Chapter 2

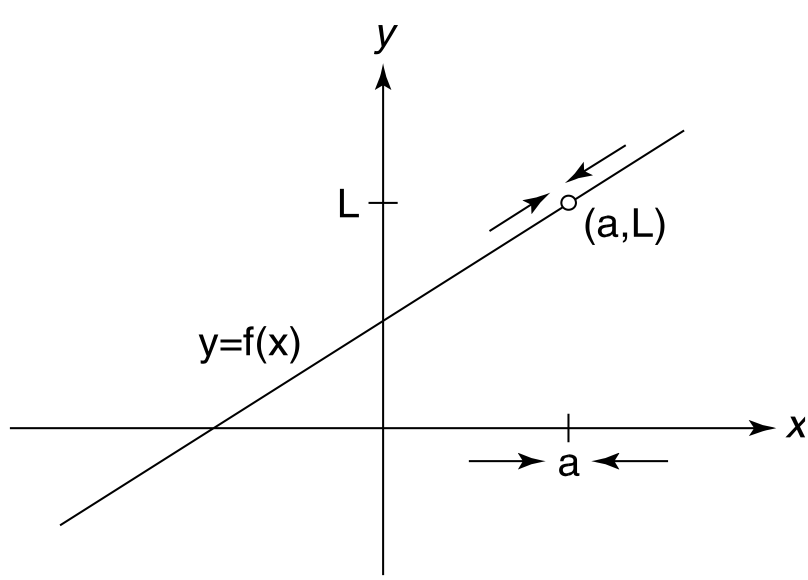
LIMITS

The concept of the limit of a function is essential to the study of calculus. It is used in defining some of the most important concepts in calculus—continuity, the derivative of a function, and the definite integral of a function.

Intuitive Definition

The **limit** of a function $f(x)$ describes the behavior of the function close to a particular x value. It does not necessarily give the value of the function at x. You write $\lim_{x \to a} f(x) = L$, which means that as x "approaches" a, the function $f(x)$ "approaches" the real number L (see Figure 2-1).

Figure 2-1 The limit of f(x) as x approaches a.



In other words, as the independent variable x gets closer and closer to a, the function value $f(x)$ gets closer to L. Note that this does not imply that $f(a) = L$; in fact, the function may not even be defined at a (Figure 2-2) or may equal some value different than L at a (Figure 2-3).

If the function does not approach a real number L as x approaches a, the limit does not exist; therefore, you write $\lim_{x \to a} f(x)$ DNE (Does Not Exist). Many different situations could occur in determining that the limit of a function does not exist as x approaches some value.

Figure 2-2 *f(a)* does not exist, but $\lim_{x \to a} f(x)$ does.

