Chapter One

Introduction to Electrostatics

1.1 Introduction

In this chapter, we begin our discussion of electromagnetics with the subject of electrostatic phenomena involving time independent distributions charge and fields. Also, we introduce concepts and definitions that are important for later discussion.

- Maxwell equations in vacuum, fields and sources

The equations governing electromagnetic phenomena are the Maxwell equations:

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned} \qquad \text{........(A1)}$$

where for external sources in vacuum,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \text{ and } \mathbf{B} = \mu_0 \mathbf{H}$$

The first two equations in Eq.(A1) then become:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho / \epsilon_0 \\
\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{c^2 \partial t} &= \mu_0 \mathbf{J}
\end{aligned}$$

Implicit in the Maxwell equations is the continuity equation for charge density and current density:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$ ....(A2)

This follows from combining the time derivative of the first equation, in Eq. (A1), with the divergence of the second equation.

Also, essential for consideration of charged particle motion is the Lorentz force equation:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$ ....(A3)

which gives the force acting on a point charge (q) in the presence of electromagnetic fields.

1.2 Coulomb's Law

All of electrostatics stems from the quantitative statement of **Coulomb's law** which concerning the force acting between charged bodies at rest with respect to each other.

The **force** between two small charged bodies separated by a distance large compared to their dimensions in air:

(1) varied directly as the magnitude of each charge

(2) varied inversely as the square of the distance between them

(3) was directed along the line joining the charges

(4) was attractive if the bodies were oppositely charged and repulsive if the bodies had the same type of charge.

If ***F*** is the force on a point charge q_1. located at $\overline{x}_1$, due to another point charge q_2, located at $\overline{x}_2$, as shown in Fig. (1.1). The Coulomb's law can be written as:

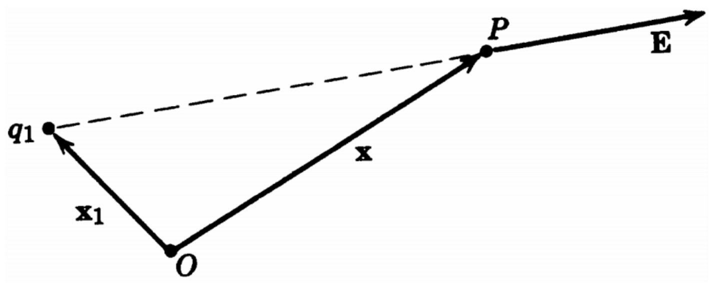

Fig. (1.1)

$$\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2) = k\, q_1 q_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \quad \text{...... (1.1)}$$

where $k = \frac{1}{4\pi\epsilon_o}$ and $\epsilon_o = 8.854 \times 10^{-12}\ (\frac{farad}{meter})$

1.3 Electric Field

The **electric field** can be defined as the force per unit charge acting at a given point. It is a vector function of position and given as:

$$\overline{\boldsymbol{F}} = q\overline{\boldsymbol{E}} \quad \text{...... (1.2)}$$

where $\overline{\boldsymbol{F}}$ is the force, $\overline{\boldsymbol{E}}$ the electric field, and q the charge.

The electric field at the point $\overline{\boldsymbol{x}}$ due to a point charge q_1 at the point $\overline{\boldsymbol{x}}_{\boldsymbol{1}}$ can be obtained directly:

$$\mathbf{E}(\mathbf{x}) = k\, q_1 \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3} \quad \text{...... (1.3)}$$

The electric field at point $\overline{\boldsymbol{x}}$ due to a system of point charges q_i, located at $\overline{\boldsymbol{x}}_{\boldsymbol{i}}, (\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{n})$ as the vector sum: (linear superposition of forces due to many charges):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3} \quad \text{...... (1.4)}$$

The sum in Eq. (4) is replaced by an integral when the charges can be described by a charge density $\rho(\bar{\acute{x}})$:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, \mathrm{d}^3 x \quad \text{...... (1.5)}$$

Where, $d^3\acute{x} = d\acute{x}\, d\acute{y}\, d\acute{z}$, is a three-dimensional volume element at $\boldsymbol{x'}$.

$$\mathrm{d}^3 x' = \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z'$$

$$\Delta q = \rho(\mathbf{x}') \Delta x \Delta y \Delta z$$

and, Δq is the charge in a small volume $\Delta x\ \Delta y\ \Delta z$ at the point $\boldsymbol{\acute{x}}$

A discrete set of point charges can be described with delta functions:

$$\rho(\mathbf{x}) = \sum_{i=1}^{n} q_i \delta(\mathbf{x} - \mathbf{x}_i) \quad \text{......(1.6)}$$

Eq. (1.6) represents a distribution of ***n*** point charges q_i, located at the points $\bar{\boldsymbol{x}}_i, (\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \boldsymbol{n})$.

H.W. (1)

Using the properties of the delta function, prove Eq. (1.4).

Hint: substitute the charge density in Eq. (1.6) into (1.5).

