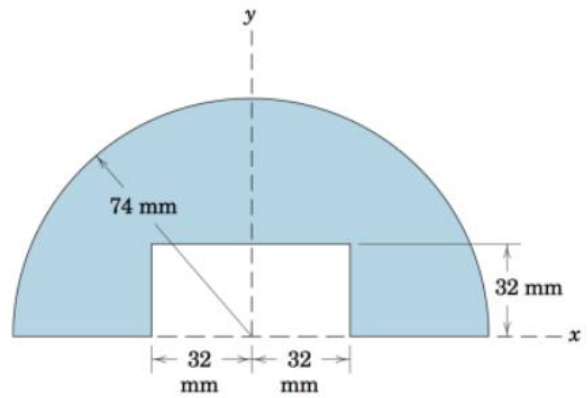
5/61 By inspection, state the quadrant in which the centroid of the shaded area is located. Then determine the coordinates of the centroid. The plate center is *M*.
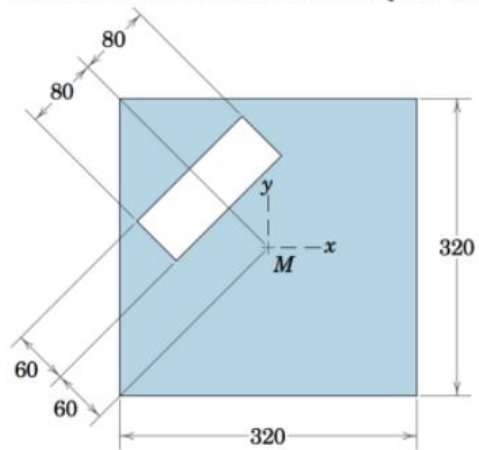
Dimensions in millimeters

Q   Locate the centroid of the plane area shown.

y

$x = ky^2$

$b = 4$ in.

x

$a = 8$ in.

y

10 in.

Parabola

Vertex

3 in.

16 in.

x

y

Parabola

Vertex

3 m

$r = 1.8$ m

x

4.5 m

4.5 m

y

Parabola

200 mm

x

240 mm   240 mm
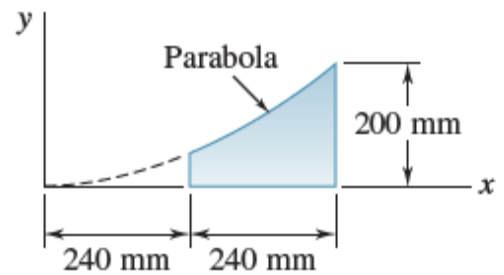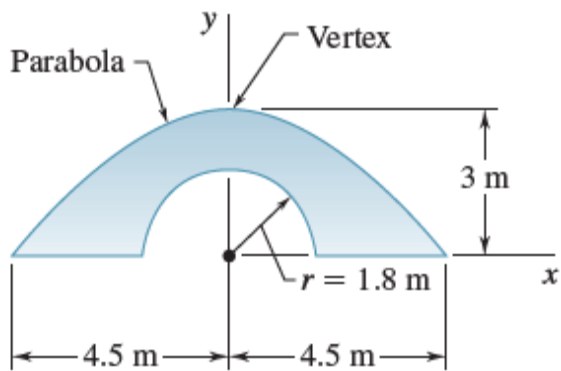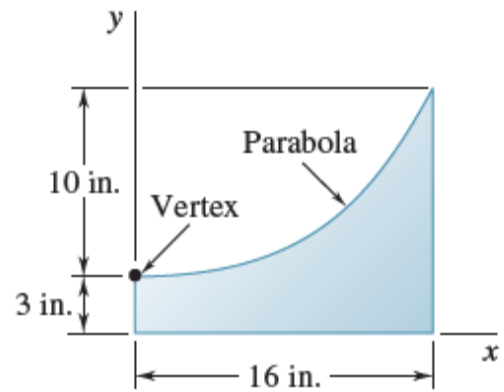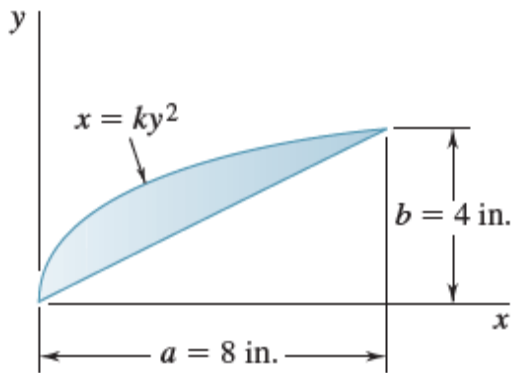
CHAPTER SEVEN