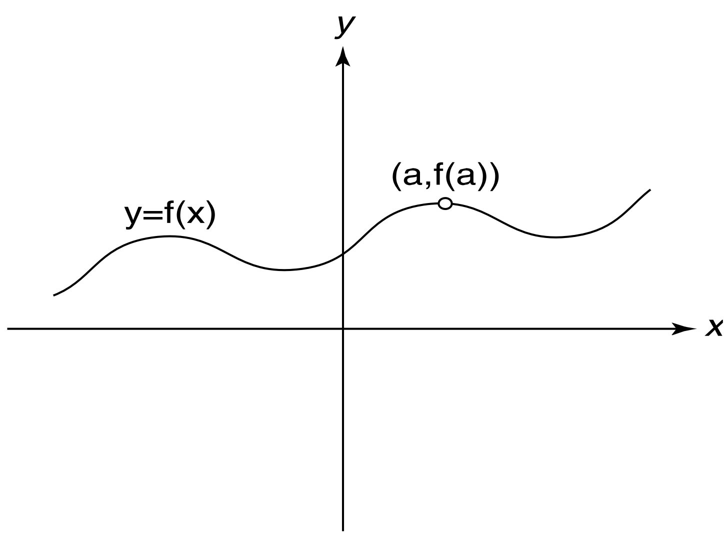
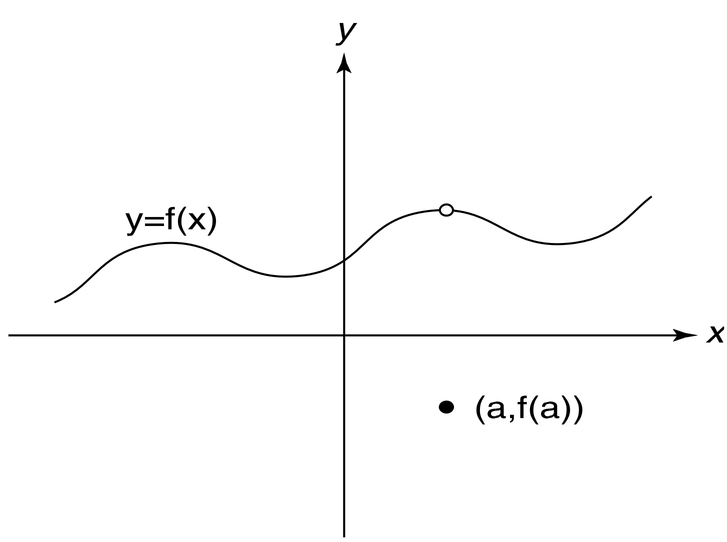


Figure 2-3 *f(a)* and $\lim_{x \to a} f(x)$ are not equal.



Evaluating Limits

Limits of functions are evaluated using many different techniques such as recognizing a pattern, simple substitution, or using algebraic simplifications. Some of these techniques are illustrated in the following examples.

Example 2-1: Find the limit of the sequence: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$

Because the value of each fraction gets slightly larger for each term, while the numerator is always one less than the denominator, the fraction values will get closer and closer to 1; hence, the limit of the sequence is 1.

Example 2-2: Evaluate $\lim_{x \to 2}(3x - 1)$.

As x approaches 2, $3x$ approaches 6, and $3x - 1$ approaches 5; hence, $\lim_{x \to 2}(3x - 1) = 5$.

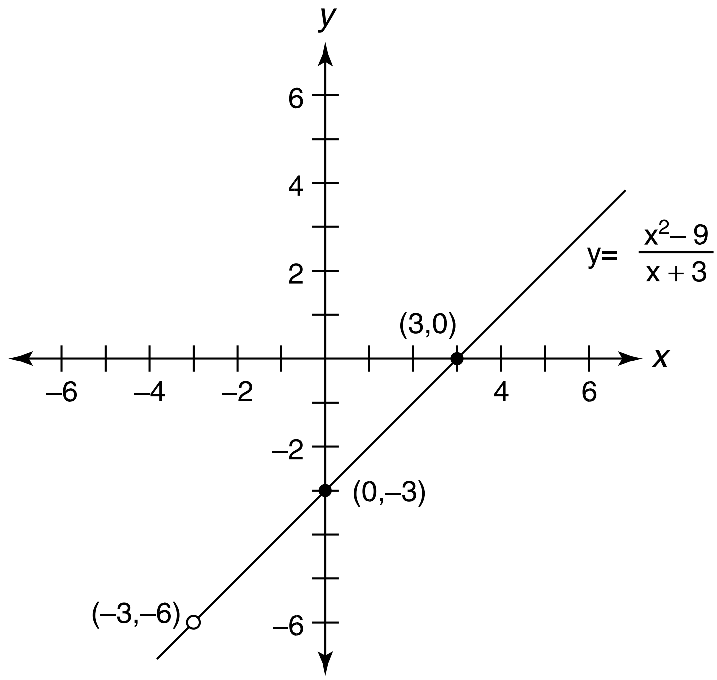
Example 2-3: Evaluate $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$.

