Optimization

Fourth Class

2020 - 2021

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Chapter Two

Line Search

Lecture 6

II: Quadratic Interpolation Method with Two points:

Given two points α_1 *and* α_2 and one function value $\Phi(\alpha_1)$ $(or\ \Phi(\alpha_2))$ and two derivative values $\Phi'(\alpha_1)$ *and* $\Phi'(\alpha_2)$.

Construct the quadratic interpolation function with the following conditions:

$$q(\alpha_1) = a{\alpha_1}^2 + b\alpha_1 + c = \Phi(\alpha_1) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (36)$$

$$q'(\alpha_1) = 2a\alpha_1 + b = \Phi'(\alpha_1) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (37)$$

$$q'(\alpha_2) = 2a\alpha_2 + b = \Phi'(\alpha_2) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (38)$$

Write $\Phi_1 = \Phi(\alpha_1)$, $\Phi'_1 = \Phi'(\alpha_1)$ *and* $\Phi'_2 = \Phi'(\alpha_2)$.

Then above equations (36-38) becomes:

$$a{\alpha_1}^2 + b\alpha_1 + c = \Phi_1 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (39)$$

$$2a\alpha_1 + b = \Phi'_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (40)$$

$$2a\alpha_2 + b = \Phi'_2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (41)$$

We want to find the coefficients a *and* b.

Subtracting Equations (40) and (41), we have

$$\therefore a = \frac{\Phi_1' - \Phi_2'}{2(\alpha_1 - \alpha_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

From equations (40), we have

$$b = \Phi_1' - 2a\alpha_1 = \Phi_1' - \frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}\alpha_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

Let $\bar{\alpha}$ be the minimizer of the quadratic function $\Phi(\alpha) = a\alpha^2 + b\alpha + c$.

$$\therefore \Phi'(\bar{\alpha}) = 0 \ \rightarrow 2a\bar{\alpha} + b = 0 \ \rightarrow \ \bar{\alpha} = -\frac{b}{2a}\ .$$

$$\therefore \bar{\alpha} = -\frac{\Phi_1' - \frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}\alpha_1}{\frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}} = -\left[\frac{\Phi_1'}{\frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}} - \frac{\frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}\alpha_1}{\frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}}\right] = \alpha_1 - \frac{\Phi_1'}{\frac{\Phi_1' - \Phi_2'}{(\alpha_1 - \alpha_2)}}\ .$$

$$\therefore \bar{\alpha} = \alpha_1 - \frac{\Phi_1'(\alpha_1 - \alpha_2)}{\Phi_1' - \Phi_2'} = \alpha_1 - \frac{\alpha_1 - \alpha_2}{\Phi_1' - \Phi_2'}\Phi_1' \quad \ldots\ldots\ldots\ldots (44)$$

Therefore, the iteration scheme is

$$\alpha_{k+1} = \alpha_k - \frac{\alpha_k - \alpha_{k-1}}{\Phi'_k - \Phi'_{k-1}}\Phi'_k \quad \ldots\ldots\ldots\ldots (45)$$

<u>Note (15):</u>

We call the quadratic interpolation method (I) the quadratic interpolation formula, and the quadratic interpolation method (II) the secant formula.

<u>Theorem (7):</u>

1: Let $\Phi: R \to R$ be three times continuously differentiable function.

2: Let α^* be such that $\Phi'(\alpha^*) = 0$ and $\Phi''(\alpha^*) \neq 0$.

Then the sequence $\{\alpha_k\}$ generated by Equation (45) converges to α^* with order $\frac{1+\sqrt{5}}{2} \cong 1.618$ of converge rate.

Note (16):

Theorem (7) tells us that the secant method has super linear convergence.

III: Quadratic Interpolation Method with Three Points

Given three distinct points $\alpha_1 , \alpha_2 \ and \ \alpha_3$ and their function values.

The required interpolation conditions are:

$$q(\alpha_i) = a{\alpha_i}^2 + b\alpha_i + c = \Phi(\alpha_i), i = 1, 2, 3. \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (46)$$

Let $\Phi_i = \Phi(\alpha_i), i = 1, 2, 3.$

Then we have the following system of equations:

$$q(\alpha_1) = a{\alpha_1}^2 + b\alpha_1 + c = \Phi_1 \ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (47)$$

$$q(\alpha_2) = a{\alpha_2}^2 + b\alpha_2 + c = \Phi_2 \ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (48)$$

$$q(\alpha_3) = a{\alpha_3}^2 + b\alpha_3 + c = \Phi_3 \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . \quad (49)$$

