



Optimization

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Chapter Six



Constrained Optimization

Lecture 1

1: Statement of Constrained Optimization Problem

An optimization problem or mathematical programming problem can be stated as follows:

$$\text{Find } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ which minimizes } f(X) \text{ subject to the constraints ... (1)}$$

$$g_j(X) \leq 0, j = 1, 2, \dots, m \dots\dots\dots (2)$$

And

$$h_j(X) = 0, j = m + 1, m + 2, \dots, p, \dots\dots\dots (3)$$

Where X is an n – dimensional vector called the design vector, $f(X)$ is called the objective function and $g_j(X)$ and $h_j(X)$ are, respectively, the inequality and the equality constraints.



The number of variables n and the number of constraints p need not be related in any way. Thus p could be less than, equal to or greater than n in a general mathematical problem.

In some problems, the value of p might be zero which means that there are no constraints on the problem. Such type of problems are called unconstrained optimization problems.

Those problems for which p is not equal to zero are known as constrained optimization problems.



In this section, we will consider the optimization of continuous functions subject to equality constraints:

Minimize $f(X)$ subject to

..... (4)

$g_j(X) = 0, j = 1, 2, \dots, m$ (5)

Where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Here m is less than or equal to n . Otherwise (if $m > n$) the problem becomes over defined and in general, there will be no solution.



There are several methods available for solution of this problem. One of them is the method of Lagrange multipliers which is discussed below.

2-1: Solution by the Method of Lagrange Multipliers

In the Lagrange multiplier method, we introduce one additional variable to the problem for each constraint. Thus if the original problem has n variables and m equality constraints, we add m additional variables to the problem so that the final number of unknowns are $(n + m)$.

The basic feature of the method will be initially given for a simple problem of two variables with one constraint. The extension of the method to a general problem of n variables with m constraints will be given later.



Now consider the problem

Minimize $f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$.

Define a function L , known as Lagrange function as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2), \dots\dots\dots (6)$$

Where λ is a quantity, called the Lagrange multiplier.

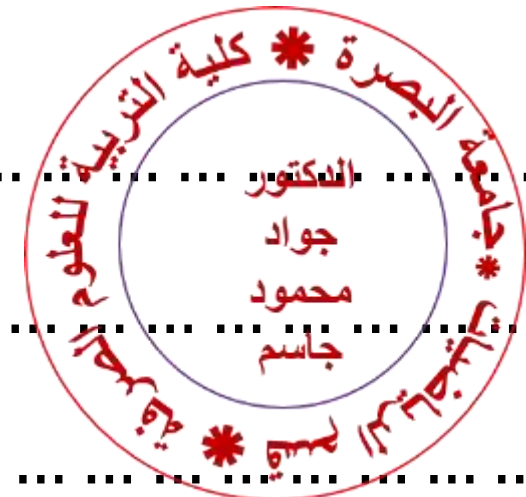
If the partial derivatives of Equation (6) with respect to x_1, x_2 and λ is set equal to zero, the necessary conditions given by the following

equations

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0 \dots\dots\dots (7)$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0 \dots\dots\dots (8)$$

$$\frac{\partial L}{\partial \lambda} = g(x_1, x_2) = 0 \dots\dots\dots (9)$$



Which are to be satisfied at an extreme point (x_1^*, x_2^*) .

Example (1):

Minimize $f(x, y) = x^{-1}y^{-2}$ subject to
 $g(x, y) \equiv x^2 + y^2 - 9 = 0, x > 0, y > 0.$

Solution:

The Lagrange function is

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y), \lambda > 0.$$

$$\therefore L(x, y, \lambda) = x^{-1}y^{-2} + \lambda(x^2 + y^2 - 9).$$

The necessary conditions are

$$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0 \text{ and } \frac{\partial L}{\partial \lambda} = 0.$$

Thus, we have

$$-x^{-2}y^{-2} + 2\lambda x = 0 \dots\dots\dots (A)$$

$$-2x^{-1}y^{-3} + 2\lambda y = 0 \dots\dots\dots (B)$$

And



$$x^2 + y^2 - 9 = 0 \dots\dots\dots (C)$$

Now, from (A), we have $2\lambda = x^{-3}y^{-2} \ (x \neq 0) \dots\dots\dots (D)$

From (B), we have $2\lambda = 2x^{-1}y^{-4} \ (y \neq 0) \dots\dots\dots (E)$

Now, from (D) and (E), we have

$$x^{-3}y^{-2} = 2x^{-1}y^{-4} \rightarrow x^{-2} = 2y^{-2} \rightarrow \frac{1}{x^2} = \frac{2}{y^2} \rightarrow y^2 = 2x^2 \dots\dots (F)$$

Now, from (C) and (F), we have

$$x^2 + 2x^2 - 9 = 0 \rightarrow 3x^2 = 9 \rightarrow x = \pm\sqrt{3} \text{ and } y = \pm\sqrt{6} .$$

Since $\lambda > 0$, we take $x = \sqrt{3}$ and $y = \sqrt{6}$ and hence $\lambda = \frac{1}{36\sqrt{6}}$.

$$\therefore (x^*, y^*) = (\sqrt{3}, \sqrt{6}).$$