MOMENTS OF INERTIA

7.1 Introduction

The integral $\int y^2 dA$ is sometimes referred to as the "second moment" of the area about an axis (the x axis), but more often it is called the ***moment of inertia of the area***. The word "inertia" is used here since the formulation is similar to the mass moment of inertia, $\int y^2 dm$. Although for an area this integral has no physical meaning, it often arises in formulas used in fluid mechanics, mechanics of materials, structural mechanics, and mechanical design, and so the engineer needs to be familiar with the methods used to determine the moment of inertia.

$p = \gamma y$
γ is the specific weight of the fluid.
dF the force acting on the differential area dA of the plate
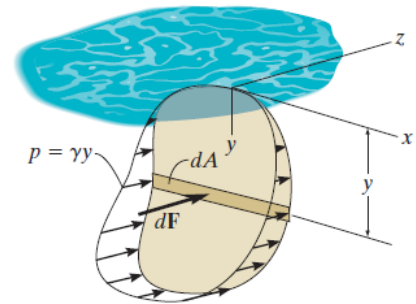$dF = p\, dA = (\gamma y)dA$.
dM The *moment* of this force about the x axis
$dM = y\, dF = \gamma y^2 dA$
$M = \gamma \int y^2 dA$.

Moment of Inertia. By definition, the moments of inertia of a differential area dA about the x and y axes are $dI_x = y^2\, dA$ and $dI_y = x^2\, dA$, respectively, Fig. 7-2. For the entire area A the ***moments of inertia*** are determined by integration; i.e.,
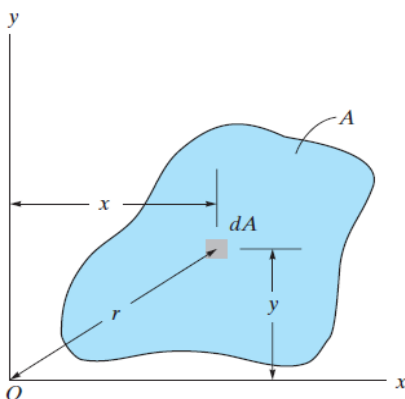
$$I_x = \int_A y^2\, dA$$
$$I_y = \int_A x^2\, dA$$

(7-1)

We can also formulate this quantity for dA about the "pole" O or z axis, Fig. 7–2. This is referred to as the ***polar moment of inertia***. It is defined as $dJ_O = r^2\, dA$, where r is the perpendicular distance from the pole (z axis) to the element dA. For the entire area the *polar moment of inertia* is

$$J_O = \int_A r^2\, dA = I_x + I_y$$

(7-2)

This relation between J_O and I_x, I_y is possible since $r^2 = x^2 + y^2$, Fig. 7–2.

From the above formulations it is seen that I_x, I_y, and J_O will *always* be *positive* since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4, mm^4, or ft^4, $in.^4$.

