Optimization

Fourth Class

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Chapter Three

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The Steepest Descent and Newton Methods

Lecture 5
Solved Problems

<u>Example (1):</u>

Compute the minimizer point for the function $f(X) = x_1^2 + 2x_2^2 - 3x_1 - 2x_2$ by steepest descent method.

Take $X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the accuracy 0.001.

<u>Solution:</u>

1: Compute the gradient vector as follows:

$$g(X) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 3 \\ 4x_2 - 2 \end{bmatrix}.$$

2: Compute the gradient vector at $X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$g_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \ \|g_0\| = \sqrt{1+4} = \sqrt{5} = 2.2361 > 0.001$$

3: Compute X_1 and $f(X_1)$ as follows:

$$X_1 = X_0 - \alpha g_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-\alpha \\ 1-2\alpha \end{bmatrix}.$$

$$f(X_1) = (2-\alpha)^2 + 2(1-2\alpha)^2 - 3(2-\alpha) - 2(1-2\alpha)$$

4: Compute α as follows:

Set $\dfrac{df}{d\alpha} = 0$.

$$-2(2-\alpha) - 8(1-2\alpha) + 3 + 4 = 0 \rightarrow$$

$$18\alpha - 5 = 0 \rightarrow \alpha = \frac{5}{18}.$$

Since $\frac{d^2f}{d\alpha^2} = 18 > 0 \rightarrow \alpha = \frac{5}{18}$ is a minimizer for $f(X_1)$.

5: Compute X_1, g_1 and $\|g_1\|$ as follows:

$$X_1 = \begin{bmatrix} 2-\alpha \\ 1-2\alpha \end{bmatrix} = \begin{bmatrix} 2-\frac{5}{18} \\ 1-2\left(\frac{5}{18}\right) \end{bmatrix} = \begin{bmatrix} \frac{31}{18} \\ \frac{8}{18} \end{bmatrix}.$$

$$g_1 = g(X_1) = \begin{bmatrix} 2\left(\frac{31}{18}\right)-3 \\ 4\left(\frac{8}{18}\right)-2 \end{bmatrix} = \begin{bmatrix} \frac{8}{18} \\ \frac{-4}{18} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

$$\|g_1\| = \sqrt{\frac{16}{81}+\frac{4}{81}} = \sqrt{\frac{20}{81}} = 0.4969 > 0.001.$$

6: Compute X_2 and $f(X_2)$ as follows:

$$X_2 = X_1 - \alpha g_1 = \begin{bmatrix} \frac{31}{18} \\ \frac{8}{18} \end{bmatrix} - \alpha \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix} = \begin{bmatrix} \frac{31}{18} - \frac{4}{9}\alpha \\ \frac{8}{18} + \frac{2}{9}\alpha \end{bmatrix}$$

$$f(X_2) = \left(\frac{31}{18} - \frac{4}{9}\alpha\right)^2 + 2\left(\frac{8}{18} + \frac{2}{9}\alpha\right)^2 - 3\left(\frac{31}{18} - \frac{4}{9}\alpha\right) - 2\left(\frac{8}{18} + \frac{2}{9}\alpha\right)$$

7: Compute α as follows:

Set $\frac{df}{d\alpha} = 0$,

$$2\left(-\frac{4}{9}\right)\left(\frac{31}{18} - \frac{4}{9}\alpha\right) + 4\left(\frac{2}{9}\right)\left(\frac{8}{18} + \frac{2}{9}\alpha\right) + \frac{12}{9} - \frac{4}{9} = 0 \rightarrow$$

$$\rightarrow \frac{48}{81}\alpha - \frac{20}{81} = 0 \;\rightarrow \frac{48}{81}\alpha = \frac{20}{81} \;\rightarrow \alpha = \frac{20}{48} = \frac{5}{12}\,.$$

Since $\dfrac{d^2f}{d\alpha^2} = \dfrac{48}{81} > 0 \;\rightarrow \alpha = \dfrac{5}{12}$ **is a minimizer for** $f(X_2)$.

