University of Basrah
College of Education for Pure Sciences
Department of Mathematics

Chapter one

Functions

Sets of Numbers:

$\mathbb{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$ Natural Numbers.

$\mathbb{Z} = \{\dots, -\mathbf{3}, -\mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \dots\}$ Integer Numbers.

$\mathbb{Q} = \{\frac{\boldsymbol{a}}{\boldsymbol{b}} : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}, \boldsymbol{b} \neq \mathbf{0}\}$ Rational Numbers.

$\mathbb{IQ} = \boldsymbol{\pi}, \boldsymbol{e}, \sqrt{\mathbf{2}}, \sqrt{\mathbf{3}}, \sqrt{\mathbf{5}}, \dots$ Irrational Numbers.

$\mathbb{R} = \mathbb{Q} \cup \mathbb{IQ}$ Real Numbers

Definition: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, we define the Intervals as follows:

(1) Open Intervals $(a,b) = \{x \in R : a < x < b\}$

(2) Closed Intervals $[a,b] = \{x \in R : a \le x \le b\}$

(3) Half open, half closed Intervals $[a,b) = \{x \in R : a \le x < b\}$ and $(a,b] = \{x \in R : a < x \le b\}$.

(4) $[a,\infty) = \{x \in R : x \ge a\}$, $(-\infty,a] = \{x \in R : x \le a\}$, $(-\infty,a) = \{x \in R : x < a\}$ and $(-\infty,\infty) = \{x \in R : x \text{ is a real number}\} = R$.

Remark: If A and B are intervals, then

1. The union of A and B is denoted by $A \cup B$ and defined as the interval whose members belong to A or B (or both).
2. The intersection of A and B is denoted by $A \cap B$ and defined as the interval whose members belong to both A and B.

Example: Let $A = [0,5]$ *and* $B = [1,7]$ then $A \cup B = [0,7]$ and $A \cap B = [1,5]$.

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Functions

Definition: A relation $f : X \to Y$ is called a function if and only if for each element $x \in X$ there exist a unique element $y \in Y$ such that $y = f(x)$.

Note:

(1) x is the independent variable (input value of f) and y is dependent variable (output value of f at x)

(2) The set X of all possible input values is called the domain of f and it's denoted by D_f

(3) The set Y is called the co-domain of the function.

(4) The set of all possible output values $f(x)$ as x varies throughout D_x is called the range of f and it's denoted by R_f. (Note that $R_f \subseteq Y$.)

Examples: Find the Domain and the Range of the following:

(1) $y = x + 5$

$D_f = \{x \in R : -\infty < x < \infty\} = R$

To find the Range, we represent x in terms of y. $x = y - 5$

$R_f = \{y \in R : -\infty < y < \infty\} = R$

(2) $y = x^2$ $D_f = \{x \in R : -\infty < x < \infty\} = R$

$x = \sqrt{y}$ $\Rightarrow R_f = \{y \in R : y \ge 0\} = [0, \infty)$

(3) $y = \dfrac{1}{x+2}$ (set the denominator = ***0)***

$x + 2 = 0, \quad x = -2$

$\Rightarrow D_f = \{x \in \Re : x \ne -2\} = \Re / \{-2\}$

$x = \dfrac{1 - 2y}{y}$

$\Rightarrow R_f = \{y \in \Re : y \ne 0\} = \Re / \{0\}$

(4) $y = \sqrt{x + 9}$ $x \in [0,7]$ $\Rightarrow D_f = [0,7]$

Put $x = 0$ in the function we get $y = 3$

Put $x = 7$ in the function we get $y = 4$ then $R_f = [3,4]$

(5) $f(x) = \dfrac{1}{\sqrt{2 - x}} + 5$

(6) $y = \dfrac{3x}{x^2 - 5x + 6}$

Algebraic Combination of Function

If ***f*** and ***g*** are two functions with domains D_f *and* D_g respectively, then

(1) ***(f + g) (x) = f(x) + g(x)*** ***with*** $D_{f+g} = D_f \cap D_g$

(2) ***(f – g) (x) = f(x) – g(x)*** ***with*** $D_{f-g} = D_f \cap D_g$

(3) ***(f.g) (x) = f(x) . g(x)*** ***with*** $D_{f.g} = D_f \cap D_g$

(4) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$ ***with*** $D_{f/g} = D_f \cap D_g$ and $g(x) \neq 0$

Examples*:* *1)* If $f(x) = x^2 - 5x - 6$ and $g(x) = 3x^2 + 4$ then find $(f+g)(x)$, $(f-g)(x), (g-f)(x), (f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains.

Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

Function	Formula	Domain
$f+g$	$(f+g)(x) = 4x^2 - 5x - 2$	$\mathbb{R}$
$f-g$	$(f-g)(x) = -2x^2 - 5x - 10$	$\mathbb{R}$
$g-f$	$(g-f)(x) = 2x^2 + 5x + 10$	$\mathbb{R}$
$f.g$	$(f.g)(x) = 3x^4 - 15x^3 - 14x^2 - 20x - 24$	$\mathbb{R}$
f/g	$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 5x - 6}{3x^2 + 4}$	$\mathbb{R}$
g/f	$\left(\frac{g}{f}\right)(x) = \frac{3x^2 + 4}{x^2 - 5x - 6}$	$(-\infty, -1) \cup (-1, 6) \cup (6, \infty)$

2) If $f(x) = \sqrt{x}$ *and* $gx) = \sqrt{1-x}$ then find $(f+g)(x), (f-g)(x), (g-f)(x)$, $(f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains.

Solution: $D_f = \{x \in \mathbb{R}: x \geq 0\} = \mathbb{R}^+ \cup \{0\} = [0, \infty)$, $D_g = \{x \in \mathbb{R}: x \leq 1\} = (-\infty, 1]$

Function	Formula	Domain
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1]$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f.g$	$(f.g)(x) = \sqrt{x(1-x)}$	$[0,1]$
f/g	$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{1-x}} = \sqrt{\frac{x}{1-x}}$	$[0,1)$
g/f	$\left(\frac{g}{f}\right)(x) = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}}$	$(0,1]$

3) If $f(x) = 2$ *and* $gx) = x^2$ then find $(f + g)(x), (f - g)(x), (g - f)(x)$, $(f.g)(x), (f/g)(x)$ *and* $(g/f)(x)$. Moreover, find their domains and ranges.