Now, solve the Equations (47) – (49) to find the coefficients $a, b \ and \ c$ by Cramer′s rule as follows:

$$a = \frac{\Delta_1}{\Delta} \quad and \quad b = \frac{\Delta_2}{\Delta} \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (50)$$

where

$$\Delta = \begin{vmatrix} {\alpha_1}^2 & \alpha_1 & 1 \\ {\alpha_2}^2 & \alpha_2 & 1 \\ {\alpha_3}^2 & \alpha_3 & 1 \end{vmatrix} = {\alpha_1}^2 \begin{vmatrix} \alpha_2 & 1 \\ \alpha_3 & 1 \end{vmatrix} - \alpha_1 \begin{vmatrix} {\alpha_2}^2 & 1 \\ {\alpha_3}^2 & 1 \end{vmatrix} + \begin{vmatrix} {\alpha_2}^2 & \alpha_2 \\ {\alpha_3}^2 & \alpha_3 \end{vmatrix}$$

$$= {\alpha_1}^2(\alpha_2 - \alpha_3) - \alpha_1\left({\alpha_2}^2 - {\alpha_3}^2\right) + \left({\alpha_2}^2\alpha_3 - \alpha_2{\alpha_3}^2\right)$$

$$= {\alpha_1}^2(\alpha_2 - \alpha_3) - \alpha_1(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3) + \alpha_2\,\alpha_3(\alpha_2 - \alpha_3)$$

$$= (\alpha_2 - \alpha_3)[{\alpha_1}^2 - \alpha_1(\alpha_2 + \alpha_3) + \alpha_2\,\alpha_3]$$

$$= (\alpha_2 - \alpha_3)[{\alpha_1}^2 - \alpha_1\alpha_2 - \alpha_1\,\alpha_3 + \alpha_2\,\alpha_3]$$

$$= (\alpha_2 - \alpha_3)[\alpha_1(\alpha_1 - \alpha_2) - \alpha_3(\alpha_1 - \alpha_2)]$$

$$= (\alpha_2 - \alpha_3)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)$$

$$\therefore \Delta = -(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \ldots\ldots\ldots\ldots \quad (51)$$

$$\Delta_1 = \begin{vmatrix} \boldsymbol{\Phi}_1 & \alpha_1 & 1 \\ \boldsymbol{\Phi}_2 & \alpha_2 & 1 \\ \boldsymbol{\Phi}_3 & \alpha_3 & 1 \end{vmatrix} = \boldsymbol{\Phi}_1 \begin{vmatrix} \alpha_2 & 1 \\ \alpha_3 & 1 \end{vmatrix} - \alpha_1 \begin{vmatrix} \boldsymbol{\Phi}_2 & 1 \\ \boldsymbol{\Phi}_3 & 1 \end{vmatrix} + \begin{vmatrix} \boldsymbol{\Phi}_2 & \alpha_2 \\ \boldsymbol{\Phi}_3 & \alpha_3 \end{vmatrix}$$

$$= \boldsymbol{\Phi}_1(\alpha_2 - \alpha_3) - \alpha_1(\boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_3) + (\boldsymbol{\Phi}_2\alpha_3 - \alpha_2\boldsymbol{\Phi}_3)$$

$$= (\alpha_2 - \alpha_3)\boldsymbol{\Phi}_1 - \alpha_1\boldsymbol{\Phi}_2 + \alpha_1\boldsymbol{\Phi}_3 + \alpha_3\boldsymbol{\Phi}_2 - \alpha_2\boldsymbol{\Phi}_3$$

$$\therefore \quad \Delta_1 = (\alpha_2 - \alpha_3)\boldsymbol{\Phi}_1 + (\alpha_3 - \alpha_1)\boldsymbol{\Phi}_2 + (\alpha_1 - \alpha_2)\boldsymbol{\Phi}_3 \quad (52)$$

$$\Delta_2 = \begin{vmatrix} {\alpha_1}^2 & \boldsymbol{\Phi}_1 & 1 \\ {\alpha_2}^2 & \boldsymbol{\Phi}_2 & 1 \\ {\alpha_3}^2 & \boldsymbol{\Phi}_3 & 1 \end{vmatrix} = {\alpha_1}^2 \begin{vmatrix} \boldsymbol{\Phi}_2 & 1 \\ \boldsymbol{\Phi}_3 & 1 \end{vmatrix} - \boldsymbol{\Phi}_1 \begin{vmatrix} {\alpha_2}^2 & 1 \\ {\alpha_3}^2 & 1 \end{vmatrix} + \begin{vmatrix} {\alpha_2}^2 & \boldsymbol{\Phi}_2 \\ {\alpha_3}^2 & \boldsymbol{\Phi}_3 \end{vmatrix}$$