Substituting –3 for x yields 0/0, which is meaningless. Factoring first and simplifying, you find that

$$\begin{aligned}\lim_{x \to -3} \frac{x^2 - 9}{x + 3} &= \lim_{x \to -3} \frac{(x+3)(x-3)}{x+3} \\ &= \lim_{x \to -3}(x - 3) \\ &= -6\end{aligned}$$

The graph of $(x^2 - 9)/(x + 3)$ would be the same as the graph of the linear function $y = x - 3$ with the single point (–3,–6) removed from the graph (see Figure 2-4).

Figure 2-4 The graph of $y = (x^2 - 9)/(x + 3)$.



Example 2-4: Evaluate $\lim_{x \to 3} \dfrac{\dfrac{x}{x+2} - \dfrac{3}{5}}{x-3}$.

Substituting 3 for x yields 0/0, which is meaningless. Simplifying the compound fraction, you find that

$$\lim_{x \to 3} \frac{\frac{x}{x+2} - \frac{3}{5}}{x-3} = \lim_{x \to 3} \frac{\frac{x}{x+2} - \frac{3}{5}}{x-3} \cdot \frac{5(x+2)}{5(x+2)}$$

$$= \lim_{x \to 3} \frac{5x - 3(x+2)}{5(x+2)(x-3)}$$

$$= \lim_{x \to 3} \frac{2x - 6}{5(x+2)(x-3)}$$

$$= \lim_{x \to 3} \frac{2(x-3)}{5(x+2)(x-3)}$$

$$= \lim_{x \to 3} \frac{2}{5(x+2)}$$

$$= \frac{2}{25}$$

Example 2-5: Evaluate $\lim_{x \to 0} \frac{x}{x+5}$.

Substituting 0 for x yields $0/5 = 0$; hence, $\lim_{x \to 0} x/(x+5) = 0$.

Example 2-6: Evaluate $\lim_{x \to 0} \frac{x+5}{x}$.

Substituting 0 for x yields 5/0, which is meaningless; here, $\lim_{x \to 0} (x+5)/x$ DNE. (Remember, infinity is not a real number.)

One-sided Limits

For some functions, it is appropriate to look at their behavior from one side only. If x approaches a from the right only, you write

$$\lim_{x \to a^+} f(x)$$

or if x approaches a from the left only, you write

$$\lim_{x \to a^-} f(x)$$

It follows, then, that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Example 2-7: Evaluate $\lim_{x \to 0^+} \sqrt{x}$.

Because x is approaching 0 from the right, it is always positive; $\sqrt{x}$ is getting closer and closer to zero, so $\lim_{x \to 0^+} \sqrt{x} = 0$. Although substituting 0 for x would yield the same answer, the next example illustrates why this technique is not always appropriate.

Example 2-8: Evaluate $\lim_{x \to 0^-} \sqrt{x}$.

Because x is approaching 0 from the left, it is always negative, and $\sqrt{x}$ does not exist. In this situation, $\lim_{x \to 0^-} \sqrt{x}$ DNE. Also, note that $\lim_{x \to 0} \sqrt{x}$ DNE because $\lim_{x \to 0^+} \sqrt{x} = 0 \neq \lim_{x \to 0^-} \sqrt{x}$.

Example 2-9: Evaluate

$$\text{(a)} \lim_{x \to 2^-} \frac{|x-2|}{x-2}$$

$$\text{(b)} \lim_{x \to 2^+} \frac{|x-2|}{x-2}$$

$$\text{(c)} \lim_{x \to 2} \frac{|x-2|}{x-2}$$

(a) As x approaches 2 from the left, $x - 2$ is negative, and $|x-2| = -(x-2)$; hence,

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$$

(b) As x approaches 2 from the right, $x - 2$ is positive, and $|x-2| = x-2$; hence;

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \frac{(x-2)}{x-2} = 1$$

(c) Because $\lim_{x \to 2^-} \frac{|x-2|}{x-2} \neq \lim_{x \to 2^+}$, $\lim_{x \to 2} \frac{|x-2|}{x-2}$ DNE