8: Compute X_2, g_2 **and** $\|g_2\|$ **as follows:**

$$X_2 = \begin{bmatrix} \frac{31}{18} - \frac{4}{9}\alpha \\ \frac{8}{18} + \frac{2}{9}\alpha \end{bmatrix} = \begin{bmatrix} \frac{31}{18} - \frac{4}{9}\left(\frac{5}{12}\right) \\ \frac{8}{18} + \frac{2}{9}\left(\frac{5}{12}\right) \end{bmatrix} = \begin{bmatrix} \frac{83}{54} \\ \frac{27}{54} \end{bmatrix} = \begin{bmatrix} \frac{83}{54} \\ \frac{1}{2} \end{bmatrix},$$

$$g_2 = \begin{bmatrix} 2\left(\frac{83}{54}\right) - 3 \\ 4\left(\frac{1}{2}\right) - 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{54} \\ 0 \end{bmatrix}, \|g_2\| = \frac{4}{54} = 0.074 > 0.001$$

9: Compute X_3 and $f(X_3)$ as follows:

$$X_3 = X_2 - \alpha g_2 = \begin{bmatrix} \frac{83}{54} \\ \frac{1}{2} \end{bmatrix} - \alpha \begin{bmatrix} \frac{4}{54} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{83}{54} - \frac{4}{54}\alpha \\ \frac{1}{2} \end{bmatrix},$$

$$f(X_3) = \left(\frac{83}{54} - \frac{4}{54}\alpha\right)^2 + 2\left(\frac{1}{2}\right)^2 - 3\left(\frac{83}{54} - \frac{4}{54}\alpha\right) - 2\left(\frac{1}{2}\right)$$

10: Compute α as follows:

Set $\frac{df}{d\alpha} = 0$,

$$2\left(-\frac{4}{54}\right)\left(\frac{83}{54} - \frac{4}{54}\alpha\right) + \frac{12}{54} = 0 \rightarrow -8\left(\frac{83}{54} - \frac{4}{54}\alpha\right) + 12 = 0$$

$$-2\left(\frac{83}{54} - \frac{4}{54}\alpha\right) + 3 = 0 \rightarrow \frac{8}{54}\alpha - \frac{4}{54} = 0 \rightarrow \alpha = \frac{1}{2}.$$

Since $\frac{d^2 f}{d\alpha^2} = \frac{8}{54} > 0 \;\rightarrow \alpha = \frac{1}{2}$ **is a minimizer for** $f(X_3)$.

11: Compute X_3, g_3 **and** $\|g_3\|$ **as follows:**

$$X_3 = \begin{bmatrix} \frac{83}{54} - \frac{4}{54}\alpha \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{83}{54} - \frac{4}{54}\left(\frac{1}{2}\right) \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{81}{54} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 2\left(\frac{3}{2}\right) - 3 \\ 4\left(\frac{1}{2}\right) - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \|g_3\| = 0 < 0.001\,.$$

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Then the minimizer of the given function is $X_3 = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$.

Example (2):

Compute the minimizer point for the function $f(X)=x_1^2+2x_2^2-3x_1-2x_2$ by Newton method with line search. Take $X_0=\begin{bmatrix}2\\1\end{bmatrix}$, and the accuracy 0.001.

Solution:

1: Compute the gradient vector of the function as follows:

$$g(X)=\begin{bmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\end{bmatrix}=\begin{bmatrix}2x_1-3\\ 4x_2-2\end{bmatrix}.$$

Then $g_0=g(X_0)=\begin{bmatrix}1\\2\end{bmatrix}$.

2: Compute the Hessian matrix as follows:

$$G(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

3: Compute $G^{-1}(X) = \frac{1}{8}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$

4: Compute the direction s_0 **as follows:**

$$s_0 = -G^{-1} g_0 = -\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

5: Compute α as follows:

$$X_1 = X_0 + \alpha s_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{2}\alpha \\ 1 - \frac{1}{2}\alpha \end{bmatrix},$$

$$f(X_1) = (2 - \tfrac{1}{2}\alpha)^2 + 2(1 - \tfrac{1}{2}\alpha)^2 - 3\left(2 - \tfrac{1}{2}\alpha\right) - 2\left(1 - \tfrac{1}{2}\alpha\right)$$

Set $\frac{df}{d\alpha} = 0 \;\rightarrow\; -\left(2 - \frac{1}{2}\alpha\right) - 2\left(1 - \frac{1}{2}\alpha\right) + \frac{3}{2} + 1 = 0 \;\rightarrow$

$$\frac{1}{2}\alpha + \alpha - \frac{3}{2} = 0 \;\rightarrow \frac{3}{2}\alpha - \frac{3}{2} = 0 \;\rightarrow \alpha = 1.$$

Since $\frac{d^2 f}{d\alpha^2} = \frac{3}{2} > 0 \;\rightarrow \alpha = 1$ **is the minimizer of** $f(X_1)$.

6: Compute X_1, g_1 and $\|g_1\|$ $as\ follows$:

$$X_1 = \begin{bmatrix} 2 - \frac{1}{2}\alpha \\ 1 - \frac{1}{2}\alpha \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \|g_1\| = 0 < 0.001.$$

Then the minimizer point of the given function is $\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$.

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