$$= {\alpha_1}^2(\boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_3) - \boldsymbol{\Phi}_1\left({\alpha_2}^2 - {\alpha_3}^2\right) + ({\alpha_2}^2\boldsymbol{\Phi}_3 - {\alpha_3}^2\boldsymbol{\Phi}_2)$$

$$= {\alpha_1}^2\boldsymbol{\Phi}_2 - {\alpha_1}^2\boldsymbol{\Phi}_3 + \boldsymbol{\Phi}_1\left({\alpha_3}^2 - {\alpha_2}^2\right) + {\alpha_2}^2\boldsymbol{\Phi}_3 - {\alpha_3}^2\boldsymbol{\Phi}_2$$

$$\therefore \quad \Delta_2 = \boldsymbol{\Phi}_1\left({\alpha_3}^2 - {\alpha_2}^2\right) + \boldsymbol{\Phi}_2\left({\alpha_1}^2 - {\alpha_3}^2\right) + \boldsymbol{\Phi}_3\left({\alpha_2}^2 - {\alpha_1}^2\right) \quad (53)$$

$$\therefore \quad a = \frac{\Delta_1}{\Delta} = \frac{(\alpha_2 - \alpha_3)\Phi_1 + (\alpha_3 - \alpha_1)\Phi_2 + (\alpha_1 - \alpha_2)\Phi_3}{-(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)} \quad \ldots\ldots \quad (54)$$

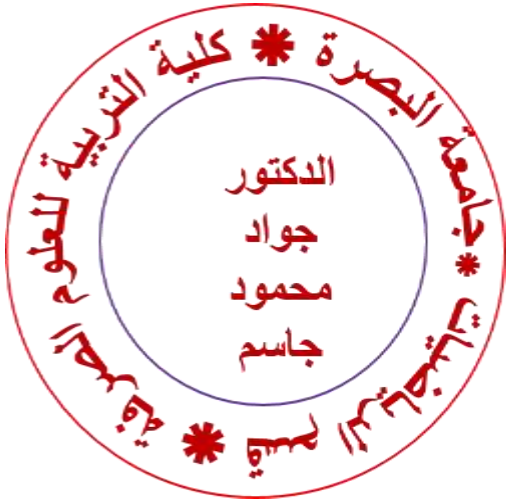
And

$$b = \frac{\Delta_2}{\Delta} = \frac{\Phi_1({\alpha_3}^2 - {\alpha_2}^2) + \Phi_2({\alpha_1}^2 - {\alpha_3}^2) + \Phi_3({\alpha_2}^2 - {\alpha_1}^2)}{-(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (55)$$

Let $\bar{\alpha}$ be the minimizer of the quadratic function

$\Phi(\alpha) = a\alpha^2 + b\alpha + c.$

$$\therefore \Phi'(\bar{\alpha}) = 0 \ \rightarrow 2a\bar{\alpha} + b = 0 \ \rightarrow \ \bar{\alpha} = -\frac{b}{2a} \ .$$



$$\therefore \bar{\alpha} = -\frac{\dfrac{\Phi_1({\alpha_3}^2 - {\alpha_2}^2) + \Phi_2({\alpha_1}^2 - {\alpha_3}^2) + \Phi_3({\alpha_2}^2 - {\alpha_1}^2)}{-(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}}{2\dfrac{(\alpha_2 - \alpha_3)\Phi_1 + (\alpha_3 - \alpha_1)\Phi_2 + (\alpha_1 - \alpha_2)\Phi_3}{-(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}}$$

$$= -\frac{\Phi_1({\alpha_3}^2 - {\alpha_2}^2) + \Phi_2({\alpha_1}^2 - {\alpha_3}^2) + \Phi_3({\alpha_2}^2 - {\alpha_1}^2)}{2\ [(\alpha_2 - \alpha_3)\Phi_1 + (\alpha_3 - \alpha_1)\Phi_2 + (\alpha_1 - \alpha_2)\Phi_3]}$$

$$\therefore \bar{\alpha} = \frac{\Phi_1(\alpha_2{}^2-\alpha_3{}^2)+\Phi_2(\alpha_3{}^2-\alpha_1{}^2)+\Phi_3(\alpha_1{}^2-\alpha_2{}^2)}{2[\Phi_1\,(\alpha_2-\alpha_3)+\Phi_2\,(\alpha_3-\alpha_1)+\Phi_3(\alpha_1-\alpha_2)]} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (56)$$

Note (17):

Equation (56) is called the quadratic interpolation formula with three points.

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