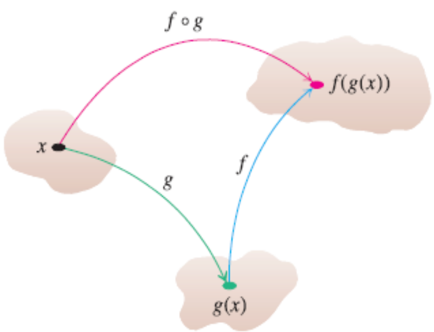
Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

Function	Formula	Domain	Rang
$f + g$	$(f + g)(x) = 2 + x^2$	$\mathbb{R}$	$[2, \infty)$
$f - g$	$(f - g)(x) = 2 - x^2$	$\mathbb{R}$	$(-\infty, 2)$
$g - f$	$(g - f)(x) = x^2 - 2$	$\mathbb{R}$	$[-2, \infty)$
$f.g$	$(f.g)(x) = 2x^2$	$\mathbb{R}$	$[0, \infty)$
f/g	$\left(\frac{f}{g}\right)(x) = \frac{2}{x^2}$	$(-\infty, 0) \cup (0, \infty)$	$(0, \infty)$
g/f	$\left(\frac{g}{f}\right)(x) = \frac{x^2}{2}$	$\mathbb{R}$	$(0, \infty)$

Composite Function

If the range of the function $g(x)$ is contained in the domain of the function $f(x)$, then the composition $f \circ g$ is the function defined by $(f \circ g)(x) = f\big(g(x)\big)$.
The domain of $f \circ g$ consists of the number x in the domain of g for which $g(x)$ lies in the domain of f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$



Examples:

1) If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, then find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

Solution: $D_f = [0, \infty)$, $D_g = \mathbb{R}$

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[\mathbf{-1}, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[\mathbf{0}, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$	$[\mathbf{0}, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$

2) If $f(x) = x^2 - 3$ and $g(x) = x + 1$, then find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$

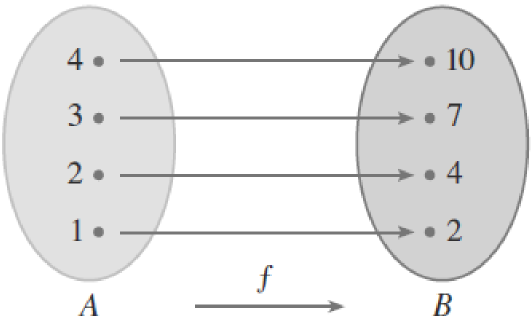
Solution: $D_f = \mathbb{R}$, $D_g = \mathbb{R}$

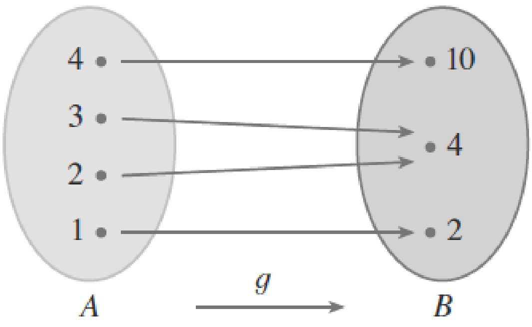
Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = (g(x))^2 - 3 = x^2 + 2x - 2$	$\mathbb{R}$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = x^2 - 2$	$\mathbb{R}$
(c) $(f \circ f)(x) = f(f(x)) = (f(x))^2 - 3 = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$	$\mathbb{R}$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$

The Inverse of the functions:

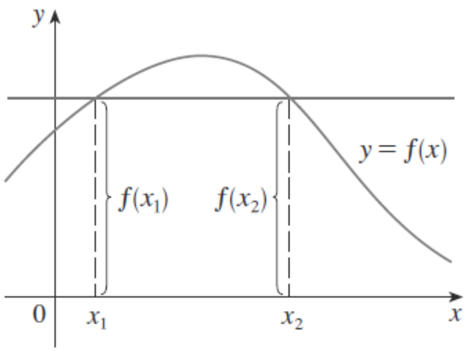
Definition : A function is said to be one – to - one function if and only if there is no two elements of the domain have the same image in the range.

i.e. ***if*** $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ***or*** ***if*** $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$





Horizontal Line Test: A function is one-to-one if and only if there is no horizontal line can intersect its graph more than once.



Definition: Let f be a one-to-one function with domain A and range B (bijective function), then its inverse function f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y, \forall y \in B$ or $f^{-1}\big(f(x)\big) = f\big(f^{-1}(x)\big) = x$ and $D_{f^{-1}} = R_f$, $D_f = R_{f^{-1}}$.

How to find the inverse function of a one-to-one function *f*

Step 1: Write $y = f(x)$.

Step 2: Solve this equation for x in terms of y (if possible).

Step 3: To express f^{-1} as a function of x, interchange x *and* y.
The resulting equation is $y = f^{-1}(x)$.

CAUTION

The -1 in f^{-1} is not a power number. Thus $f^{-1}(x)$ dose not mean $\frac{1}{f(x)}$

Example: Find the inverse of f for $f(x) = x^3 + 2$

Solution: According to the above algorithm, we first write
$y = x^3 + 2$, then we solve this equation for x:

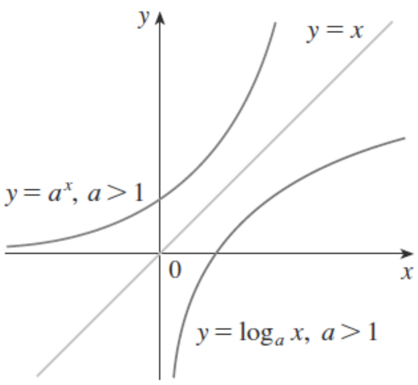
$x^3 = y - 2 \Longrightarrow x = \sqrt[3]{y - 2}$

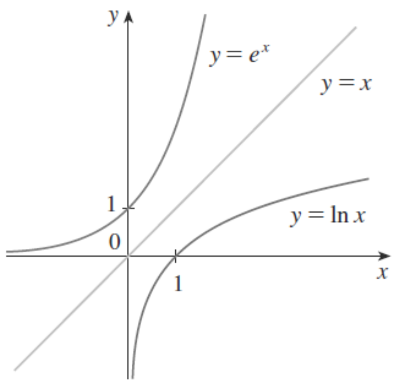
Finally, we iterchange x *and* y:

$y = \sqrt[3]{x - 2}$

$\therefore f^{-1}(x) = \sqrt[3]{x - 2}$

❖ The graph of f^{-1} is obtained by reflecting the graph of f around the line $y = x$.



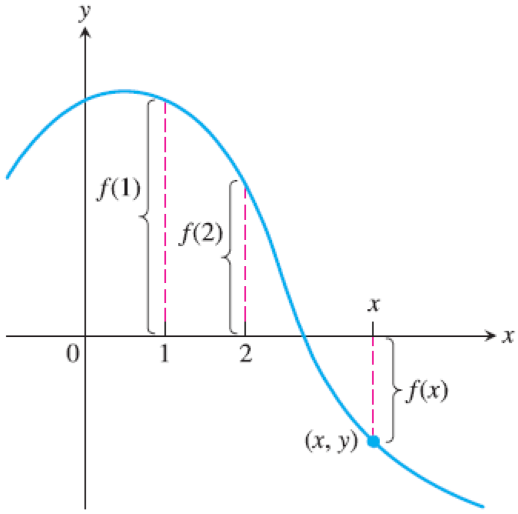


Graph of Functions:

Definition: Let $f(x)$ be a function with domain D_f. The set of all points (x, y) in the plane with x in D_f and $y = f(x)$ is called the graph of $f(x)$.

