Optimization

Fourth Class

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Chapter Six

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Constrained Optimization

Lecture 5

Example (3):

Minimize $f(x, y, z) = x^2 + y^2 + z^2 + 40x + 20y - 3000$

Subject to

$g_1(x, y, z) \equiv (x - 50) \geq 0$

$g_2(x, y, z) \equiv (x + y - 100) \geq 0$

$g_3(x, y, z) \equiv (x + y + z - 150) \geq 0$

Solution:

The Kuhn – Tucker conditions can be stated as

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} + \lambda_3 \frac{\partial g_3}{\partial x_i} = 0 \text{ , } i = 1, 2, 3.$$

Then we have the following equations

$$2x + 40 + \lambda_1 + \lambda_2 + \lambda_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

$$2y + 20 + \lambda_2 + \lambda_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (41)$$

$$2z + \lambda_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

Also,

$$\lambda_j g_j = 0 \,, j = 1, 2, 3.$$

Then we have the following equations

$$\lambda_1(x - 50) = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (43)$$

$$\lambda_2(x + y - 100) = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \quad (44)$$

$$\lambda_3(x + y + z - 150) = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \quad (45)$$

Also,

$$g_j \geq 0 \,, j = 1, 2, 3.$$

Then we have the following inequalities

$$(x - 50) \geq 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (46)$$

$$(x + y - 100) \geq 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (47)$$

$$(x + y + z - 150) \geq 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (48)$$

Also,

$\lambda_j \leq 0 \,, j = 1, 2, 3.$

Then we have the following inequalities

$\lambda_1 \leq 0$ … (49)

$\lambda_2 \leq 0$ … (50)

$\lambda_3 \leq 0$ … (51)

Now from (43), we have two cases either $\lambda_1 = 0$ **or** $x - 50 = 0 \rightarrow x = 50.$

Case 1: $\lambda_1 = 0.$

$\therefore$**The Equations (40), (41) and (42) yields**

$x = -20 - \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_3$ … (52)

$y = -10 - \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_3$ … (53)

$z = -\frac{1}{2}\lambda_3$ … (54)

Now, we use (52), (53), (54), (44) and (45), we have

$$\lambda_2(-130-\lambda_2-\lambda_3)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

$$\lambda_3\left(-180-\lambda_2-\frac{3}{2}\lambda_3\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

Therefore four possible solutions of Equations (55) and (56), they are

$$A: \lambda_2=0 \ \ and \ -180-\lambda_2-\frac{3}{2}\lambda_3=0 \, .$$

$$B: \lambda_3=0 \ and \ -130-\lambda_2-\lambda_3=0 \, .$$

$$C: \lambda_2=0 \ \ and \ \lambda_3=0 \, .$$

$$D: -180-\lambda_2-\frac{3}{2}\lambda_3=0 \ \ and \ -130-\lambda_2-\lambda_3=0 \, .$$

Now, from case A, we have

$$\lambda_2=0 \, , \lambda_3=-120 \, .$$

∴From Equations (52), (53) and (54), we have

$$x=40, y=50, z=60.$$

This solution is not satisfy the constraints (46) and (47).

Now take case B, we have

$\lambda_3 = 0 \; and \; \lambda_2 = -130$.

∴From Equations (52), (53) and (54), we have

$x = 45, y = 55, z = 0$.

This solution is not satisfy the constraints (46) and (47).

Now take case C, we have

$\lambda_2 = 0 \;, \lambda_3 = 0$.

∴From Equations (52), (53) and (54), we have

$x = -20, y = -10, z = 0$.

This solution is not satisfy the constraints (46) and (47).

Now take case D, we have

$\lambda_2 = -30 \;, \lambda_3 = -100$.

∴From Equations (52), (53) and (54), we have

$x = 45 \;, y = 55, z = 50$.

This solution is not satisfy the constraint (46).

<u>Case 2: $x = 50$.</u>

$\therefore$The Equations (40), (41) and (42) yields

$$\lambda_3 = -2z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (57)$$

$$\lambda_2 = -20 - 2y - 2z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (58)$$

$$\lambda_1 = -120 + 2y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

From Equations (47), (48), (57), (58) and (59) yields

$$(-2z)(x + y + z - 150) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (60)$$

$$(-20 - 2y + 2z)(x + y - 100) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

Again there are four possible solutions for (60) and (61), they are:

A: $-20 - 2y + 2z = 0\,,\, x + y + z - 150 = 0$.

B: $-20 - 2y + 2z = 0\,,\, -2z = 0$.

C: $x + y - 100 = 0\,,\, -2z = 0$.

D: $x + y - 100 = 0, x + y + z - 150 = 0$.

Now, take case A, we have:

$x = 50, y = 45, z = 55$.

This solution is not satisfy the constraint (47).

Now, take case B, we have:

$x = 50, y = -10, z = 0$.

This solution is not satisfy the constraints (47) and (48).

Now, take case C, we have:

$x = 50, y = 50, z = 0$.

This solution is not satisfy the constraint (48).

Now, take case D, we have:

$x = 50, y = 50, z = 50$.

This solution can be seen to satisfy all the constraints.

The values of λ_1, λ_2 *and* λ_3 corresponding to this solution are

$\lambda_1 = -20, \lambda_2 = -20$ *and* $\lambda_3 = -100$.

Therefore the optimum solution is $x^* = 50, y^* = 50$ *and* $z^* = 50$.

<u>H.W.</u>

Maximize $f(x,y) = 8x + 4y + xy - x^2 - y^2$ **subject to**

$g_1(x,y) \equiv 2x + 3y \leq 24$

$g_2(x,y) \equiv -5x + 12y \leq 24$

$g_3(x,y) \equiv y \leq 5$

By applying the Kuhn – Tucker conditions.

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