Fig. 7-2

7.2 Parallel-Axis Theorem for an Area

The *parallel-axis theorem* can be used to find the moment of inertia of an area about *any axis* that is parallel to an axis passing through the centroid and about which the moment of inertia is known. To develop this theorem, we will consider finding the moment of inertia of the shaded area shown in Fig. 7–3 about the x axis. To start, we choose a differential element dA located at an arbitrary distance y' from the *centroidal x'* axis. If the distance between the parallel x and x' axis is d_y, then the moment of inertia of dA about the x axis is $dIx = (y' + d_y)^2 \, dA$. For the entire area,
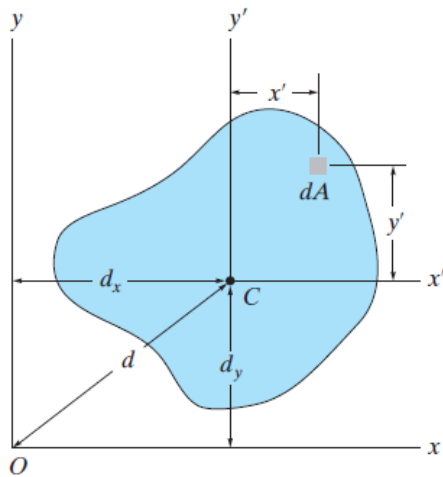
$$I_x = \int_A (y' + d_y)^2 \, dA$$

$$= \int_A y'^2 \, dA + 2d_y \int_A y' \, dA + d_y^2 \int_A dA$$

The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$. The second integral is zero since the x' axis passes through the area's centroid C; i.e., $\int y' \, dA = \bar{y}' \int dA = 0$ since $\bar{y}' = 0$. Since the third integral represents the total area A, the final result is therefore

$$\boxed{I_x = \bar{I}_{x'} + Ad_y^2}$$

A similar expression can be written for I_y; i.e.,

$$\boxed{I_y = \bar{I}_{y'} + Ad_x^2}$$

And finally, for the polar moment of inertia, since $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$ and $d^2 = d_x^2 + d_y^2$, we have

$$\boxed{J_O = \bar{J}_C + Ad^2}$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes.*

7.3 Radius of Gyration of an Area

The radius of gyration of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are known, the radii of gyration are determined from the formulas

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

The form of these equations is easily remembered since it is similar to that for finding the moment of inertia for a differential area about an axis. For example, $Ix = k_x{}^2 A$; whereas for a differential area, $dI_x = y^2\, dA$.

Important Points

- The moment of inertia is a geometric property of an area that is used to determine the strength of a structural member or the location of a resultant pressure force acting on a plate submerged in a fluid. It is sometimes referred to as the second moment of the area about an axis, because the distance from the axis to each area element is squared.

- If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

7.3 Procedure of Analysis
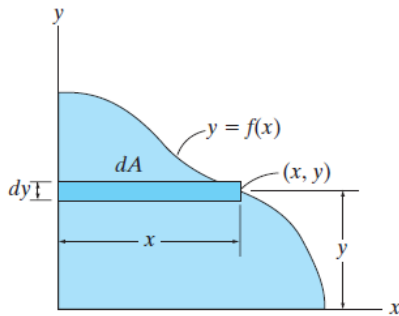
(a)
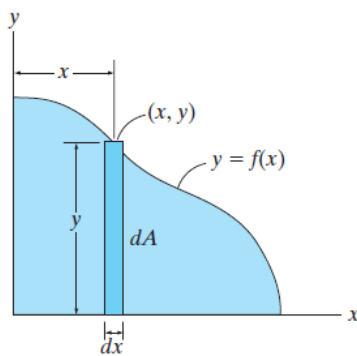
(b)

Fig. 10–4

Procedure for Analysis

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the *arbitrary point* (x, y).

Case 1.

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is computed. This situation occurs when the rectangular element shown in Fig. 10–4a is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 10–4b. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

Case 2.

- The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is computed; however, Eq. 10–1 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 10–4a is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y. See Examples 10.2 and 10.3.

EXAMPLE 10.1

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x'–y' plane and passing through the centroid C.

