



Optimization
Fourth Class
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Chapter Two



Line Search

Lecture 5

5: Interpolation Methods

Interpolation methods are the other approach of line search. This class of methods approximates the function $\Phi(\alpha) = f(X + \alpha d)$ by fitting a quadratic or cubic polynomial in α to known data and choosing a new α – *value* and known points.

First: Quadratic Interpolation Methods

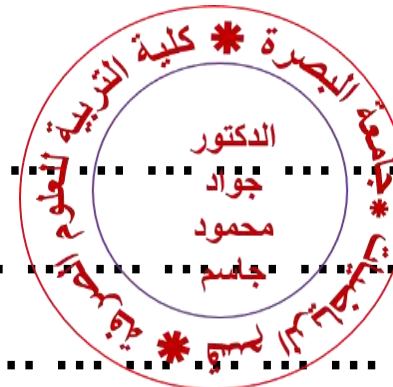
I: Quadratic Interpolation with Two Points:

Given two points α_1, α_2 and their function values $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ and the derivative $\Phi'(\alpha_1)$ or $(\Phi'(\alpha_2))$. Construct the quadratic interpolation function $q(\alpha) = a\alpha^2 + b\alpha + c$ with interpolation conditions

$$q(\alpha_1) = a\alpha_1^2 + b\alpha_1 + c = \Phi(\alpha_1) \dots\dots\dots (23)$$

$$q(\alpha_2) = a\alpha_2^2 + b\alpha_2 + c = \Phi(\alpha_2) \dots\dots\dots (24)$$

$$q'(\alpha_1) = 2a\alpha_1 + b = \Phi'(\alpha_1) \dots\dots\dots (25)$$



Write $\Phi_1 = \Phi(\alpha_1)$, $\Phi_2 = \Phi(\alpha_2)$, $\Phi'_1 = \Phi'(\alpha_1)$ and $\Phi'_2 = \Phi'(\alpha_2)$

Then we have the following equations

$$a\alpha_1^2 + b\alpha_1 + c = \Phi_1 \dots\dots\dots (26)$$

$$a\alpha_2^2 + b\alpha_2 + c = \Phi_2 \dots\dots\dots (27)$$

$$2a\alpha_1 + b = \Phi'_1 \dots\dots\dots (28)$$

Now, we want to find the coefficients of the quadratic function a and b .

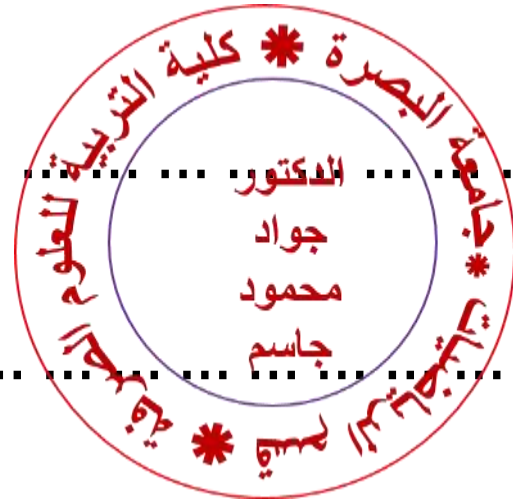
From Equation (28), we have

$$b = \Phi'_1 - 2a\alpha_1 \dots\dots\dots (29)$$

Subtracting (26) and (27) yields

$$a(\alpha_1^2 - \alpha_2^2) + b(\alpha_1 - \alpha_2) = \Phi_1 - \Phi_2 \dots\dots\dots (30)$$

Put the value of b , Equation (29) in Equation (30), we have



$$\begin{aligned}
& a(\alpha_1^2 - \alpha_2^2) + (\Phi'_1 - 2a\alpha_1)(\alpha_1 - \alpha_2) = \Phi_1 - \Phi_2 \rightarrow \\
& a(\alpha_1^2 - \alpha_2^2) + \Phi'_1(\alpha_1 - \alpha_2) - 2a\alpha_1(\alpha_1 - \alpha_2) = \Phi_1 - \Phi_2 \rightarrow \\
& a(\alpha_1^2 - \alpha_2^2) - 2a\alpha_1(\alpha_1 - \alpha_2) = \Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2) \rightarrow \\
& a[(\alpha_1^2 - \alpha_2^2) - 2\alpha_1(\alpha_1 - \alpha_2)] = \Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2) \rightarrow
\end{aligned}$$

$$\begin{aligned}
\therefore a &= \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1^2 - \alpha_2^2) - 2\alpha_1(\alpha_1 - \alpha_2)} = \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) - 2\alpha_1(\alpha_1 - \alpha_2)} = \\
& \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2 - 2\alpha_1)}
\end{aligned}$$

$$\therefore a = \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)(-\alpha_1 + \alpha_2)} = \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)}$$

$$\therefore a = \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)^2} \dots \dots \dots (31)$$



