Optimization

Fourth Class

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Chapter Four

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Conjugate Direction Methods

Lecture 4

Example:

Given the objective $function\ f(X) = x_1^2 + 2x_2^2$. Compute the first two iterations for

1: $Fletcher\ Reeves\ Method$.

2: $Dixon\ Method$.

3: $Polak\ and\ Ribiere\ Method$.

Using the initial point $X_0 = [1, 1]^T$.

Solution:

1: $Fletcher\ Reeves\ method$.

First, we compute the gradient vector $g(X)$ as follows:

$$g(X) = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}]^T = [2x_1, 4x_2]^T.$$

$$\therefore g_0 = g(X_0) = [2, 4]^T.$$

$$\therefore p_0 = -g_0 = [-2, -4]^T.$$

We find $\alpha\ and\ X_1$ as follows:

$$X_1 = X_0 + \alpha p_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1-2\alpha \\ 1-4\alpha \end{bmatrix}.$$

$$\therefore f(X_1) = (1-2\alpha)^2 + 2(1-4\alpha)^2.$$

Now, we set $\frac{df(X_1)}{d\alpha} = 0 \rightarrow 2(1-2\alpha)(-2) + 4(1-4\alpha)(-4) = 0 \rightarrow$

$-4 + 8\alpha - 16 + 64\alpha = 0 \rightarrow 72\alpha = 20 \rightarrow \alpha = \frac{20}{72} = 0.2777.$

Since $\frac{d^2 f(X_1)}{d\alpha^2} = 72 > 0 \rightarrow \alpha = 0.2777$ is a minimizer of $f(X_1)$.

$$\therefore X_1 = \begin{bmatrix} 1-2\alpha \\ 1-4\alpha \end{bmatrix} = \begin{bmatrix} 0.4446 \\ -0.1108 \end{bmatrix}.$$

We compute g_1 as follows:

$$g_1 = g(X_1) = [0.8892, -0.4432]^T.$$

We compute β_0 *and* p_1 as follows:

$$\beta_0 = \frac{{g_1}^T g_1}{{g_0}^T g_0}.$$

$$g_1{}^T g_1 = [0.8892, -0.4432]\begin{bmatrix} 0.8892 \\ -0.4432 \end{bmatrix} = (0.8892)^2 + (-0.4432)^2$$

$$= 0.9871$$

$$g_0{}^T g_0 = [2, 4]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4 + 16 = 20.$$

$$\therefore \beta_0 = \frac{g_1{}^T g_1}{g_0{}^T g_0} = \frac{0.9871}{20} = 0.0494.$$

$$\therefore p_1 = -g_1 + \beta_0 p_0$$

$$= \begin{bmatrix} -0.8892 \\ 0.4432 \end{bmatrix} + 0.0494\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.8892 \\ 0.4432 \end{bmatrix} + \begin{bmatrix} -0.0988 \\ -0.1976 \end{bmatrix}$$

$$= \begin{bmatrix} -0.988 \\ 0.2456 \end{bmatrix}.$$

Now, we find X_2 as follows:

$$X_2 = X_1 + \alpha p_1 = \begin{bmatrix} 0.4446 \\ -0.1108 \end{bmatrix} + \alpha\begin{bmatrix} -0.988 \\ 0.2456 \end{bmatrix} = \begin{bmatrix} 0.4446 - 0.988\alpha \\ -0.1108 + 0.2456\alpha \end{bmatrix}.$$

$$\therefore f(X_2) = (0.4446 - 0.988\alpha)^2 + 2(-0.1108 + 0.2456\alpha)^2$$

$$\frac{df}{d\alpha} = 2(0.4446 - 0.988\alpha)(-0.988) + 4(-0.1108 + 0.2456\alpha)(0.2456)$$

$$= -0.87875296 + 1.952288\alpha - 0.10884992 + 0.201708774\alpha$$

$$= -0.98760288 + 2.15399677\alpha.$$

Set $\frac{df}{d\alpha} = 0 \rightarrow = -0.98760288 + 2.15399677\alpha = 0$

$$\therefore \alpha = \frac{0.98760288}{2.15399677} = 0.4585.$$

Since $\frac{d^2 f(X_1)}{d\alpha^2} > 0 \rightarrow \alpha = 0.4585$ is a minimizer of $f(X_2)$.

$$\therefore X_2 = X_1 + \alpha p_1 = \begin{bmatrix} 0.4446 - 0.988\alpha \\ -0.1108 + 0.2456\alpha \end{bmatrix} = \begin{bmatrix} -0.0084 \\ 0.0018 \end{bmatrix}.$$

2: *Dixon Method*

We know that the difference between ***Fletcher Reeves method*** and ***Dixon Method the value of*** $\boldsymbol{\beta_k}$.