$\{(x, f(x)) : x \in D_f\}$

x	1	2	…	n
$f(x)$	$f(1)$	$f(2)$	…	$f(n)$
$(x, f(x))$	$(1, f(1))$	$(2, f(2))$	…	$(n, f(n))$



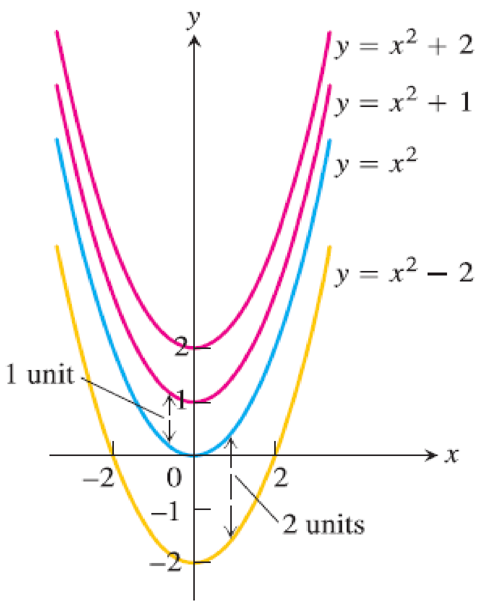
Shifting a Graph of a function:

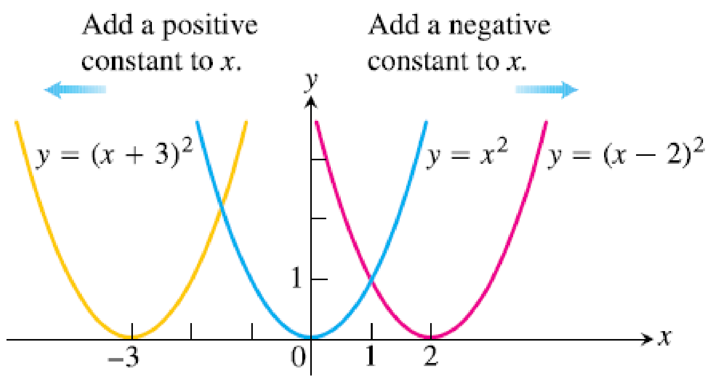
1. Vertical Shifts:

$y = f(x) + k \Longrightarrow$ Shifts $the\ graph\ of\ f \begin{cases} up\ \boldsymbol{k}\ units\ if\ k > 0 \\ down\ |\boldsymbol{k}|\ units\ if\ k < 0 \end{cases}$

2. Horizontal Shifts:

$y = f(x + h) \Longrightarrow$ Shifts $the\ graph\ of\ f \begin{cases} left\ \boldsymbol{h}\ units\ if\ h > 0 \\ right\ |\boldsymbol{h}|\ units\ if\ h < 0 \end{cases}$





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<u>Scaling and Reflecting a Graph of a Function</u>:

For $c > 1$, the graph is scaled as:

$y = cf(x)$ Stretches the graph of f vertically by a factor of c.

$y = \frac{1}{c}f(x)$ Compresses the graph of f vertically by a factor of c.

$y = f(cx)$ Compresses the graph of f horizontally by a factor of c.

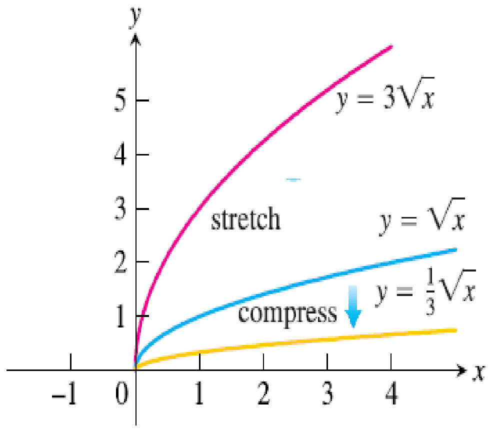
$y = f(\frac{1}{c}x)$ Stretches the graph of f horizontally by a factor of c.

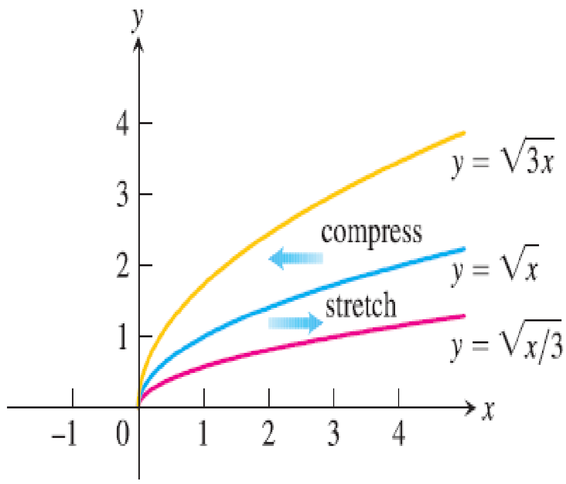
For $c = -1$, the graph is reflected as:

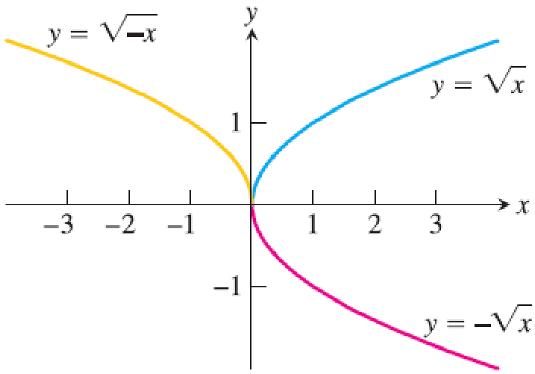
$y = -f(x)$ Reflects the graph f across the $x - axis$

$y = f(-x)$ Reflects the graph f across the $y - axis$

<u>Example</u>: Consider the function $y = \sqrt{x}$







Some Specific Types of Functions
Algebraic Functions:

1. Real valued function:
The function $f: X \to Y$ is called a real valued function if $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$.