From Equations (29) and (31), we have

$$b = \Phi'_1 + 2 \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2} \alpha_1 \dots \dots \dots (32)$$

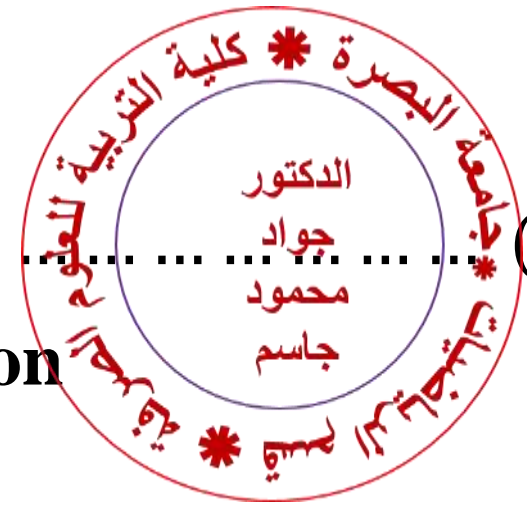
Let $\bar{\alpha}$ be the minimizer of the quadratic function

$$\Phi(\alpha) = a\alpha^2 + b\alpha + c.$$

$$\therefore \Phi'(\bar{\alpha}) = 0 \rightarrow 2a\bar{\alpha} + b = 0 \rightarrow \bar{\alpha} = -\frac{b}{2a} \dots \dots \dots (33)$$

From Equations (31), (32) and (33), we have

$$\bar{\alpha} = -\frac{\Phi'_1 + 2 \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2} \alpha_1}{2 \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)^2}} = -\frac{\frac{\Phi'_1(\alpha_1 - \alpha_2)^2 + 2\alpha_1[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]}{(\alpha_1 - \alpha_2)^2}}{2 \frac{\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)}{-(\alpha_1 - \alpha_2)^2}} \rightarrow$$



$$\begin{aligned}\bar{\alpha} &= \frac{\Phi'_1(\alpha_1 - \alpha_2)^2 + 2\alpha_1[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]}{2[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]} \\ &= \frac{\Phi'_1(\alpha_1 - \alpha_2)^2}{2[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]} + \frac{2\alpha_1[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]}{2[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]} = \frac{\Phi'_1(\alpha_1 - \alpha_2)^2}{2[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]} + \alpha_1 \\ \therefore \bar{\alpha} &= \alpha_1 + \frac{1}{2} \frac{\Phi'_1(\alpha_1 - \alpha_2)^2}{[\Phi_1 - \Phi_2 - \Phi'_1(\alpha_1 - \alpha_2)]} = \alpha_1 + \frac{1}{2} \frac{\Phi'_1(\alpha_1 - \alpha_2)}{\frac{\Phi_1 - \Phi_2}{(\alpha_1 - \alpha_2)} - \Phi'_1} = \alpha_1 - \frac{1}{2} \frac{(\alpha_1 - \alpha_2)\Phi'_1}{\Phi'_1 - \frac{\Phi_1 - \Phi_2}{(\alpha_1 - \alpha_2)}}\end{aligned}$$

$$\therefore \bar{\alpha} = \alpha_1 - \frac{1}{2} \frac{(\alpha_1 - \alpha_2)\Phi'_1}{\Phi'_1 - \frac{\Phi_1 - \Phi_2}{(\alpha_1 - \alpha_2)}} \dots \dots \dots (34)$$

Then we get the following iterative formula:

$$\alpha_{k+1} = \alpha_k - \frac{1}{2} \left[\frac{(\alpha_k - \alpha_{k-1})\Phi'_k}{\Phi'_k - \frac{\Phi_k - \Phi_{k-1}}{(\alpha_k - \alpha_{k-1})}} \right] \dots \dots \dots (35)$$

