Properties of Logarithmic Functions :- خواص الدوال اللوغاريتمية

اذا كانت $b>0$ ، $b \neq 1$ ، $a>0$ ، $c>0$

و $r$ اي عدد حقيقي ، فان

① $\log_b(ac) = \log_b(a) + \log_b(c)$

② $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$

③ $\log_b a^r = r \log_b a$

④ $\log_b(1) = 0$

⑤ $\log_b x$ هنا يكون غير معرف عندما $x<0$

⑥ $\log_b b = 1$

⑦ $\log_b\left(\frac{1}{c}\right) = -\log_b(c)$

⑧ $\ln(e^x) = x$ لكل قيم $x$

⑨ $e^{\ln(x)} = x$

⑩ $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

⑪ $\log_b b^x = x$ لكل قيم $x$

مثال : جد $\operatorname{Log} \dfrac{x y^5}{\sqrt{z}}$

ملاحظة : عندما لا يذكر الاساس (b) هذا يعني انه اللوغارتم عشري اي ان $b = 10$

$$\operatorname{Log} \frac{x y^5}{\sqrt{z}} = \operatorname{Log}(x y^5) - \operatorname{Log}(\sqrt{z})$$

$$= \operatorname{Log} x + \operatorname{Log} y^5 - \operatorname{Log}(z^{\frac{1}{2}})$$

$$= \operatorname{Log} x + 5 \operatorname{Log} y - \frac{1}{2} \operatorname{Log} z$$

مثال :- جد $\frac{1}{3} \ln(x) - \ln(x^2-1) + 2\ln(x+3)$

الحل

$$\frac{1}{3} \ln(x) - \ln(x^2-1) + 2\ln(x+3)$$

$$= \ln(x)^{\frac{1}{3}} - \ln(x^2-1) + \ln(x+3)^2$$

$$= \ln(x)^{\frac{1}{3}} + \ln(x+3)^2 - \ln(x^2-1)$$

$$= \ln\left((x)^{\frac{1}{3}}(x+3)^2\right) - \ln(x^2-1)$$

$$= \ln\left(\frac{\sqrt[3]{x}\,(x+3)^2}{x^2-1}\right)$$

مثال : جد قيمة $x$ لكل مما يأتي :

① $\text{Log}\, x = 2$

الحل

$$\text{Log}\, x = 2 \Rightarrow \text{Log}\, x = \frac{\ln(x)}{\ln(10)} = 2 \Rightarrow$$

$$\Rightarrow \ln(x) = 2\ln(10) \Rightarrow \ln(x) = \ln(10)^2 \Rightarrow$$

$$\Rightarrow \ln(x) = \ln(100)$$ بأخذ exp للطرفين

$$\Rightarrow e^{\ln(x)} = e^{\ln(100)}$$

$$= \boxed{x = 100}$$

② $\ln(x+1) = 5$

بأخذ exp للطرفين

$$= e^{\ln(x+1)} = e^{5}$$

$$= x+1 = e^5 \Rightarrow \boxed{x = e^5 - 1}$$

③ $5^x = 7$

بأخذ ln للطرفين

$$\Rightarrow \ln(5^x) = \ln(7)$$

$$\Rightarrow x\ln(5) = \ln(7)$$

$$x = \frac{\ln(7)}{\ln(5)}$$

④ $\dfrac{e^{x} - e^{-x}}{2} = 1$

الحل $= e^{x} - e^{-x} = 2$

$= e^{x} - \dfrac{1}{e^{x}} = 2$

$= \dfrac{e^{2x} - 1}{e^{x}} = 2$

$\Rightarrow e^{2x} - 1 = 2e^{x} \Rightarrow e^{2x} - 2e^{x} - 1 = 0$ $\boxed{\pm 1}$

$\Rightarrow e^{2x} - 2e^{x} - 1 - 1 + 1 = 0$

$e^{2x} - 2e^{x} + 1 - 2 = 0$

$(e^{x} - 1)(e^{x} - 1) - 2 = 0$

$\therefore (e^{x} - 1)^{2} = 2$ بالجذر

$e^{x} - 1 = \pm\sqrt{2} \Rightarrow e^{x} = \pm\sqrt{2} + 1$ بأخذ $\ln$

$\ln e^{x} = \ln(\sqrt{2} + 1)$ + فقط

$\boxed{\therefore x = \ln(1 + \sqrt{2})}$

⑤ $x^{\log x} = 100x$ H.W

# Hyperbolic Functions الدوال الزائدية :

① $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$ where $D_f = R$ , $R_f = R$

② $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$ where $D_f = R$ , $R_f = [1, \infty)$

③ $\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \dfrac{\sinh(x)}{\cosh(x)}$ wher $D_f = R$ , $R_f = (-1, 1)$