Infinite Limits

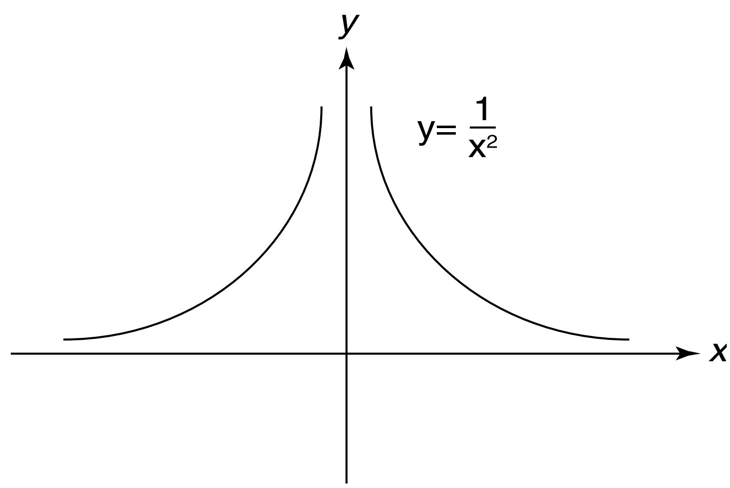
Some functions "take off" in the positive or negative direction (increase or decrease without bound) near certain values for the independent variable. When this occurs, the function is said to have an *infinite limit;* hence, you write $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$. Note also that the function has a vertical asymptote at $x = a$ if either of the above limits hold true.

In general, a fractional function will have an infinite limit if the limit of the denominator is zero and the limit of the numerator is not zero. The sign of the infinite limit is determined by the sign of the quotient of the numerator and the denominator at values close to the number that the independent variable is approaching.

Example 2-10: Evaluate $\lim_{x \to 0} \frac{1}{x^2}$.

As x approaches 0, the numerator is always positive and the denominator approaches 0 and is always positive; hence, the function increases without bound and $\lim_{x \to 0} 1/x^2 = +\infty$. The function has a vertical asymptote at $x = 0$ (see Figure 2-5).

Figure 2-5 The graph of $y = 1/x^2$.



Example 2-11: Evaluate $\lim_{x \to 2^-} \frac{x+3}{x-2}$.

As x approaches 2 from the left, the numerator approaches 5, and the denominator approaches 0 through negative values; hence, the function decreases without bound and $\lim_{x \to 2^-} (x+3)/(x-2) = -\infty$. The function has a vertical asymptote at $x = 2$.

Example 2-12: Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{x^3}\right)$.

Rewriting $1/x^2 - 1/x^3$ as an equivalent fractional expression $(x-1)/x^3$, the numerator approaches –1, and the denominator approaches 0 through positive values as x approaches 0 from the right; hence, the function decreases without bound and $\lim_{x \to 0} \left(1/x^2 - 1/x^3\right) = -\infty$. The function has a vertical asymptote at $x = 0$.

A word of caution: Do not evaluate the limits individually and subtract because $\pm\infty$ are not real numbers. Using this example,

$$\lim_{x \to 0^+}\left(\frac{1}{x^2}-\frac{1}{x^2}\right) \neq \lim_{x \to 0^+}\frac{1}{x^2} - \lim_{x \to 0^+}\frac{1}{x^3} = (+\infty) - (+\infty)$$

which is meaningless.

Limits at Infinity

Limits at infinity are used to describe the behavior of functions as the independent variable increases or decreases without bound. If a function approaches a numerical value L in either of these situations, write

$$\lim_{x \to +\infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

and $f(x)$ is said to have a horizontal asymptote at $y = L$. A function may have different horizontal asymptotes in each direction, have a horizontal asymptote in one direction only, or have no horizontal asymptotes.