SOLUTION (CASE 1)

Part (a). The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

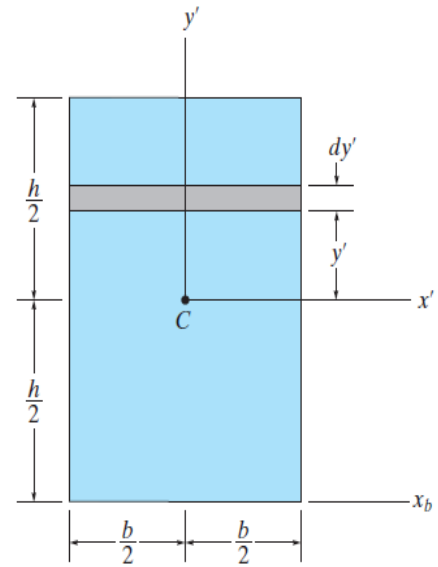$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad\qquad Ans.$$

Fig. 10–5

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

$$I_{x_b} = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \qquad\qquad Ans.$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,
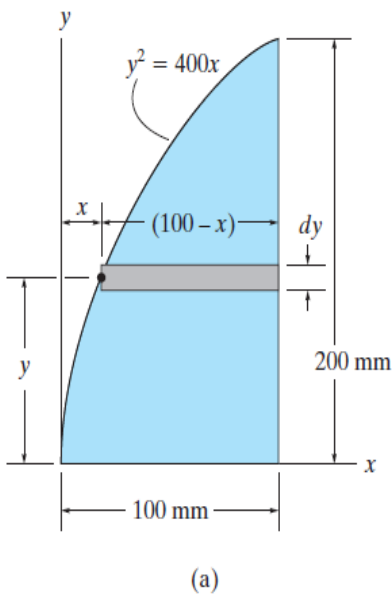
$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10–2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad\qquad Ans.$$

Example 2

(a)

(b)

Fig. 10–6

Determine the moment of inertia for the shaded area shown in Fig. 10–6a about the x axis.

SOLUTION I (CASE 1)
A differential element of area that is *parallel* to the x axis, as shown in Fig. 10–6a, is chosen for integration. Since this element has a thickness dy and intersects the curve at the *arbitrary point* (x, y), its area is $dA = (100 - x)\, dy$. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

$$I_x = \int_A y^2\, dA = \int_0^{200\ \text{mm}} y^2(100 - x)\, dy$$

$$= \int_0^{200\ \text{mm}} y^2\left(100 - \frac{y^2}{400}\right) dy = \int_0^{200\ \text{mm}} \left(100y^2 - \frac{y^4}{400}\right) dy$$

$$= 107(10^6)\ \text{mm}^4 \qquad\qquad\qquad\qquad Ans.$$

SOLUTION II (CASE 2)
A differential element *parallel* to the y axis, as shown in Fig. 10–6b, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the x axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 10.1. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 10–6b, $b = dx$ and $h = y$, and thus $d\bar{I}_{x'} = \frac{1}{12}dx\, y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the x axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\, \tilde{y}^2 = \frac{1}{12}dx\, y^3 + y\, dx \left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\, dx$$

(This result can also be concluded from part (b) of Example 10.1.) Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

$$I_x = \int dI_x = \int_0^{100\ \text{mm}} \frac{1}{3}y^3\, dx = \int_0^{100\ \text{mm}} \frac{1}{3}(400x)^{3/2}\, dx$$

