Optimization

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By

Dr. Jawad Mahmoud Jassim

Dept. of Math.

Education College for Pure Sciences

University of Basrah

Iraq

Chapter Six

Constrained Optimization

Lecture 1

1: Statement of Constrained Optimization Problem

An optimization problem or mathematical programming problem can be stated as follows:

$$\text{Find } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ which minimizes } f(X) \text{ subject to the constraints } \ldots \quad (1)$$

$$g_j(X) \leq 0 \,, j = 1, 2, \ldots, m \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

And

$$h_j(X) = 0 \,, j = m+1, m+2, \ldots, p, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Where $X \ is \ an \ n - dimensional$ vector called the design vector, $f(X)$ is called the objective function and $g_j(X) \ and \ h_j(X)$ are, respectively, the inequality and the equality constraints.

The number of variables n and the number of constraints p need not be related in any way. Thus p could be less than, equal to or greater than n in a general mathematical problem.

In some problems, the value of p might be zero which means that there are no constraints on the problem. Such type of problems are called unconstrained optimization problems.

Those problems for which p is not equal to zero are known as constrained optimization problems.

In this section, we will consider the optimization of continuous functions subject to equality constraints:

Minimize $f(X)$ subject to

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$g_j(X) = 0 \,, j = 1, 2, \ldots, m \ \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Here m *is less than or equal to* n. Otherwise (*if* $m > n$) the problem becomes over defined and in general, there will be no solution.

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There are serval methods available for solution of this problem. One of them is the method of Lagrange multipliers which is discussed below.

2-1: Solution by the Method of Lagrange Multipliers

In the Lagrange multiplier method, we introduce one additional variable to the problem for each constraint. Thus if the original problem has n variables and m equality constraints, we add m additional variables to the problem so that the final number of unknowns are $(n + m)$.

The basic feature of the method will be initially given for a simple problem of two variables with one constraint. The extension of the method to a general problem of n variables with m constraints will be given later.

Now consider the problem

Minimize $f(x_1, x_2)$ **subject to** $g(x_1, x_2) = 0$.

$Define\ a\ function\ L$, **known as Lagrange function as**

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Where $\lambda\ is\ a\ quantity, called\ the\ Lagrange\ multiplier$.

If the partial derivatives of Equation (6) with respect to $x_1, x_2\ and\ \lambda$ **is set equal to zero, the necessary conditions given by the following equations**

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$\frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Which are to be satisfied at an extreme point $({x_1}^*, {x_2}^*)$.

Example (1):

Minimize $f(x,y) = x^{-1}y^{-2}$ subject to

$g(x,y) \equiv x^2 + y^2 - 9 = 0\ , x > 0, y > 0.$

Solution:

The Lagrange function is

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)\ , \lambda > 0.$$

$$\therefore L(x,y,\lambda) = x^{-1}y^{-2} + \lambda\left(x^2 + y^2 - 9\right).$$

The necessary conditions are

$$\frac{\partial L}{\partial x} = 0\ , \frac{\partial L}{\partial y} = 0\ and\ \frac{\partial L}{\partial \lambda} = 0.$$

Thus, we have

$$-x^{-2}y^{-2} + 2\lambda x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (A)$$

$$-2x^{-1}y^{-3} + 2\lambda y = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .\ (B)$$

And

$$x^2 + y^2 - 9 = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (C)$$

Now, from (A), we have $2\lambda = x^{-3}y^{-2}$ $(x \neq 0)$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots$ (D)

From (B), we have $2\lambda = 2x^{-1}y^{-4}$ $(y \neq 0)$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$ (E)

Now, from (D) and (E), we have

$$x^{-3}y^{-2} = 2x^{-1}y^{-4} \;\rightarrow\; x^{-2} = 2y^{-2} \;\rightarrow\; \frac{1}{x^2} = \frac{2}{y^2} \;\rightarrow\; y^2 = 2x^2 \quad \dots\dots \quad (F)$$

Now, from (C) and (F), we have

$$x^2 + 2x^2 - 9 = 0 \;\rightarrow 3x^2 = 9 \;\rightarrow x = \mp\sqrt{3} \;\; and \; y = \mp\sqrt{6} \;.$$

$$Since \; \lambda > 0 \,, we \; take \; x = \sqrt{3} \;\; and \; y = \sqrt{6} \;\; and \; hence \; \lambda = \frac{1}{36\sqrt{6}} \,.$$

$$\therefore (x^*, y^*) = \left(\sqrt{3}\,, \sqrt{6}\right).$$