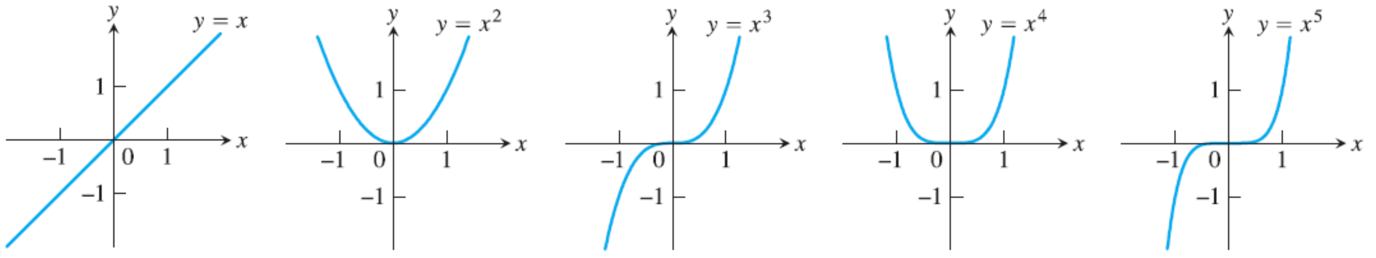
2. Constant function:
The function $f: \mathbb{R} \to \mathbb{R}$, which is defined by $y = f(x) = c, \forall x \in \mathbb{R}$, where c is constant in $\mathbb{R}$, is called a constant function. $D_f = \mathbb{R}$ and $R_f = \{c\}$.
Example: Consider the function $y = f(x) = 2$

x	-2	-1	0	1	2
$f(x)$	2	2	2	2	2
$(x, f(x))$	$(-2,2)$	$(-1,2)$	$(0,2)$	$(1,2)$	$(2,2)$

3. Identity function:
The function $f: \mathbb{R} \to \mathbb{R}$, which is defined by $y = f(x) = x, \forall x \in \mathbb{R}$, is called the identity function. $D_f = \mathbb{R}$ and $R_f = \mathbb{R}$.

x	-2	-1	0	1	2
$f(x)$	-2	-1	0	1	2
$(x, f(x))$	$(-2,-2)$	$(-1,-1)$	$(0,0)$	$(1,1)$	$(2,2)$

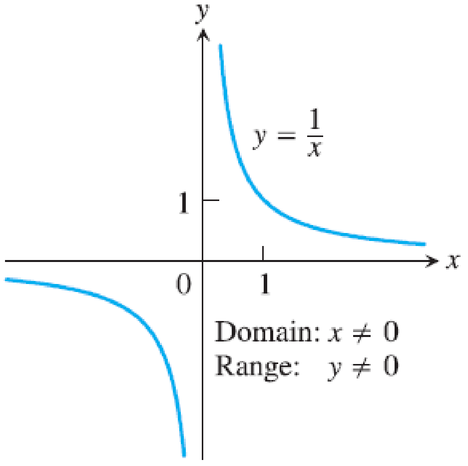
4. Power function:
The function $f: X \to Y$, which is defined by $y = f(x) = x^a, \forall x \in X$, is called the power function.
a) If $a = n \in \mathbb{N}$: $graph\ for\ a = 1,2,3,4,5$
$a = 1 \to f(x) = x$ (Identity function)
$a = 2 \to f(x) = x^2 \to D_f = \mathbb{R}$ and $R_f = [0, \infty)$.
$a = 3 \to f(x) = x^3 \to D_f = \mathbb{R}$ and $R_f = \mathbb{R}$.
$a = 4 \to f(x) = x^4 \to D_f = \mathbb{R}$ and $R_f = [0, \infty)$.
$a = 5 \to f(x) = x^5 \to D_f = \mathbb{R}$ and $R_f = \mathbb{R}$.

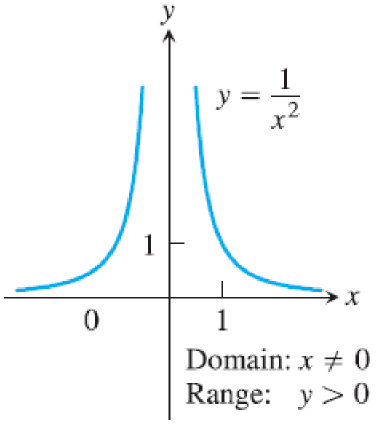


b) If $a = -n \in \mathbb{N}$: $graph\ for\ a = -1, -2$

$a = -1 \rightarrow f(x) = x^{-1} \rightarrow f(x) = \frac{1}{x} \rightarrow D_f = \mathbb{R} / \{0\}$ and $R_f = \mathbb{R} / \{0\}$.

$a = -2 \rightarrow f(x) = x^{-2} \rightarrow f(x) = \frac{1}{x^2} \rightarrow D_f = \mathbb{R} / \{0\}$ and $R_f = (0, \infty)$.





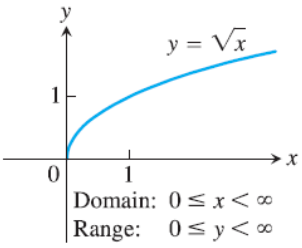
c) If $a \in \mathbb{Q}$: $graph\ for\ a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$

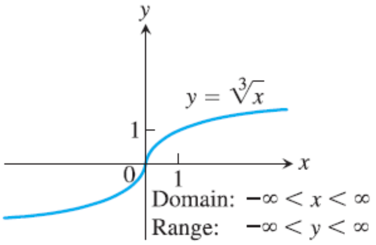
$a = \frac{1}{2} \rightarrow f(x) = x^{\frac{1}{2}} = \sqrt{x} \rightarrow D_f = [0, \infty)$ and $R_f = [0, \infty)$.

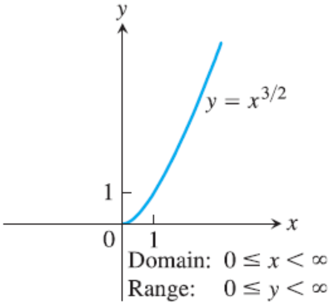
$a = \frac{1}{3} \rightarrow f(x) = x^{\frac{1}{3}} = \sqrt[3]{x} \rightarrow D_f = \mathbb{R}$ and $R_f = \mathbb{R}$.

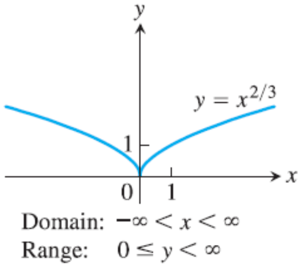
$a = \frac{3}{2} \rightarrow f(x) = x^{\frac{3}{2}} = \sqrt{x^3} \rightarrow D_f = [0, \infty)$ and $R_f = [0, \infty)$.

$a = \frac{2}{3} \rightarrow f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2} \rightarrow D_f = \mathbb{R}$ and $R_f = [0, \infty)$.









5. <u>Polynomial function</u>:

A polynomial function is any function can be written in the form

$$f(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, n \in \mathbb{N}$$

with the coefficients $a_n, a_{n-1}, \cdots, a_2, a_1, a_0 \in \mathbb{R}$ and $D_f = \mathbb{R}$.

with $a_n \neq 0,\ \ n \geq 0, (n$ is the degree of polynomial)

if $n = 0 \rightarrow f(x) = P_0(x) = a_0$ (constant function)

if $n = 1 \rightarrow f(x) = P_1(x) = a_1 x + a_0$ (linear function)

if $n \geq 2 \rightarrow f(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ (nonlinear function)

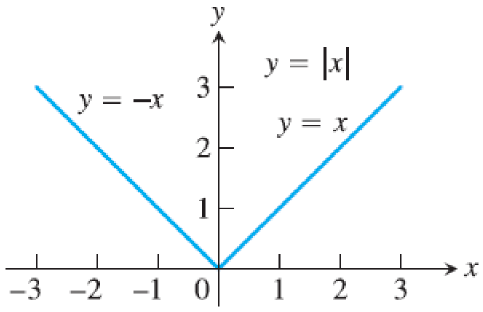
6. Absolute value function:

The function $f: \mathbb{R} \to \mathbb{R}$, which is defined by

$$y = f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is called the absolute value function (the modulus function).