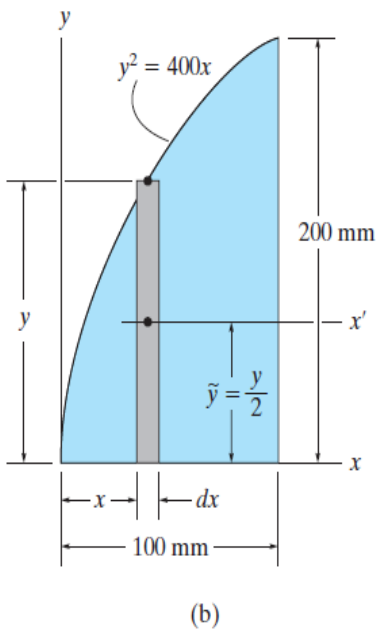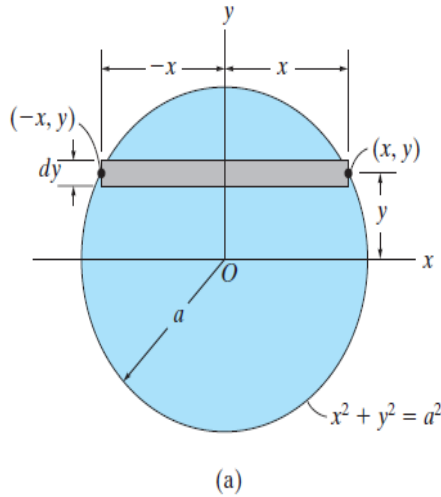$$= 107(10^6)\ \text{mm}^4 \qquad\qquad\qquad\qquad Ans.$$

EXAMPLE 10.3

Determine the moment of inertia with respect to the x axis for the circular area shown in Fig. 10–7a.

(a)

SOLUTION I (CASE 1)

Using the differential element shown in Fig. 10–7a, since $dA = 2x\,dy$, we have

$$I_x = \int_A y^2\,dA = \int_A y^2(2x)\,dy$$

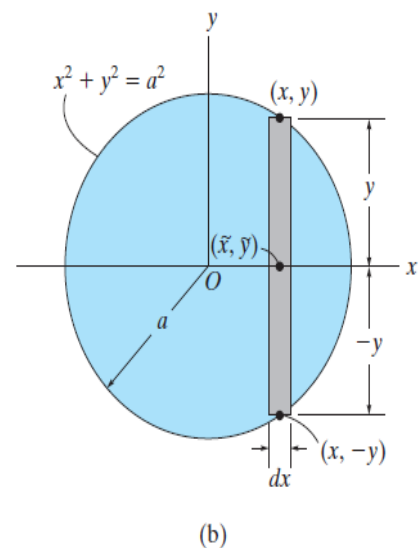$$= \int_{-a}^{a} y^2\left(2\sqrt{a^2 - y^2}\right)dy = \frac{\pi a^4}{4} \qquad Ans.$$

SOLUTION II (CASE 2)

When the differential element shown in Fig. 10–7b is chosen, the centroid for the element happens to lie on the x axis, and since $\bar{I}_{x'} = \frac{1}{12}bh^3$ for a rectangle, we have

$$dI_x = \frac{1}{12}dx(2y)^3$$

$$= \frac{2}{3}y^3\,dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a}\frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4} \qquad Ans.$$

NOTE: By comparison, Solution I requires much less computation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.

(b)

Fig. 10–7

7-4 Moments of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the *algebraic sum* of the moments of inertia of all its parts.

Procedure for Analysis

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the reference axis. For the calculation of $\bar{I}$, use the table on the inside back cover.

Summation.

- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.

- If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.
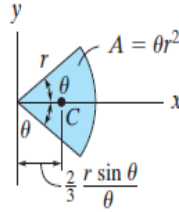
Geometric Properties of Line and Area Elements

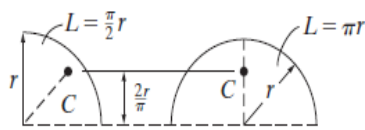| Centroid Location	Centroid Location	Area Moment of Inertia
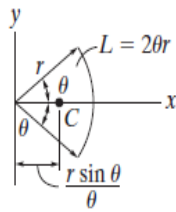
Circular arc segment