Dirac delta function

In one dimension, the Dirac delta function written as $\delta(x-a)$, and having the properties:

1. $\delta(x-a)=0$ for $x \neq a$ in 1d ,
2. $\int \delta(x-a)\,\mathrm{d}x=1$ if the region of integration includes $x=a$, and is zero otherwise
3. $\int f(x)\,\delta(x-a)\,\mathrm{d}x=f(a)$
4. $\int f(x)\,\delta'(x-a)\,\mathrm{d}x=-f'(a)$
5. $\delta(f(x))=\sum_i \left|\frac{\mathrm{d}f}{\mathrm{d}x}(x_i)\right|^{-1} \delta(x-x_i)$
6. $\delta(\mathbf{x}-\mathbf{X})=\delta(x_1-X_1)\,\delta(x_2-X_2)\,\delta(x_3-X_3)$ with Cartesian coordinates in 3d
7. $\int_{\Delta V} \delta(\mathbf{x}-\mathbf{X})\,\mathrm{d}^3x=\begin{cases} 1 & \text{if } \Delta V \text{ constains } \mathbf{x}=\mathbf{X} \\ 0 & \text{if } \Delta V \text{ does not constain } \mathbf{x}=\mathbf{X} \end{cases} \Rightarrow [\delta(\mathbf{x}-\mathbf{X})]=\frac{1}{V}$

1.4 Gauss's Law

Gauss's law is sometimes more useful and leads to a differential equation for electric field (***E***). Fig. (1.2) shows a point charge *q* and a closed surface *S*.

Let: ***r*** be the distance from the charge to a point on the surface,

$\hat{\boldsymbol{n}}$ be the outwardly directed unit normal to the surface at that point,

$d\bar{\boldsymbol{a}}$ be an element of surface area.

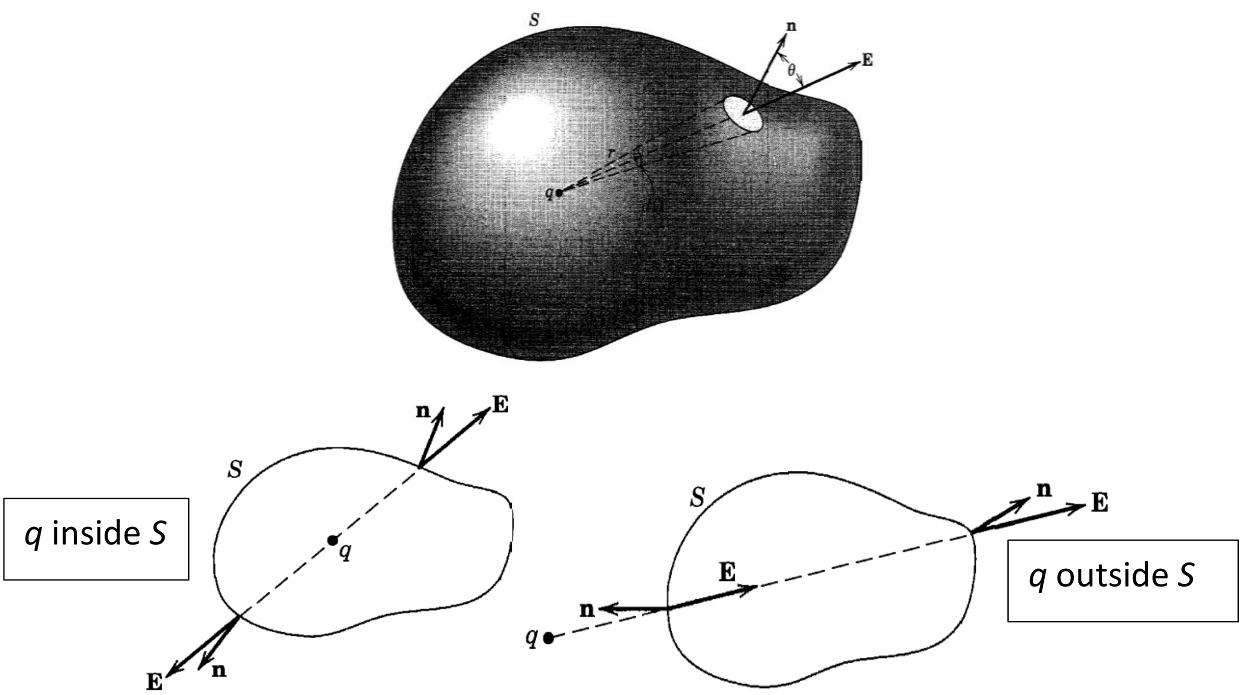

Fig. (1.2)

If the electric field ***E*** at the point on the surface due to the charge q makes an angle (θ) with the unit normal $\hat{\boldsymbol{n}}$, then the normal component of ***E*** times the area element $d\bar{\boldsymbol{a}}$ is:

$$\mathbf{E}\cdot\mathbf{n}\, \mathrm{d}a = \frac{q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\,\mathrm{d}a \qquad \text{...... (1.7)}$$

$$= \frac{q}{4\pi\epsilon_0}\,\mathrm{d}\Omega \qquad \text{...... (1.8)}$$

$$r^2 \, d\Omega = \cos\theta \, d a$$

where $d\Omega$ is the element of solid angle.

