Optimization

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Chapter Six

Constrained Optimization

Lecture 2

Theorem (1):

A necessary condition for a function $f(X)$ subject to the constraints $g_j(X) = 0\ , j = 1, 2, \dots, m\ to\ have$ a local minimizer at a point X^* is that the first partial derivatives of the Lagrange function defined by $L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m)\ with\ respect\ to\ each\ of\ its\ arguments\ must\ be\ zero$.

H.W.

Minimize $f(x, y, z) = -xyz\ subject\ to$

$$g_1(x, y, z) \equiv x^2 + y^2 = 1\ , g_2(x, y, z) \equiv x + z = 1.$$

Theorem (2):

A sufficient condition for $f(X)$ subject to the constraints $g_j(X) = 0$, $j = 1, 2, \dots, m$ *to have* a local minimizer at a point X^* is that the matrix, Q, defined by

$$Q = \frac{\partial^2 L}{\partial x_i \partial x_l} \quad , i = 1, 2, \dots, n \, ; l = 1, 2, \dots, n \qquad (10)$$

evaluated at $X = X^*$ *must be positive definite for all values for which the constraints are satisfied.*

Note (1):

If Q which is defined in (10) negative definite, then X^* will be a local constrained maximizer of $f(X)$.

<u>Note (2):</u>

It has been shown by Hancock that a necessary condition for $\boldsymbol{the\ matrix\ Q}$, defined by (10) to be positive definite is that each root of the polynomial $\boldsymbol{z_i}$ defined by the following equation be positive .

$$
\begin{vmatrix}
(L_{11}-z) & L_{12} & L_{13} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\
L_{21} & (L_{22}-z) & L_{23} & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
L_{n1} & L_{n2} & L_{n3} & \cdots & (L_{nn}-z) & g_{1n} & g_{2n} & \cdots & g_{mn} \\
g_{11} & g_{12} & g_{13} & \cdots & g_{1n} & 0 & 0 & \cdots & 0 \\
g_{21} & g_{22} & g_{23} & \cdots & g_{2n} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
g_{m1} & g_{m2} & g_{m3} & & g_{mn} & 0 & 0 & & 0
\end{vmatrix} = 0
$$

(11)

Where

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \, \partial x_j}(X^*, \lambda) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (12)$$

And

$$g_{ii} = \frac{\partial g_i}{\partial x_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad (13)$$

Note: 3

If each root of the polynomial z_i defined by the equation (11) negative then the matrix Q in Theorem (2) be negative definite. In this case the point X^* will be a local constrained maximizer of $f(X)$.

<u>Note (4):</u>

Equation (11), on expansion, leads to $(n-m)th$ order polynomial in z.

If the some of the roots of this polynomial are positive while the others are negative, the point X^* is not an extreme point.