$L = 2\theta r$

$\dfrac{r \sin \theta}{\theta}$

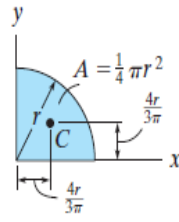**Circular sector area**

$A = \theta r^2$

$\dfrac{2}{3}\dfrac{r \sin \theta}{\theta}$

$I_x = \dfrac{1}{4}r^4\left(\theta - \dfrac{1}{2}\sin 2\theta\right)$

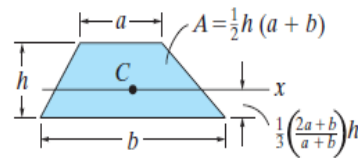$I_y = \dfrac{1}{4}r^4\left(\theta + \dfrac{1}{2}\sin 2\theta\right)$

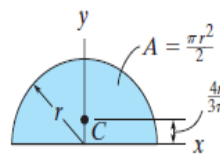
Quarter and semicircle arcs

$L = \dfrac{\pi}{2}r$

$L = \pi r$

$\dfrac{2r}{\pi}$

Quarter circle area

$A = \dfrac{1}{4}\pi r^2$

$\dfrac{4r}{3\pi}$

$\dfrac{4r}{3\pi}$

$I_x = \dfrac{1}{16}\pi r^4$

$I_y = \dfrac{1}{16}\pi r^4$

Trapezoidal area

$A = \dfrac{1}{2}h(a+b)$

$\dfrac{1}{3}\left(\dfrac{2a+b}{a+b}\right)h$

Semicircular area

$A = \dfrac{\pi r^2}{2}$

$\dfrac{4r}{3\pi}$

$I_x = \dfrac{1}{8}\pi r^4$

$I_y = \dfrac{1}{8}\pi r^4$

Semiparabolic area

$A = \dfrac{2}{3}ab$

$\dfrac{3}{5}a$

$\dfrac{3}{8}b$

Circular area

$A = \pi r^2$
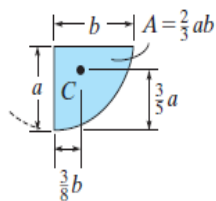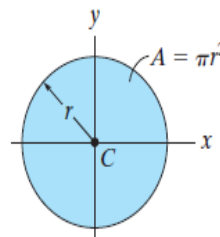
$I_x = \dfrac{1}{4}\pi r^4$

$I_y = \dfrac{1}{4}\pi r^4$

Exparabolic area

$A = \dfrac{1}{3}ab$

$\dfrac{3}{4}a$

$\dfrac{3}{10}b$

Rectangular area

$A = bh$
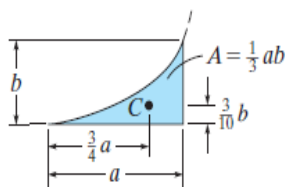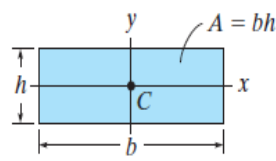
$I_x = \dfrac{1}{12}bh^3$

$I_y = \dfrac{1}{12}hb^3$

Parabolic area

$A = \dfrac{4}{3}ab$

$\dfrac{2}{5}a$

Triangular area

$A = \dfrac{1}{2}bh$

$\dfrac{1}{3}h$
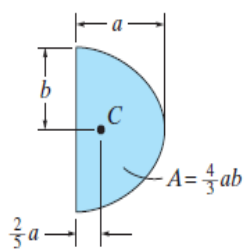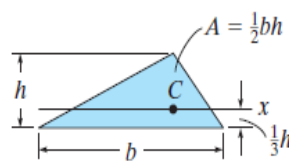
$I_x = \dfrac{1}{36}bh^3$

