Optimization

Fourth Class

2020 - 2021

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Chapter Two

Line Search

Lecture 8

Second: Cubic Interpolation Method:

The cubic interpolation method approximates the objective function $\Phi(\alpha)$ by $a\ cubic\ polynomial\ P(\alpha)$, four interpolation conditions are required. For example, we may use function values at four points or function values at three points and a derivative at one point or function values and derivative values at two points. Note that in general, the cubic interpolation has better convergence than the quadratic interpolation but that it needs computing of derivatives and more expensive computation. In the following discuss the cubic interpolation method with two points.

$We\ are\ given\ two\ points\ a\ and\ b, the\ function\ values\ \Phi(a)$ $and\ \Phi(b), and\ their\ derivatives\ \Phi'(a)\ and\ \Phi'(b)$ $to\ be\ construct\ a\ cubic\ polynomial\ of\ the\ form$

$$P(\alpha) = c_1(\alpha - a)^3 + c_2(\alpha - a)^2 + c_3(\alpha - a) + c_4 \quad \dots \dots \dots \dots \dots \quad (57)$$

Where $c_1,\ c_2, c_3$ *and* c_4 are the coefficients of the polynomial which are chosen such that

$$P(a) = c_4 = \Phi(a) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (58)$$

$$P'(a) = c_3 = \Phi'(a) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (59)$$

$$P(b) = c_1(b - a)^3 + c_2(b - a)^2 + c_3(b - a) + c_4 = \Phi(b) \dots \dots \dots \quad (60)$$

$$P'(b) = 3c_1(b - a)^2 + 2c_2(b - a) + c_3 = \Phi'(b) \dots \dots \dots \dots \dots \dots \dots \quad (61)$$

From the sufficient condition of the minimizer, we have

$$P'(\alpha) = 3c_1(\alpha - a)^2 + 2c_2(\alpha - a) + c_3 = 0 \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (62)$$

And

$$P''(\alpha) = 6c_1(\alpha - a) + 2c_2 > 0 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (63)$$

Solving Equation (62), yields

$$3c_1(\alpha - a)^2 + 2c_2(\alpha - a) + c_3 = 0$$

$$\alpha - a = \frac{-2c_2 \mp \sqrt{4c_2^2 - 12c_1c_3}}{6c_1} = \frac{-2c_2 \mp 2\sqrt{c_2^2 - 3c_1c_3}}{6c_1} = \frac{-c_2 \mp \sqrt{c_2^2 - 3c_1c_3}}{3c_1}$$

$$, c_1 \neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (64)$$

In order to guarantee the condition (63) holding, we only take

$$\alpha - a = \frac{-c_2 + \sqrt{c_2^2 - 3c_1c_3}}{3c_1}, c_1 \neq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (65)$$

The formula (65) can be written as

$$\alpha - a = \frac{-c_2 + \sqrt{c_2^2 - 3c_1c_3}}{3c_1} = \left(\frac{-c_2 + \sqrt{c_2^2 - 3c_1c_3}}{3c_1}\right)\left(\frac{-c_2 - \sqrt{c_2^2 - 3c_1c_3}}{-c_2 - \sqrt{c_2^2 - 3c_1c_3}}\right)$$

$$= \frac{c_2^2-(c_2^2-3c_1c_3)}{3c_1\left(-c_2-\sqrt{c_2^2-3c_1c_3}\,\right)} = \frac{3c_1c_3}{3c_1\left(-c_2-\sqrt{c_2^2-3c_1c_3}\,\right)} = \frac{c_3}{-c_2-\sqrt{c_2^2-3c_1c_3}}$$

$$\therefore \alpha - a = -\frac{c_3}{c_2+\sqrt{c_2^2-3c_1c_3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (66)$$

Let $\overline{\alpha}$ be the minimizer of $P(\alpha)$, then

$$\overline{\alpha} = a - \frac{c_3}{c_2+\sqrt{c_2^2-3c_1c_3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (67)$$

The minimizer in Equation (67) *is represented by* c_1, c_2 *and* c_3. We hope to represent $\overline{\alpha}$ by $\Phi(a), \Phi'(a), \Phi(b)$ *and* $\Phi'(b)$

Let

$$\boldsymbol{s = 3\,\frac{\Phi(b)-\Phi(a)}{b-a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(68)}$$

$$\boldsymbol{z = s - \Phi'(a) - \Phi'(b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(69)}$$

$$\boldsymbol{w^2 = z^2 - \Phi'(a)\Phi'(b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(70)}$$

Now, from Equations (58) and (60), we have

$$\boldsymbol{s = 3\left[\frac{c_1(b-a)^3+c_2(b-a)^2+c_3(b-a)+c_4-c_4}{b-a}\right]}$$

$$\boldsymbol{s = 3c_1(b-a)^2+3c_2(b-a)+3c_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(71)}$$

Now, from Equations (59), (61) and (71), we have

$$\boldsymbol{z = 3c_1(b-a)^2+3c_2(b-a)+3c_3 - c_3 - 3c_1(b-a)^2}$$

$$\boldsymbol{-2c_2(b-a) - c_3}$$

$$\boldsymbol{z = c_2(b-a) + c_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(72)}$$

