*Chapter One*

*Introduction to Electrostatics*

## 1.7 Poisson and Laplace Equations

The behaviour of an electrostatic field can be described by the two differential equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{...... (1.21)}$$

$$\nabla \times \mathbf{E} = 0 \quad \text{...... (1.22)}$$

The electrostatic field is the gradient of a scalar potential:

$$\mathbf{E} = -\nabla \Phi \quad \text{...... (1.23)}$$

From Eqs. (1.21) and (l.23) we get the Poisson equation:

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \text{...... (1.24)}$$

In regions of space where there is no charge density, the scalar potential satisfies the Laplace equation:

$$\nabla^2 \Phi = 0 \quad \text{...... (1.25)}$$

From Eq. (1.14) the scalar potential is given by:

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'} \quad \text{...... (1.14)}$$

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## *H.W. (3)*

Does Eq. (1.14) satisfy Poisson's equation?

Also, we can write the formal equation:

$\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$ or, more generally

$$\boxed{\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\,\delta(\mathbf{x}-\mathbf{x}')} \qquad \text{...... (1.26)}$$

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### 1.8 Green's Theorem

In order to handle the boundary conditions, we develop some new mathematical tools using simple applications of the divergence theorem:

$$\int_V \nabla\cdot\mathbf{A}\,d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}\,da \quad \text{and} \quad \mathbf{A} = \phi\nabla\psi \qquad \text{...... (1.27)}$$

The vector field ***A*** defined in the volume *V* bounded by the closed surface *S*. Where $\phi$ *and* $\psi$ are arbitrary scalar fields. We have:

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi \qquad \text{...... (1.28)}$$

$$\phi\nabla\psi\cdot\mathbf{n} = \phi\frac{\partial\psi}{\partial n} \qquad \text{...... (1.29)}$$

Where $(\partial/\partial n)$ is the normal derivative at the surface *S* (directed outwards from inside the volume *V*). When Eqs. (1.28) and (1.29) are substituted into Eq. (1.27) (divergence theorem), we get:

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, d^3x = \oint_S \phi \frac{\partial \psi}{\partial n}\, da \qquad \text{...... (1.30)}$$

Eq. (1.30) called Green's first identity.

If we write down Eq. (1.30) again with $\phi$ *and* $\psi$ interchanged, and then subtract it from Eq. (1.30), the $(\nabla\phi \cdot \nabla\psi)$ terms cancel, and we obtain Green's second identify or Green's theorem:

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d^3x = \oint_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) da \qquad \text{...... (1.31)}$$

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## *H.W. (4)*

Convert the Poisson differential equation into an integral equation, by choosing the following particular relations:

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \quad \psi = \frac{1}{R} = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad \Rightarrow \quad \nabla^2 \psi = -4\pi\,\delta(\mathbf{x} - \mathbf{x}')$$

And find scalar potential at:

1- If the point ***x*** lies within the volume *V*.

2- If the point ***x*** lies outside the surface *S*.

3- If the surface *S* goes to infinity.

4- If $\rho(x) \acute{=} 0$ .

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