Optimization

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Chapter Five

Quasi – Newton Methods

Lecture 2

Theorem (2):

If $\{H_k\}$ is generated from Algorithm (2) then H_k is symmetric positive definite for all $k \geq 0$.

Proof:

We use the mathematical induction.

We know that H_0 is symmetric positive definite by hypothesis.

Suppose that H_k is symmetric positive definite for some $k > 0$.

Then there exists a symmetric positive definite matrix M_k

$such\ that\ H_k = {M_k}^2$.

Let $v \in R^n$ *be a nonzero vector and let* $p = M_k v$ *and* $q = M_k y_k$.

$Then\ by\ the\ formula$ (4) H_{k+1} *is symmetric since* H_k, and

$$v^T H_{k+1} v = v^T H_k v + \frac{v^T s_k s_k^T v}{s_k^T y_k} - \frac{v^T (H_k y_k)(H_k y_k)^T v}{y_k^T H_k y_k} \quad .$$

We want to prove that $v^T H_{k+1} v > 0$.

Now, we have:

$$v^T H_k y_k = v^T M_k M_k y_k = p^T q.$$

$$v^T H_k v = v^T M_k M_k v = p^T p\ .$$

$$y_k^T H_k y_k = y_k^T M_k M_k y_k = q^T q\ .$$

Therefore

$$\begin{aligned} v^T H_{k+1} v &= v^T H_k v + \frac{v^T s_k s_k^T v}{s_k^T y_k} - \frac{v^T (H_k y_k)(H_k y_k)^T v}{y_k^T H_k y_k} \\ &= p^T p + \frac{(v^T s_k)^2}{s_k^T y_k} - \frac{(p^T q)^2}{q^T q} \\ &= p^T p - \frac{\left(p^T q\right)^2}{q^T q} + \frac{(v^T s_k)^2}{s_k^T y_k} \\ &= \frac{(p^T p)(q^T q) - \left(p^T q\right)^2}{q^T q} + \frac{(v^T s_k)^2}{s_k^T y_k} \end{aligned}$$

$\therefore$**By Schwarz inequality and the definitions of** $\boldsymbol{p\ and\ q, we\ have}$

$$(p^T p)(q^T q) - (p^T q)^2 > 0,$$

$$unless\ v = \lambda y_k\ for\ some\ \lambda \neq 0, when\ v^T s_k = \lambda y_k^T s_k.$$

But

$$s_k^T y_k = s_k^T g_{k+1} - s_k^T g_k = -s_k^T g_k\ \left((because\ s_k\ perpendicular\ to\ g_{k+1})\right).$$

Since

$$s_k = \alpha_k p_k = -\alpha_k H_k g_k\ .$$

Therefore

$$s_k^T y_k = -s_k^T g_k = \alpha_k {g_k}^T H_k g_k\ .$$

Since H_k is positive definite, therefore ${g_k}^T H_k g_k > 0$.

Also, since H_k is positive definite, therefore $-H_k g_k$ is downhill direction for f at X_k so $\alpha_k > 0$.

Therefore

$s_k^T y_k > 0$, where$v^T s_k \neq 0$ if $v = \lambda y_k$.

Therefore $v^T H_{k+1} v > 0$.

$\therefore H_{k+1}$ is positive definite matrix and hence by induction H_k is symmetric positive definite matrix for all $k \geq 0$.

Lemma (1):

If M is an $n \times n$ matrix with n linearly independent eigenvectors u_i , $(i = 1, 2, \cdots, n)$ *each corresponding* to eigenvalue unity.

Then M is the $n \times n$ identity matrix I .

Proof:

Since u_i is eigenvectors for *the matrix* M.

$\therefore\ Mu_i = u_i\ , i = 1, 2, \cdots, n$.

Since $u_i\ , i = 1, 2, \cdots, n$ are linearly independent.

$\therefore$ *For* $v \in R^n$, then there exist real numbers $c_i, , i = 1, 2, \cdots, n$ such that

$v = \sum_{i=1}^{n} c_i u_i$.

$\therefore Mv = \sum_{i=1}^{n} c_i M u_i = \sum_{i=1}^{n} c_i u_i = v$.

Therefore $M = I$.