$D_f = \mathbb{R}$ and $R_f = \mathbb{R}^+ \cup \{0\} = [0, \infty)$.



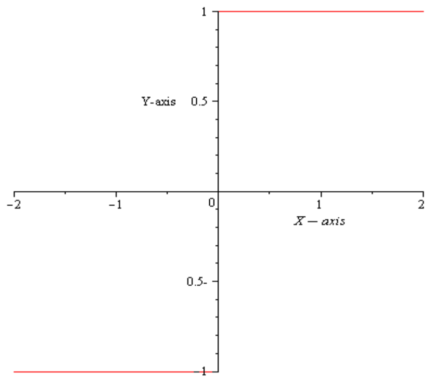
7. Sign function

The function $f: \mathbb{R} \to \mathbb{R}$, which is defined by

$$f(x) = sgn(x) = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 0, \ x = 0 \end{cases} = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$$

is called the sign function.

$D_f = \mathbb{R}$ and $R_f = \{-1, 0, 1\}$



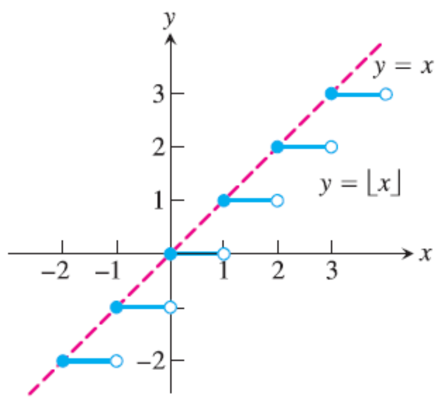
8. The Greatest integer function)

The function $f: \mathbb{R} \to \mathbb{R}$, which defined by

$$f(x) = \lfloor x \rfloor = \begin{cases} \vdots \\ -2 & if -2 \leq x < -1 \\ -1 & if -1 \leq x < 0 \\ 0 & if\ 0 \leq x < 1 \\ 1 & if\ 1 \leq x < 2 \\ 2 & if\ 2 \leq x < 3 \\ \vdots \end{cases}$$

is called the greatest integer function.

$D_f = \mathbb{R}$ and $R_f = \mathbb{Z}$.



Remark: Another name of the gratest integer function is the **Floor Function**.

Remark: There is another function called **Celling Function** denoted by $\lceil x \rceil$ that maps x to the least integer, greater than or equal to x.

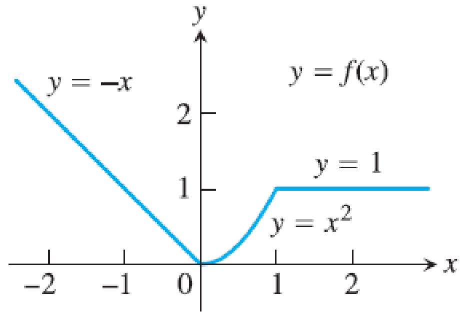
9. The piecewise function

A piecewise function is a function that is described by using different formulas on different part of its domain.

Example: Find the domain and the range and sketch the graph of the function

$$f(x) = \begin{cases} -x \ \ if \ x < 0 \\ x^2 \ if \ 0 \le x \le 1 \\ 1 \ \ if \ x > 1 \end{cases}$$

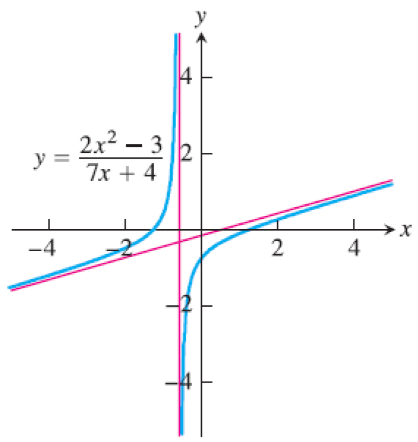
Solution: $D_f = \mathbb{R}$ and $R_f = \mathbb{R}^+ \cup \{0\} = [0, \infty)$.

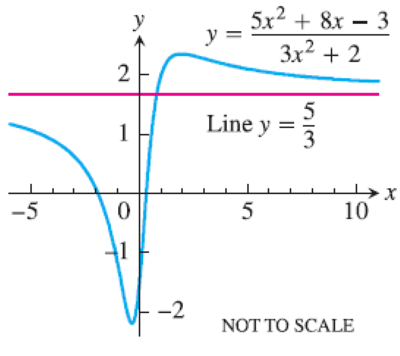


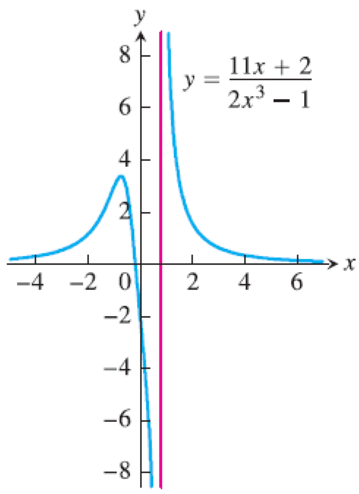
10. Odd function: The function $f: X \to Y$ is an odd function if $f(-x) = -f(x)$.

11. Even function: The function $f: X \to Y$ is an even function if $f(-x) = f(x)$.

12. Rational function: A rational function is a quotient or ratio $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. The graphs of several rational functions are sbown here below.







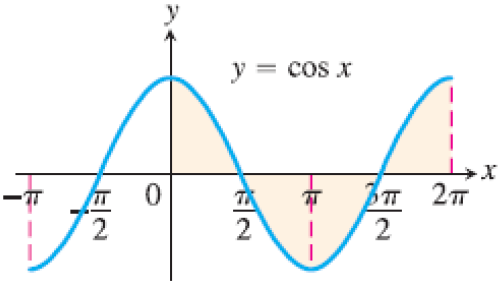
Transcendental Functions:

1. Trigonometric Functions:

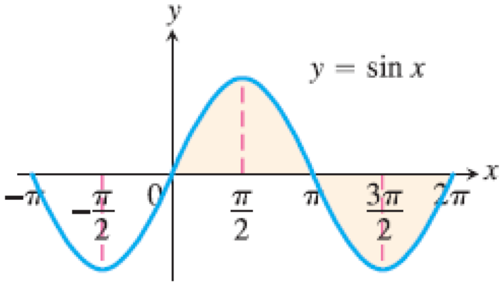
Definition: A function $f(x)$ is a periodic function if there is a positive number p such that $f(x+p)=f(x)$ for every value of x. The positive number p is called the period of the function f.

$$\left.\begin{array}{l} sin(x+2\pi)=\sin(x), cos(x+2\pi)=cos(x) \\ sec(x+2\pi)=\sec(x), csc(x+2\pi)=\csc(x) \end{array}\right\} period\ 2\pi.$$