④ $\coth(x) = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \dfrac{1}{\tanh(x)} = \dfrac{\cosh(x)}{\sinh(x)}$

where $D_f = R / \{0\}$

$R_f = R / (-1, 1)$

⑤ $\mathrm{sech}(x) = \dfrac{2}{e^{x} + e^{-x}} = \dfrac{1}{\cosh(x)}$ where $D_f = R$

$R_f = (0, 1]$

⑥ $\mathrm{csch}(x) = \dfrac{2}{e^{x} - e^{-x}} = \dfrac{1}{\sinh(x)}$ where $D_f = R / \{0\}$

$R_f = R / \{0\}$

⑦ $\cosh^2(x) - \sinh^2(x) = 1$

⑦ مبرهنة

$$\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2 = 1$$

$$\frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$$

$$\frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = 1$$

$$= \frac{4}{4} = 1$$

8) $1-\tanh^2(x) = \operatorname{sech}^2(x)$

9) $\coth^2(x) - 1 = \operatorname{csch}^2(x)$

ملاحظات حول الدوال الزائدية :-

دالة فردية 1) $\sinh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2} = \dfrac{-(e^{x} - e^{-x})}{2} = -\sinh(x)$

دالة زوجية 2) $\cosh(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \dfrac{e^{x} + e^{-x}}{2} = \cosh(x)$

3) $\tanh(-x) = -\tanh(x)$ دالة فردية

4) $\coth(-x) = -\coth(x)$ دالة فردية

5) $\operatorname{sech}(-x) = \operatorname{sech}(x)$ دالة زوجية

6) $\operatorname{csch}(-x) = -\operatorname{csch}(x)$ دالة فردية

# Properties of Hyperbolic Functions

# خواص الدوال الزائدية :

1. $\sinh(x \mp y) = \sinh(x)\cosh(y) \mp \sinh(y)\cosh(x)$

2. $\cosh(x \mp y) = \cosh(x)\cosh(y) \mp \sinh(x)\sinh(y)$

3. $\tanh(x \mp y) = \dfrac{\tanh(x) \mp \tanh(y)}{1 \mp \tanh(x)\tanh(y)}$

4. $\sinh(2x) = 2\sinh(x)\cosh(x)$

5. $\cosh(2x) = \sinh^2(x) + \cosh^2(x)$

وباستفادة من المتطابقة $\cosh^2(x) - \sinh^2(x) = 1$ نجد

$\cosh(2x) = 2\sinh^2(x) + 1$

or $= 2\cosh^2(x) - 1$

6. $\sinh^2(x) = \dfrac{\cosh(2x) - 1}{2}$

7. $\cosh^2(x) = \dfrac{\cosh(2x) + 1}{2}$

مثال :- اذا كانت $\cosh(x) = 5$ ، أن $x > 0$ ، جد

$\sinh(x)$ , $\tanh(x)$ , $\coth(x)$ , $\operatorname{sech}(x)$ and $\operatorname{csch}(x)$

الحل :

$\because \cosh^2(x) - \sinh^2(x) = 1$

$(5)^2 - \sinh^2(x) = 1$

$\therefore \sinh^2(x) = (5)^2 - 1$

$\sinh^2(x) = 25 - 1$

$\sinh^2(x) = 24$ بالجذر

$\therefore \boxed{\sinh(x) = \sqrt{24}}$

$\because \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{\sqrt{24}}{5}$

$\because \coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{5}{\sqrt{24}}$

$\because \operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{1}{5}$

$\because \operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{1}{\sqrt{24}}$

مثال :- اثبت أن

$\cosh(x) + \sinh(x) = e^x$

الحل :

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{e^x + \cancel{e^{-x}} + e^x - \cancel{e^{-x}}}{2}$$

$$\Rightarrow \frac{\cancel{2}e^x}{\cancel{2}} = e^x$$

مثال :- اثبت ان $\cosh(x) - \sinh(x) = e^{-x}$ H.W

مثال : اثبت أن $\tanh\left(\frac{1}{2}\ln(x)\right) = \frac{x-1}{x+1}$

الحل :

$$\tanh\left(\frac{1}{2}\ln(x)\right) = \frac{e^{\frac{1}{2}\ln(x)} - e^{-\frac{1}{2}\ln(x)}}{e^{\frac{1}{2}\ln(x)} + e^{-\frac{1}{2}\ln(x)}}$$

$$= \frac{e^{\ln(x)^{\frac{1}{2}}} - e^{\ln(x)^{-\frac{1}{2}}}}{e^{\ln(x)^{\frac{1}{2}}} + e^{\ln(x)^{-\frac{1}{2}}}}$$

$$= \frac{e^{\ln(\sqrt{x})} - e^{\ln\left(\frac{1}{\sqrt{x}}\right)}}{e^{\ln(\sqrt{x})} + e^{\ln\left(\frac{1}{\sqrt{x}}\right)}}$$

$$= \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$= \frac{\frac{x-1}{\cancel{\sqrt{x}}}}{\frac{x+1}{\cancel{\sqrt{x}}}}$$

$$= \frac{x-1}{x+1}$$

مثال: اثبت أن $\tanh(x+y) = \dfrac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$