$$\therefore \beta_0 = -\frac{{g_0}^T g_0}{{p_0}^T g_0}\ .$$

$${g_0}^T g_0 = 20 \text{ (as in the first part).}$$

$${p_0}^T g_0 = [-2, -4]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = -4 - 16 = -20.$$

$$\therefore \beta_0 = -\frac{{g_0}^T g_0}{{p_0}^T g_0} = -\frac{20}{-20} = 1.$$

$$\therefore p_1 = -g_1 + \beta_0 p_0 = \begin{bmatrix} -0.8892 \\ 0.4432 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -2.8892 \\ -3.5568 \end{bmatrix}.$$

$$\therefore X_2 = X_1 + \alpha p_1 = \begin{bmatrix} 0.4446 \\ -0.1108 \end{bmatrix} + \alpha\begin{bmatrix} -2.8892 \\ -3.5568 \end{bmatrix} = \begin{bmatrix} 0.4446 - 2.8892\alpha \\ -0.1108 - 3.5568\alpha \end{bmatrix}.$$

$$\therefore f(X_2) = (0.4446 - 2.8892\alpha)^2 + 2(-0.1108 - 3.5568\alpha)^2$$

$$\frac{df}{d\alpha} = 2(0.4446 - 2.8892\alpha)(-2.8892) + 4(-0.1108 - 3.5568\alpha)(-3.5568)$$

$$= -2.56907664 + 16.69495328\alpha + 1.57637376 + 50.60330496\alpha$$

$$= -0.99270288 + 67.29825824\alpha$$

Set $\frac{df}{d\alpha} = 0 \rightarrow -0.99270288 + 67.29825824\alpha = 0 \rightarrow \alpha = 0.0148.$

Since $\frac{d^2 f(X_1)}{d\alpha^2} > 0 \rightarrow \alpha = 0.0148$ **is a minimizer of** $f(X_2)$.

$$\therefore X_2 = \begin{bmatrix} 0.4446 - 2.8892\alpha \\ -0.1108 - 3.5568\alpha \end{bmatrix} = \begin{bmatrix} 0.4018 \\ -0.1634 \end{bmatrix}.$$

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3: *Polak and Ribiere Method*.

We know that the difference between *Fletcher Reeves Method* and *Polak and Ribiere Method the value of* β_k.

$$\therefore \beta_0 = \frac{{g_1}^T(g_1 - g_0)}{{g_0}^T g_0}.$$

$$\therefore g_1 - g_0 = \begin{bmatrix} 0.8892 \\ -0.4432 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1.1108 \\ -4.4432 \end{bmatrix}.$$

$$\begin{aligned} {g_1}^T(g_1 - g_0) &= [\,0.8892,\ -0.4432] \begin{bmatrix} -1.1108 \\ -4.4432 \end{bmatrix} \\ &= (0.8892)(-1.1108) + (-0.4432)(-4.4432) \\ &= -0.98772336 + 1.96922624 = 0.9815. \end{aligned}$$

$$\therefore \beta_0 = \frac{{g_1}^T(g_1 - g_0)}{{g_0}^T g_0} = \frac{0.9815}{20} = 0.0491.$$

$$\therefore p_1 = -g_1 + \beta_0 p_0 = \begin{bmatrix} -0.8892 \\ 0.4432 \end{bmatrix} + 0.0491 \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.9874 \\ 0.2468 \end{bmatrix}.$$

$$\therefore X_2 = X_1 + \alpha p_1 = \begin{bmatrix} 0.4446 \\ -0.1108 \end{bmatrix} + \alpha \begin{bmatrix} -0.9874 \\ 0.2468 \end{bmatrix} = \begin{bmatrix} 0.4446 - 0.9874\alpha \\ -0.1108 + 0.2468\alpha \end{bmatrix}$$

$$\therefore f(X_2) = (0.4446 - 0.9874\alpha)^2 + 2(-0.1108 + 0.2468\alpha)^2$$

$$\frac{df}{d\alpha} =$$

$$2(0.4446 - 0.9874\alpha)(-0.9874) + 4(-0.1108 + 0.2468\alpha)(0.2468\alpha)$$

$$= -0.87799608 + 1.94991752\alpha - 0.10938176 + 0.24364096\alpha$$

$$= -0.98737784 + 2.19355848\alpha\,.$$

Set $\frac{df}{d\alpha} = 0 \;\rightarrow\; \alpha = 0.4501.$

Since $\frac{d^2 f(X_1)}{d\alpha^2} > 0 \;\rightarrow\; \alpha = 0.4501$ is a minimizer of $f(X_2)$.

$$\therefore X_2 = X_1 + \alpha p_1 = \begin{bmatrix} 0.4446 \\ -0.1108 \end{bmatrix} + 0.4501 \begin{bmatrix} -0.9874 \\ 0.2468 \end{bmatrix} = \begin{bmatrix} 0.00017 \\ 0.00028 \end{bmatrix}.$$

H.W.:

Given the objective $function$

$$f(X) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 \, .$$

Compute the first two iterations for

1: $Fletcher\ Reeves\ Method$.

2: $Dixon\ Method$.

3: $Polak\ and\ Ribiere\ Method$.

Using the initial point $X_0 = [0,0]^T$.

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