EXAMPLE 10.4

Determine the moment of inertia of the area shown in Fig. 10–8a about the x axis.
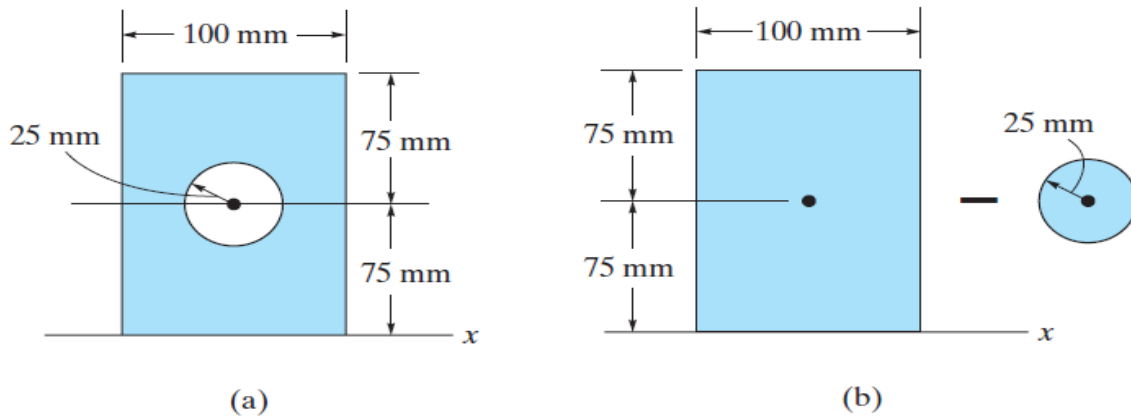

(a)   (b)

Fig. 10–8

SOLUTION

Composite Parts.   The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8b. The centroid of each area is located in the figure.

Parallel-Axis Theorem.   The moments of inertia about the x axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$; $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ mm^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ mm^4$$

Summation.   The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\ mm^4 \hspace{3cm} Ans.$$

100 mm

400 mm

C • —— x

100 mm

400 mm

—100 mm

— 600 mm —

(a)
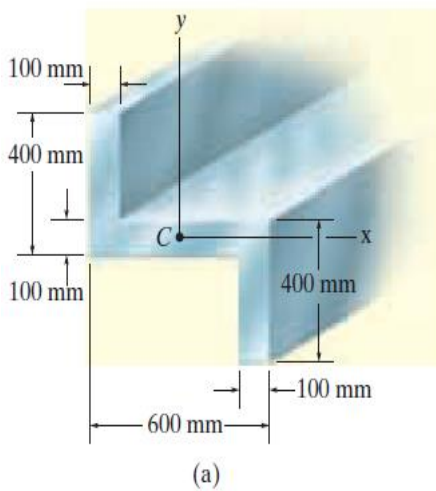
Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9a about the x and y centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas A, B, and D shown in Fig. 10–9b. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles A and D, the calculations are as follows:
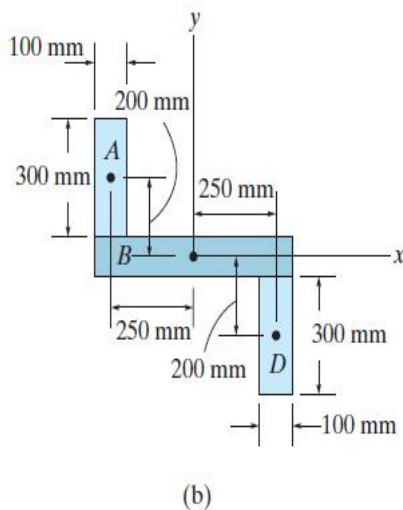
Rectangles A and D

100 mm

200 mm

A

300 mm

250 mm

B —— x

250 mm

200 mm   D

300 mm

—100 mm

(b)

Fig. 10–9

$$I_x = \bar{I}_{x'} + A d_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$

$$= 1.425(10^9) \text{ mm}^4$$

$$I_y = \bar{I}_{y'} + A d_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$

$$= 1.90(10^9) \text{ mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9) \text{ mm}^4$$