الحل:- نأخذ الطرف الأيمن

$$\frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)} = \frac{\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}} + \dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}}{1+\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\cdot\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}}$$

$$\Rightarrow \frac{\dfrac{(e^{x}-e^{-x})(e^{y}+e^{-y})+(e^{y}-e^{-y})(e^{x}+e^{-x})}{\cancel{(e^{x}+e^{-x})(e^{y}+e^{-y})}}}{\dfrac{(e^{x}+e^{-x})(e^{y}+e^{-y})+(e^{x}-e^{-x})(e^{y}-e^{-y})}{\cancel{(e^{x}+e^{-x})(e^{y}+e^{-y})}}}$$

$$\frac{e^{(x+y)}\cancel{+e^{(x-y)}}\cancel{-e^{y-x}}-e^{-(x+y)}+e^{(x+y)}\cancel{+e^{(y-x)}}\cancel{-e^{(x-y)}}-e^{-(x+y)}}{e^{(x+y)}\cancel{+e^{x-y}}\cancel{+e^{y-x}}+e^{-(x+y)}+e^{x+y}\cancel{-e^{x-y}}\cancel{-e^{y-x}}+e^{-(x+y)}}$$

$$= \frac{2e^{(x+y)}-2e^{-(x+y)}}{2e^{(x+y)}+2e^{-(x+y)}}$$

$$= \frac{e^{(x+y)}-e^{-(x+y)}}{e^{(x+y)}+e^{-(x+y)}} = \tanh(x+y)$$

مثال :- اثبت أن

$$\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^{2}(x)}$$

H.W.

## Inverse of Hyperbolic Functions: معكوس الدوال الزائدية :

① If $y = \sinh(x) \Rightarrow x = \sinh^{-1}(y)$ حيث $D_f = R$ , $R_f = R$

② If $y = \cosh(x) \Rightarrow x = \cosh^{-1}(y)$ حيث $D_f = [1, \infty]$ , $R_f = [0, \infty)$

③ If $y = \tanh(x) \Rightarrow x = \tanh^{-1}(y)$ حيث $D_f = [-1, 1)$ , $R_f = R$

④ If $y = \coth(x) \Rightarrow x = \coth^{-1}(y)$ حيث $D_f = R / [-1, 2]$ , $R_f = R / \{0\}$

⑤ If $y = \operatorname{sech}(x) \Rightarrow x = \operatorname{sech}^{-1}(y)$ حيث $D_f = (0, 1]$ , $R_f = R$

⑥ If $y = \operatorname{csch}(x) \Rightarrow x = \operatorname{csch}^{-1}(y)$ حيث $D_f = R / \{0\}$ , $R_f = R / \{0\}$

## Relations Between Functions * العلاقة بين الدوال :

① $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

② $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$

③ $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$ , $|x| < 1$

④ $\coth^{-1}(x) = \frac{1}{2} \ln\left(\frac{x + 1}{x - 1}\right)$ , $|x| > 1$

⑤ $\operatorname{sech}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), \ 0 < x \leq 1$

⑥ $\operatorname{csch}^{-1}(x) = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right), \ \forall x \in \mathbb{R}/\{0\}$

* برهن أن $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

الحل:

$y = \sinh^{-1}(x)$ نفرض أن

$\therefore x = \sinh(y) \Rightarrow x = \dfrac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - e^{-y}$

$\Rightarrow e^y - 2x - e^{-y} = 0 \qquad * e^y$

$\therefore e^{2y} - 2xe^y - 1 = 0$ هذه المعادلة يمكن ان تحل بالدستور

$e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = \dfrac{\cancel{2}(x \pm \sqrt{x^2+1})}{\cancel{2}}$

$\therefore e^y = x \pm \sqrt{x^2+1}$

$\because e^y > 0 \Rightarrow e^y = x + \sqrt{x^2+1})$ بأخذ ln للطرفين

$\ln e^y = \ln(x + \sqrt{x^2+1})$

$\therefore y = \ln(x + \sqrt{x^2+1})$

$\therefore \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

* برهن أن $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

الحل

نفرض ان

$$y = \tanh^{-1}(x) \Rightarrow x = \tanh(y) \Rightarrow x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

$$\Rightarrow e^{y} - e^{-y} = x e^{y} + x e^{-y}$$

$$\Rightarrow e^{y} - e^{-y} - x e^{y} - x e^{-y} = 0$$

$$\Rightarrow (1-x)e^{y} - (1+x)e^{-y} = 0 \qquad * \; e^{y}$$

$$\Rightarrow (1-x)e^{2y} - (1+x) = 0$$

$$\therefore e^{2y} = \frac{1+x}{1-x}$$ بأخذ ln للطرفين

$$\ln e^{2y} = \ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

برهن أن

$$\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$\boxed{\text{H.W}}$