Now, from Equations (59), (61) and (72), we have

$$\boldsymbol{w^2 = [c_2(b-a)+c_3\ \ ]^2 - c_3[3c_1(b-a)^2 + 2c_2(b-a)+c_3]}$$

$$\boldsymbol{= c_2^2(b-a)^2 + 2c_2c_3(b-a) + c_3^2 - 3c_1c_3(b-a)^2}$$

$$\boldsymbol{-2c_2c_3(b-a) - c_3^2}$$

$$\boldsymbol{= c_2^2(b-a)^2 - 3c_1c_3(b-a)^2}$$

$$\boldsymbol{w^2 = (b-a)^2\left(c_2^2 - 3c_1c_3\right)} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \text{(73)}$$

Hence, we have

$$\boldsymbol{c_3 = \Phi'(a),\ c_2^2 - 3c_1c_3 = \frac{w^2}{(b-a)^2}\ ,\ c_2 = \frac{z-c_3}{b-a} = \frac{z-\Phi'(a)}{b-a}.}$$

Use above values in Equation (67), yields

$$\boldsymbol{\bar{\alpha} = a - \frac{c_3}{c_2 + \sqrt{c_2^2 - 3c_1c_3}} = a - \frac{\Phi'(a)}{\frac{z-\Phi'(a)}{b-a} + \frac{w}{b-a}} = a - \frac{\Phi'(a)}{\frac{z-\Phi'(a)+w}{b-a}} = a - \frac{(b-a)\Phi'(a)}{z-\Phi'(a)+w}}$$

… (74)

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Now, multiply and divide above $formula\ by\ \Phi'(b)$, and use Equation (70), yields:

$$\bar{\alpha} = a - \frac{(b-a)\Phi'(a)\Phi'(b)}{[z+w-\Phi'(a)]\Phi'(b)} = a - \frac{(b-a)\left(z^2-w^2\right)}{(z+w)\Phi'(b)-\left(z^2-w^2\right)}$$

$$= a - \frac{(b-a)(z-w)(z+w)}{(z+w)\Phi'(b)-(z-w)(z+w)}$$

$$\bar{\alpha} = a - \frac{(b-a)(z-w)(z+w)}{(z+w)[\Phi'(b)-z+w]} = a + \frac{(b-a)(w-z)}{\Phi'(b)+w-z} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (75)$$

Unfortunately the formula (75) is not adequate for calculating $\bar{\alpha}$ because its denominator is possibly zero or merely very small. Fortunately it can be overcome by use of Equations (74) and (75).

Now, from Equations (74) and (75) and we use the fact that

If $\quad \frac{a}{b} = \frac{c}{d} \quad \rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

$$\left(\left(for\ example\ \frac{2}{5} = \frac{6}{15}, then\ \frac{2+6}{5+15} = \frac{8}{20} = \frac{2}{5}\right)\right).$$

Then, we have

$$-\frac{(b-a)\Phi'(a)}{z-\Phi'(a)+w}=\frac{(b-a)(w-z)}{\Phi'(b)+w-z}=\frac{-(b-a)\Phi'(a)+(b-a)(w-z)}{z-\Phi'(a)+w+\Phi'(b)+w-z}=\frac{(b-a)(-\Phi'(a)+w-z)}{\Phi'(b)-\Phi'(a)+2w}$$

$$=(b-a)\left[\frac{-\Phi'(a)+w-z}{\Phi'(b)-\Phi'(a)+2w}\right]$$

$$\therefore\quad \bar{\alpha}-a=(b-a)\left[\frac{-\Phi'(a)+w-z}{\Phi'(b)-\Phi'(a)+2w}\right]$$

Or

$$\bar{\alpha}=a+(b-a)\left[\frac{-\Phi'(a)+w-z}{\Phi'(b)-\Phi'(a)+2w}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (76)$$

$Now\ if\ we\ take\ \Phi'(a)<0\ \ and\ \ \Phi'(b)>0\ then$
$w^2=z^2-\Phi'(a)\Phi'(b)\ >0$. **Taking** $w>0$ **it follows that the denominator in Equation (76):** $\Phi'(b)-\Phi'(a)+2w\ >0$.

Equation (76) may be written as:

$$\bar{\alpha} = a + (b-a)\left[\frac{-\Phi'(a) + w - z}{\Phi'(b) - \Phi'(a) + 2w}\right]$$

$$= \ a + (b-a)\left[\frac{\Phi'(b) - \Phi'(b) - \Phi'(a) + w + w - w - z}{\Phi'(b) - \Phi'(a) + 2w}\right]$$

$$= \ a + (b-a)\left[\frac{\Phi'(b) - \Phi'(a) + 2w}{\Phi'(b) - \Phi'(a) + 2w} - \frac{\Phi'(b) + w + z}{\Phi'(b) - \Phi'(a) + 2w}\right]$$

$$= \ a + (b-a)\left[1 - \frac{\Phi'(b) + w + z}{\Phi'(b) - \Phi'(a) + 2w}\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (77)$$

<u>Note (18):</u>

The cubic interpolation method with two points has converge rate with order two.