Optimization

Fourth Class

2020 - 2021

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Chapter Two

Line Search

Lecture 5

الدكتور
جواد
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5: Interpolation Methods

Interpolation methods are the other approach of line search. This class of methods approximates the function $\Phi(\alpha) = f(X + \alpha d)$ by fitting a quadratic or cubic polynomial in α to know data and choosing a new $\alpha - value$ and known points.

First: Quadratic Interpolation Methods

I: Quadratic Interpolation with Two Points:

Given two points α_1, α_2 and their function values $\Phi(\alpha_1)$ *and* $\Phi(\alpha_2)$ and the derivative $\Phi'(\alpha_1)$ or $(\Phi'(\alpha_2))$. Construct the quadratic interpolation function $q(\alpha) = a\alpha^2 + b\alpha + c$ with interpolation conditions

$$q(\alpha_1) = a{\alpha_1}^2 + b\alpha_1 + c = \Phi(\alpha_1) \quad \ldots\ldots\ldots \quad (23)$$

$$q(\alpha_2) = a{\alpha_2}^2 + b\alpha_2 + c = \Phi(\alpha_2) \quad \ldots\ldots\ldots \quad (24)$$

$$q'(\alpha_1) = 2a\alpha_1 + b = \Phi'(\alpha_1) \quad \ldots\ldots\ldots \quad (25)$$

Write $\Phi_1 = \Phi(\alpha_1)$, $\Phi_2 = \Phi(\alpha_2)$, $\Phi_1' = \Phi'(\alpha_1)$ and $\Phi_2' = \Phi'(\alpha_2)$

Then we have the following equations

$$a{\alpha_1}^2 + b\alpha_1 + c = \Phi_1 \quad \ldots\ldots\ldots\ldots \quad (26)$$

$$a{\alpha_2}^2 + b\alpha_2 + c = \Phi_2 \quad \ldots\ldots\ldots\ldots \quad (27)$$

$$2a\alpha_1 + b = \Phi_1' \quad \ldots\ldots\ldots\ldots \quad (28)$$

Now, we want to find the coefficients of the quadratic function a and b.

From Equation (28), we have

$$b = \Phi_1' - 2a\alpha_1 \quad \ldots\ldots\ldots\ldots \quad (29)$$

Subtracting (26) and (27) yields

$$a\left({\alpha_1}^2 - {\alpha_2}^2\right) + b(\alpha_1 - \alpha_2) = \Phi_1 - \Phi_2 \quad \ldots\ldots\ldots\ldots \quad (30)$$

Put the value of b , Equation (29) in Equation (30), we have

$$a\left({\alpha_1}^2-{\alpha_2}^2\right)+(\Phi_1'-2a\alpha_1)(\alpha_1-\alpha_2)=\Phi_1-\Phi_2\ \rightarrow$$

$$a\left({\alpha_1}^2-{\alpha_2}^2\right)+\Phi_1'(\alpha_1-\alpha_2)-2a\alpha_1(\alpha_1-\alpha_2)=\Phi_1-\Phi_2\ \rightarrow$$

$$a\left({\alpha_1}^2-{\alpha_2}^2\right)-2a\alpha_1(\alpha_1-\alpha_2)=\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)\ \rightarrow$$

$$a\left[\left({\alpha_1}^2-{\alpha_2}^2\right)-2\alpha_1(\alpha_1-\alpha_2)\right]=\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)\ \rightarrow$$

$$\therefore a=\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{\left({\alpha_1}^2-{\alpha_2}^2\right)-2\alpha_1(\alpha_1-\alpha_2)}=\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)-2\alpha_1(\alpha_1-\alpha_2)}=$$

$$\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2-2\alpha_1)}$$

$$\therefore a=\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)(-\alpha_1+\alpha_2)}=\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{-(\alpha_1-\alpha_2)(\alpha_1-\alpha_2)}$$

$$\therefore a=\frac{\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)}{-(\alpha_1-\alpha_2)^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

From Equations (29) and (31), we have

$$b = \Phi_1' + 2\frac{\Phi_1 - \Phi_2 - \Phi_1'(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2}\alpha_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (32)$$

Let $\overline{\alpha}$ be the minimizer of the quadratic function

$$\Phi(\alpha) = a\alpha^2 + b\alpha + c.$$

$$\therefore \Phi'(\overline{\alpha}) = 0 \;\rightarrow\; 2a\overline{\alpha} + b = 0 \;\rightarrow\; \overline{\alpha} = -\frac{b}{2a} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (33)$$

From Equations (31), (32) and (33), we have

$$\overline{\alpha} = -\frac{\Phi_1' + 2\frac{\Phi_1 - \Phi_2 - \Phi_1'(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2}\alpha_1}{2\frac{\Phi_1 - \Phi_2 - \Phi_1'(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)^2}} = -\frac{\frac{\Phi_1'(\alpha_1 - \alpha_2)^2 + 2\alpha_1\left[\Phi_1 - \Phi_2 - \Phi_1'(\alpha_1 - \alpha_2)\right]}{(\alpha_1 - \alpha_2)^2}}{2\frac{\Phi_1 - \Phi_2 - \Phi_1'(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)^2}} \rightarrow$$

$$\bar{\alpha} = \frac{\Phi_1'(\alpha_1-\alpha_2)^2+2\alpha_1[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]}{2[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]}$$

$$= \frac{\Phi_1'(\alpha_1-\alpha_2)^2}{2[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]} + \frac{2\alpha_1[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]}{2[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]} = \frac{\Phi_1'(\alpha_1-\alpha_2)^2}{2[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]} + \alpha_1$$

$$\therefore \quad \bar{\alpha} = \alpha_1 + \frac{1}{2}\,\frac{\Phi_1'(\alpha_1-\alpha_2)^2}{[\Phi_1-\Phi_2-\Phi_1'(\alpha_1-\alpha_2)]} = \alpha_1 + \frac{1}{2}\,\frac{\Phi_1'(\alpha_1-\alpha_2)}{\frac{\Phi_1-\Phi_2}{(\alpha_1-\alpha_2)}-\Phi_1'} = \alpha_1 - \frac{1}{2}\,\frac{(\alpha_1-\alpha_2)\Phi_1'}{\Phi_1'-\frac{\Phi_1-\Phi_2}{(\alpha_1-\alpha_2)}}$$

$$\therefore \quad \bar{\alpha} = \alpha_1 - \frac{1}{2}\,\frac{(\alpha_1-\alpha_2)\Phi_1'}{\Phi_1'-\frac{\Phi_1-\Phi_2}{(\alpha_1-\alpha_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

Then we get the following iterative formula:

$$\alpha_{k+1} = \alpha_k - \frac{1}{2}\left[\frac{(\alpha_k-\alpha_{k-1})\Phi'_k}{\Phi'_k - \frac{\Phi_k-\Phi_{k-1}}{(\alpha_k-\alpha_{k-1})}}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (35)$$