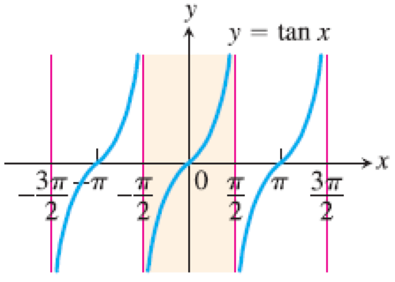
$$\left.\begin{array}{l} tan(x+\pi)=\tan(x) \\ cot(x+\pi)=cot(x) \end{array}\right\} period\ \pi.$$



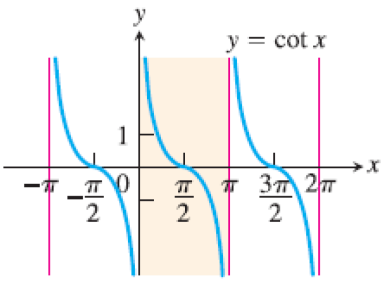
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



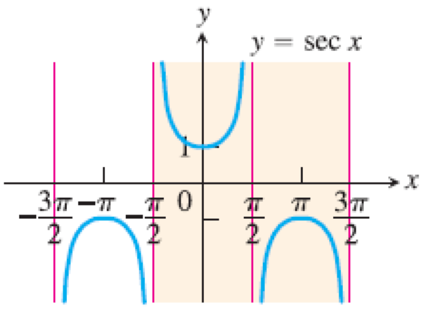
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



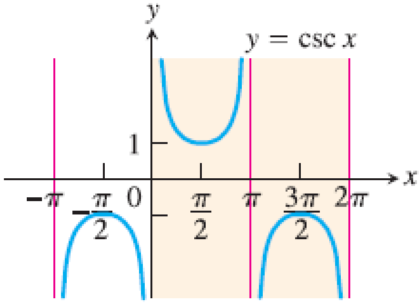
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

Some Important Identities of The Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Definition by Power series

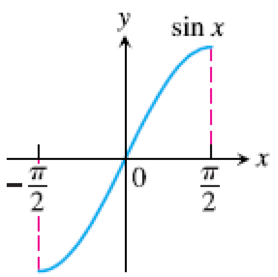
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Remark: One can define tangent, cotangent, secant and cosecant using identities above.

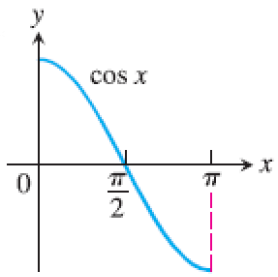
Inverse of Trigonometric Functions

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$, we make it one-to-one, so that it has an inverse $\sin^{-1} x$. Similar domain restrictions can be applied to all six trigonometric functions.

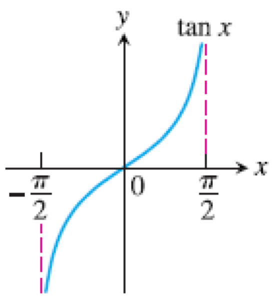
Domain restrictions that make the trigonometric functions one-to-one



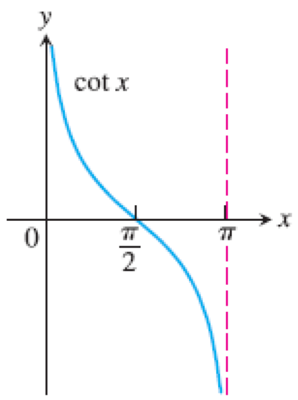
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



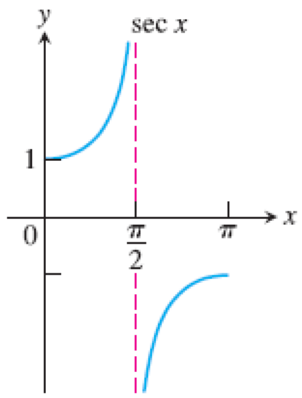
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



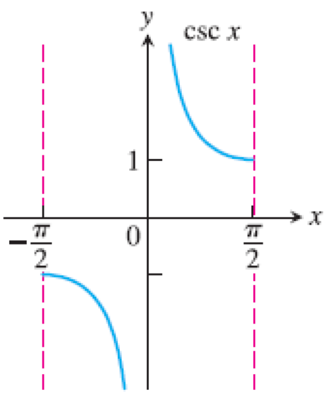
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



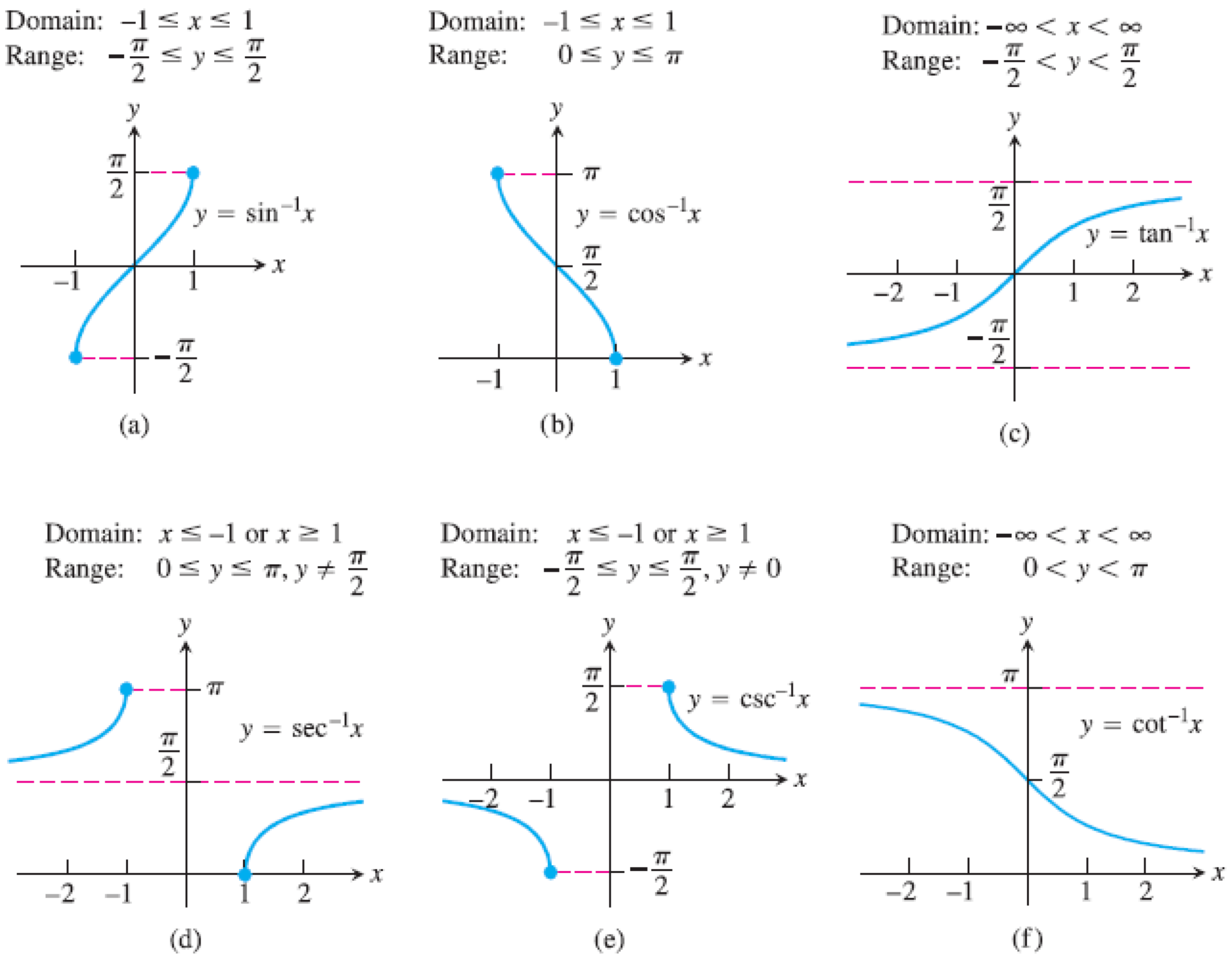
$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