If we now integrate the normal component of ***E*** over the whole surface ***<u>for a single point charge</u>***, the result is:

$$\oint_S \mathbf{E} \cdot \mathbf{n} \, da = \begin{cases} q/\epsilon_0 & \text{if } q \text{ lies } \textit{inside } S \\ 0 & \text{if } q \text{ lies } \textit{outside } S \end{cases} \quad \text{...... (1.9)}$$

<u>For a set of charges,</u> the normal component of ***E*** over the whole surface is:

$$\oint_S \mathbf{E} \cdot \mathbf{n} \, d a = \frac{1}{\epsilon_0} \sum_i q_i \quad \text{...... (1.10)}$$

<u>For a continuous charge density,</u> the normal component of ***E*** over the whole surface is:

$$\oint_S \mathbf{E} \cdot \mathbf{n} \, d a = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x}) \, d^3 x \quad \text{...... (1.11)}$$

where $\rho(\boldsymbol{x})$ is the continuous charge density and *V* is the volume enclosed by *S*.

1.5 Differential Form of Gauss's Law

Applying the divergence theorem on Eq. (1.11), the differential form of Gauss's law is:

$$\boxed{\nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} = 0} \quad \text{...... (1.12)}$$

Where the divergence theorem is defined by:

$$\oint_S \mathbf{A}\cdot\mathbf{n}\, \mathrm{d}a = \int_V \nabla\cdot\mathbf{A}\, \mathrm{d}^3 x \quad \text{...... (1.13)}$$

1.6 Another Equation of Electrostatics and the Scalar Potential

A vector field can be specified almost completely if its divergence and curl are given everywhere in space.

$$\nabla \times \nabla\Phi = 0\,, \qquad for\ all\ \Phi$$

H.W. (2)

Derive the equation of scalar potential:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \quad \text{...... (1.14)}$$

The physical interpretation of the scalar potential can explain from Fig. (1.3). This figure shows transporting a test charge *q* from one point (A) to another point (B) in the presence of an electric field ***E(x)***,

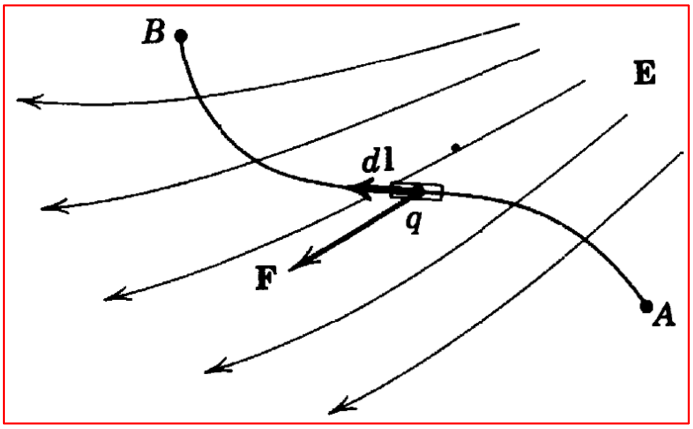

Fig. (1.3)

The work done in moving the charge from A to B is ($\overline{\boldsymbol{F}} = q\,\overline{\boldsymbol{E}}$):

$$W = -\int_A^B \mathbf{F}\cdot \mathrm{d}\,\boldsymbol{\ell} = -q\int_A^B \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} \quad \text{...... (1.15)}$$

In Eq. (1.15), the minus sign appears because that the work done on the charge against the action of the field. By using:

$$\overline{\boldsymbol{E}} = -\nabla\Phi \quad \text{...... (1.16)}$$

The work can be written:

$$= q\int_A^B \nabla\,\Phi\cdot \mathrm{d}\,\boldsymbol{\ell} = q\int_A^B \mathrm{d}\,\Phi = q\,(\Phi_B - \Phi_A) \quad \text{...... (1.17)}$$

From Eq. (1.17) the term ($q\Phi$) can be interpreted as ***the potential energy*** of the test charge in the electrostatic field. From Eqs. (1.15) and (1. 17), we have:

$$\int_A^B \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = -(\Phi_B - \Phi_A) \quad \text{...... (1.18)}$$

From Eq. (1.18), it can be seen that the line integral of the electric field between two points is independent of the path and is the negative of the potential difference between the points. If the ***path is closed***, the line integral in Eq. (1.18) becomes zero:

$$\oint \mathbf{E}\cdot \mathrm{d}\,\boldsymbol{\ell} = 0 \quad \text{...... (1.19)}$$

With the line integral of the electric field being independent of the path and the application of the Stokes's theorem on Eq.(1.19), we get:

$$\nabla\times\mathbf{E} = 0 \quad \text{...... (1.20)}$$

The Stokes's theorem is given by:

$$\oint_C \mathbf{A} \cdot \mathrm{d}\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, \mathrm{d}a$$