$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9) \text{ mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

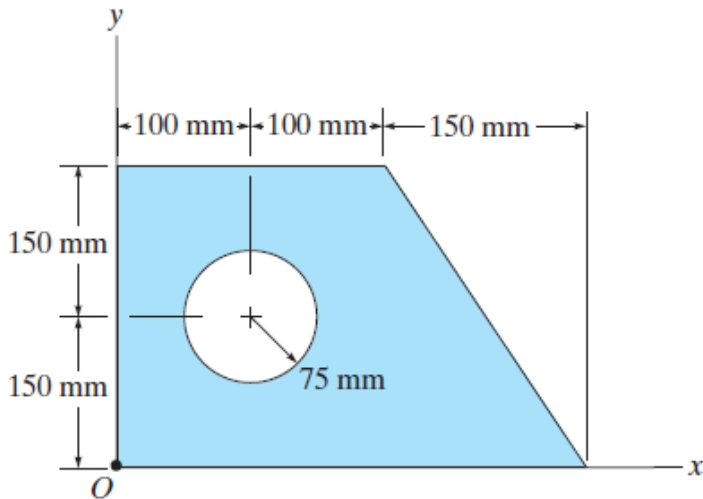$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$

$$= 2.90(10^9) \text{ mm}^4 \qquad \qquad Ans.$$

$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$

$$= 5.60(10^9) \text{ mm}^4 \qquad \qquad Ans.$$

A- Determine the moment of inertia I_x of the shaded area about the x axis.
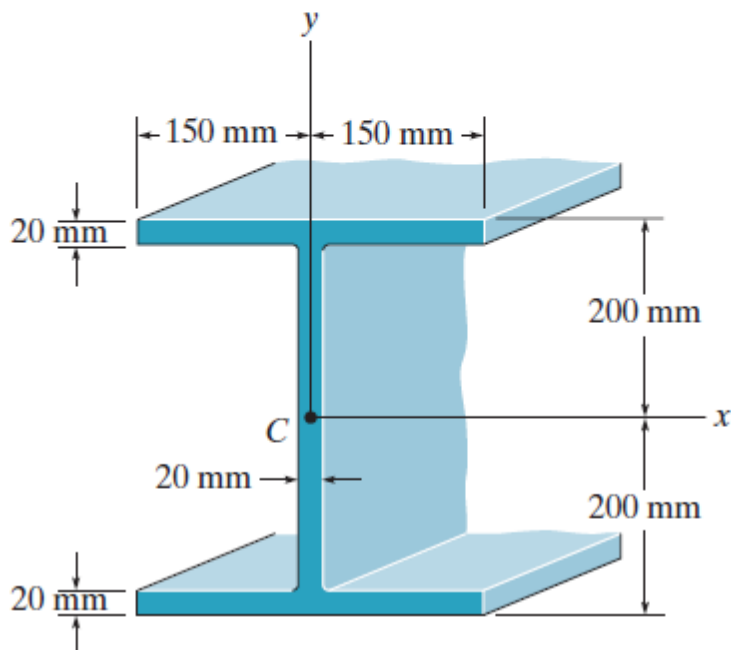
B- Determine the moment of inertia I_x of the shaded area about the y axis.

A- Determine the moment of inertia about the x axis.

B- Determine the moment of inertia about the y axis.

A- Determine the moment of inertia of the beam's cross-sectional area about the x axis.

B- Determine the moment of inertia of the beam's cross-sectional area about the y axis.

C- Determine the distance $\bar{y}$ to the centroid C of the beam's cross-sectional area and then compute the moment of inertia $\bar{I}_{x'}$ about the x' axis.

D- Determine the distance $\bar{x}$ to the centroid C of the beam's cross-sectional area and then compute the moment of inertia $\bar{I}_{y'}$ about the y' axis.