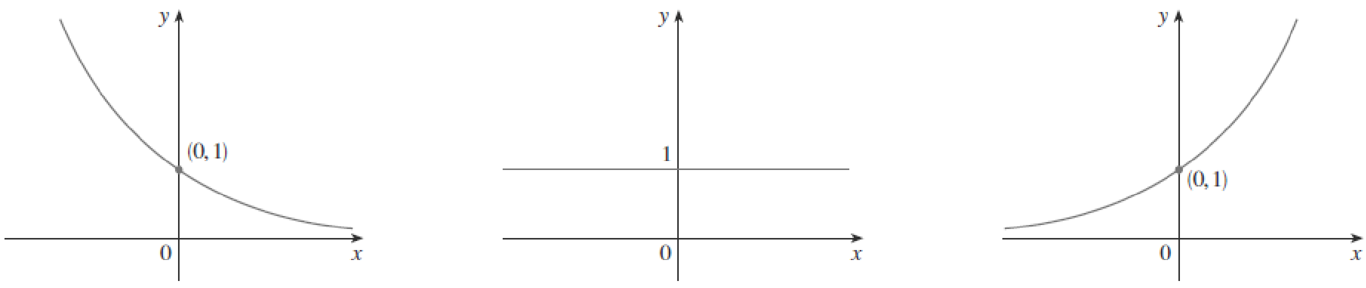
$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$



Graphs of the six basic inverse trigonometric functions.

2. <u>Exponential Functions</u>:

The function of the form $f(x) = a^x$ *where* $a > 0$ *and* $a \ne 1$, is called an exponential function. The domain of the exponential function is $\mathbb{R}$.

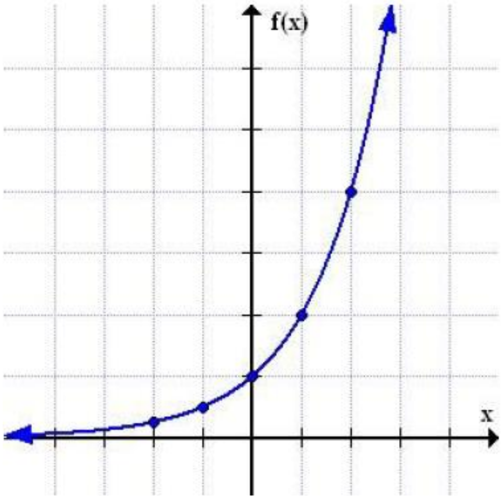


(a) $y = a^x,\ 0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x,\ a > 1$

Examples:

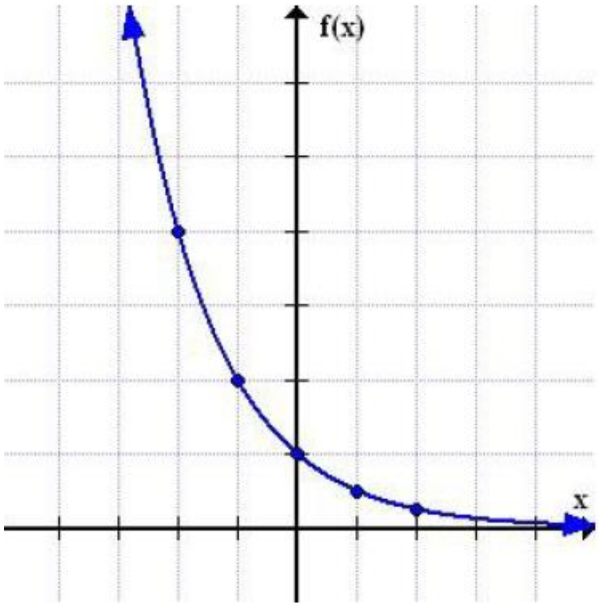
1) $y = f(x) = 2^x \to D_f = \mathbb{R}, R_f = (0, \infty)$

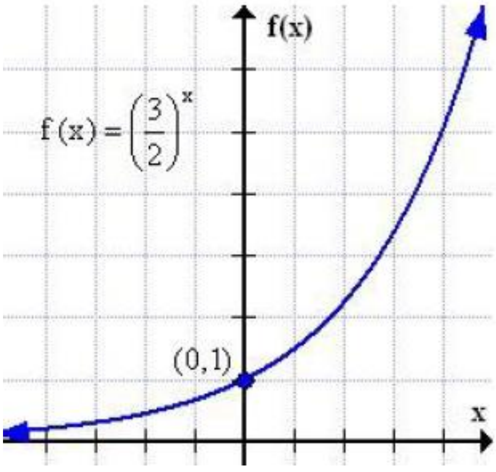
x	$f(x) = 2^x$
-2	$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$f(0) = 2^0 = 1$
1	$f(1) = 2^1 = 2$
2	$f(2) = 2^2 = 4$

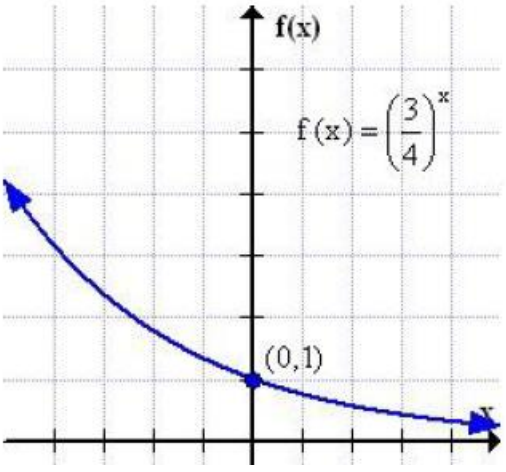


2) $y = f(x) = \left(\frac{1}{2}\right)^x \to D_f = \mathbb{R}, R_f = (0, \infty)$

x	$f(x) = \left(\frac{1}{2}\right)^x$
-2	$f(-2) = \left(\frac{1}{2}\right)^{-2} = \frac{1^{-2}}{2^{-2}} = \frac{2^2}{1^2} = 4$
-1	$f(-1) = \left(\frac{1}{2}\right)^{-1} = \frac{1^{-1}}{2^{-1}} = \frac{2^1}{1^1} = 2$
0	$f(0) = \left(\frac{1}{2}\right)^0 = 1$
1	$f(1) = \left(\frac{1}{2}\right)^1 = \frac{1^1}{2^1} = \frac{1}{2}$
2	$f(2) = \left(\frac{1}{2}\right)^2 = \frac{1^2}{2^2} = \frac{1}{4}$



