



# Optimization

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By

*Dr. Jawad Mahmoud Jassim*

**Dept. of Math.**

**Education College for Pure Sciences**

*University of Basrah*

**Iraq**





# Chapter Six



# Constrained Optimization

## Lecture 2

## Theorem (1):

A necessary condition for a function  $f(X)$  subject to the constraints  $g_j(X) = 0, j = 1, 2, \dots, m$  to have a local minimizer at a point  $X^*$  is that the first partial derivatives of the Lagrange function defined by  $L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m)$  with respect to each of its arguments must be zero.

## H.W.

Minimize  $f(x, y, z) = -xyz$  subject to

$$g_1(x, y, z) \equiv x^2 + y^2 = 1, g_2(x, y, z) \equiv x + z = 1.$$



## Theorem (2):

A sufficient condition for  $f(X)$  subject to the constraints  $g_j(X) = 0$ ,  $j = 1, 2, \dots, m$  to have a local minimizer at a point  $X^*$  is that the matrix,  $Q$ , defined by

$$Q = \frac{\partial^2 L}{\partial x_i \partial x_l}, \quad i = 1, 2, \dots, n; l = 1, 2, \dots, n \quad (10)$$

*evaluated at  $X = X^*$  must be positive definite for all values for which the constraints are satisfied.*

## Note (1):

If  $Q$  which is defined in (10) negative definite, then  $X^*$  will be a local constrained maximizer of  $f(X)$ .



## Note (2):

It has been shown by Hancock that a necessary condition for *the matrix Q*, defined by (10) to be positive definite is that each root of the polynomial  $z_i$  defined by the following equation be positive .



$$\begin{vmatrix}
 (L_{11} - z) & L_{12} & L_{13} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\
 L_{21} & (L_{22} - z) & L_{23} & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 L_{n1} & L_{n2} & L_{n3} & \cdots & (L_{nn} - z) & g_{1n} & g_{2n} & \cdots & g_{mn} \\
 g_{11} & g_{12} & g_{13} & \cdots & g_{1n} & 0 & 0 & \cdots & 0 \\
 g_{21} & g_{22} & g_{23} & \cdots & g_{2n} & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 g_{m1} & g_{m2} & g_{m3} & \cdots & g_{mn} & 0 & 0 & \cdots & 0
 \end{vmatrix} = 0$$

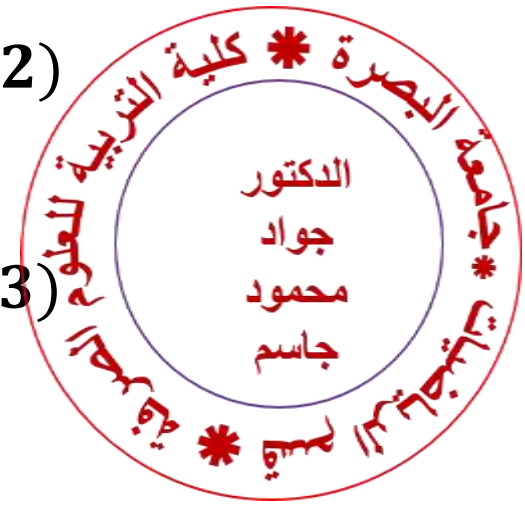
(11)

Where

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j} (X^*, \lambda) \dots \dots \dots (12)$$

And

$$g_{ii} = \frac{\partial g_i}{\partial x_i} \dots \dots \dots (13)$$



### Note: 3

If each root of the polynomial  $z_i$  defined by the equation (11) negative then the matrix  $Q$  in Theorem (2) be negative definite. In this case the point  $X^*$  will be a local constrained maximizer of  $f(X)$ .



**Note (4):**

**Equation (11), on expansion, leads to  $(n - m)$ th order polynomial in  $z$ .**

**If the some of the roots of this polynomial are positive while the others are negative, the point  $X^*$  is not an extreme point.**