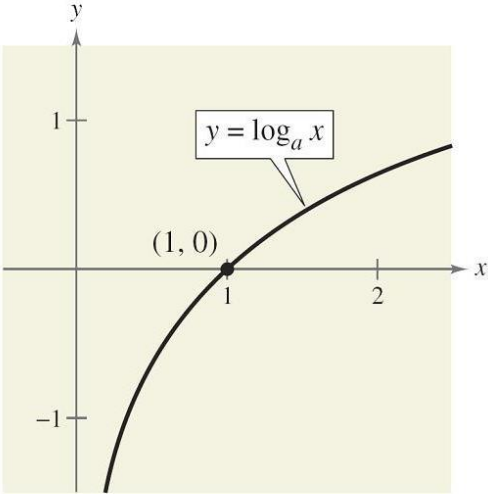




3. <u>Logarithmic Functions</u>:

The function of the form $f(x) = log_a(x)$ *where* $a > 0$ *and* $a \neq 1$, called **logarithmic function**. The domain of **logarithmic function** is $\mathbb{R}^+$.

$y = log_a(x) \Leftrightarrow a^y = x$



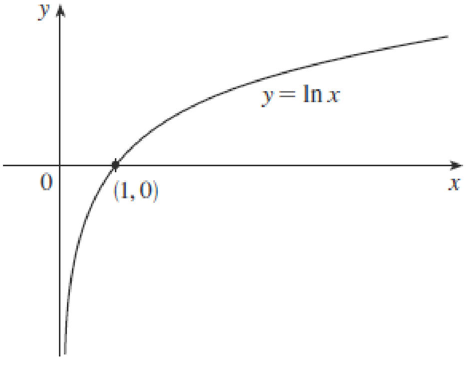
<u>Lows of logarithms</u>:

If x and y are real numbers, then

1. $log_a(x.y) = log_a(x) + log_a(y)$
2. $log_a(x/y) = log_a(x) - log_a(y)$
3. $log_a(x^r) = r.log_a(x) \quad (where\ r \in \mathbb{R})$

<u>The Natural Logarithmic Function</u>:

The function of the form $f(x) = log_e(x) = \ln(x)$, $x > 0$ called the natural logarithmic function.$D_f = \mathbb{R}^+$, $R_f = \mathbb{R}$.



Exercises 2

In exercises (1-20), find the domain and range of each the following where $\boldsymbol{y = f(x)}$

1. $y = 5x + 3$
2. $y = 2x^2 + 1$
3. $y = -7x - 4$
4. $y = 7$
5. $y = 4 - x^2$
6. $y = \sqrt{5x + 10}$
7. $y = \sqrt{x^2 - 3x}$
8. $y = 1 - \sqrt{x}$
9. $y = \sqrt{x + 9}$
10. $y = \sqrt{3x - 4}$
11. $y = \sqrt{x^2 - 4}$
12. $y = \sqrt{4 - x^2}$
13. $y = \frac{1}{7-x}$
14. $y = \frac{6}{x+2}$
15. $y = \sqrt{\frac{1}{x-2}}$
16. $y = \frac{2}{\sqrt{2x-5}}$
17. $y = \frac{x}{\sqrt{x+1}}$
18. $y = \frac{1}{1-\frac{1}{x-2}}$
19. $y = 2 + \frac{x^2}{x^2+4}$
20. $y = \frac{2}{x^2-16}$

In exercises (21-35), find the domain and the range and sketch the graph of each the following function

21. $y = \sqrt{x - 5}$
22. $y = |2x + 1|$
23. $y = \frac{3x+|x|}{x}$
24. $y = \frac{|x|}{x^2}$
25. $y = \begin{cases} x & if\ x \le 0 \\ x + 1 & if\ x > 0 \end{cases}$
26. $y = \begin{cases} 2x + 3 & if\ x < -1 \\ 3 - x & if\ x \ge -1 \end{cases}$
27. $y = \begin{cases} x + 2 & if\ x \le -1 \\ x^2 & if\ x > -1 \end{cases}$
28. $\begin{cases} -1 & if\ \ x \le -1 \\ 3x + 2 & if\ |x| < 1 \\ 7 - 2x & if\ x \ge 1 \end{cases}$
29. $y = \begin{cases} \frac{1}{x} & if\ x < 0 \\ x & if\ x \ge 0 \end{cases}$
30. $y = \frac{1}{|x|}$
31. $y = |x^2 - 1|$
32. $y = [\sin(x)]$
33. $y = [\cos(x)]$
34. $y = \sqrt{|x|}$
35. $y = \frac{x^2}{|x|}$

In exercises 36 and 37, write formulas for $\boldsymbol{f \circ g}$ *and* $\boldsymbol{g \circ f}$ and find the domain and the range of each

36. $f(x) = \sqrt{x + 1}\,, g(x) = \frac{1}{x}$
37. $f(x) = x^2, g(x) = 1 - \sqrt{x}$

38. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

39. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

University of Basrah
College of Education for Pure Sciences
Department of Mathematics

In exercises 40-54 , graph each function, not by plotting points, but by applying appropriate transformation to the graph of the standard functions.

40. $y = -\sqrt{2x+1}$ 41. $y = \sqrt{1-\frac{x}{2}}$ 42. $y = (x-1)^3 + 2$

43. $y = (1-x)^3 + 2$ 44. $y = \frac{1}{2x} - 1$ 45. $y = \frac{2}{x^2} + 1$

46. $y = -\sqrt[3]{x^2}$ 47. $y = (-2x)^{2/3}$ 48. $y = |x^2 - 2x|$

49. $y = -\sqrt{|x|}$ 50. $y = 1 + 2\cos(x)$ 51. $y = |\sin(x)|$

52. $y = \frac{1}{4}\tan\left(x - \frac{\pi}{4}\right)$ 53. $y = \begin{cases} -x & if\ x < 0 \\ e^x - 1 & if\ x \geq 0 \end{cases}$ 54. $y = 3\ln(x-2)$

In exercises 55-60, find f^{-1} and it's range of each function

55. $f(x) = \sqrt{10-3x}$ 56. $f(x) = \frac{4x-1}{2x+3}$ 57. $f(x) = e^{x^3}$

58. $f(x) = 2x^3 + 3$ 59. $f(x) = \ln(x+3)$ 60. $f(x) = \frac{1+e^x}{1-e^x}$