Evaluate 2-13: Evaluate $\lim_{x \to +\infty} \dfrac{2x^2+3}{x^2-5x-1}$.

Factor the largest power of x in the numerator from each term and the largest power of x in the denominator from each term.

You find that

$$\lim_{x \to +\infty} \frac{2x^2+3}{x^2-5x-1} = \lim_{x \to +\infty} \frac{x^2\left(2+\frac{3}{x^2}\right)}{x^2\left(1-\frac{5}{x}-\frac{1}{x^2}\right)}$$

$$= \lim_{x \to +\infty} \frac{2+\frac{3}{x^2}}{1-\frac{5}{x}-\frac{1}{x^2}}$$

$$= \frac{2+0}{1-0-0}$$

$$\longrightarrow \lim_{x \to +\infty} \frac{2x^2+3}{x^2-5x-1} = 2$$

The function has a horizontal asymptote at $y = 2$.

Example 2-14: Evaluate $\lim_{x \to +\infty} \frac{x^3-2}{5x^4-3x^3+2x}$.

Factor x^3 from each term in the numerator and x^4 from each term in the denominator, which yields

$$\lim_{x \to -\infty} \frac{x^3-2}{5x^4-3x^3+2x} = \lim_{x \to -\infty} \frac{x^3\left(1-\frac{2}{x^3}\right)}{x^4\left(5-\frac{3}{x}+\frac{2}{x^3}\right)}$$

$$= \lim_{x \to -\infty} \left(\frac{1}{x}\right)\left(\frac{1-\frac{2}{x^3}}{5-\frac{3}{x}+\frac{2}{x^3}}\right)$$

$$= (0)\left(\frac{1-0}{5-0+0}\right)$$

$$= 0$$

The function has a horizontal asymptote at $y = 0$.

Example 2-15: Evaluate $\lim_{x \to +\infty} \frac{9x^2}{x+2}$.

Factor x^2 from each term in the numerator and x from each term in the denominator, which yields

$$\lim_{x \to +\infty} \frac{9x^2}{x+2} = \lim_{x \to +\infty} \frac{x^2(9)}{x\left(1+\frac{2}{x}\right)}$$

$$= \lim_{x \to +\infty} x\left(\frac{9}{1+\frac{2}{x}}\right)$$

$$= \left[\lim_{x \to +\infty}(x)\right]\left[\frac{9}{1+0}\right]$$

$$= \left[\lim_{x \to +\infty} (x)\right][9]$$

$$\lim_{x \to +\infty} \frac{9x^2}{x+2} = +\infty$$

Because this limit does not approach a real number value, the function has no horizontal asymptote as x increases without bound.

Example 2-16: Evaluate $\lim_{x \to -\infty} \left(x^3 - x^2 - 3x\right)$.

Factor x^3 from each term of the expression, which yields

$$\lim_{x \to -\infty} (x^3 - x^2 - 3x) = \lim_{x \to -\infty} (x^3)\left(1 - \frac{1}{x} - \frac{3}{x^2}\right)$$

$$= \lim_{x \to -\infty} (x^3) \cdot \lim_{x \to -\infty} \left(1 - \frac{1}{x} - \frac{3}{x^2}\right)$$

$$= \lim_{x \to -\infty} (x^3) \cdot [1 - 0 - 0]$$

$$= \left[\lim_{x \to -\infty} (x^3)\right] \cdot [1]$$

$$\lim_{x \to -\infty} (x^3 - x^2 - 3x) = -\infty$$

As in the previous example, this function has no horizontal asymptote as x decreases without bound.

Limits Involving Trigonometric Functions

The trigonometric functions sine and cosine have four important limit properties:

$$\lim_{x \to c} \sin x = \sin c$$

$$\lim_{x \to c} \cos x = \cos c$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

You can use these properties to evaluate many limit problems involving the six basic trigonometric functions.

Example 2-17: Evaluate $\lim_{x \to 0} \frac{\cos x}{\sin x - 3}$.

Substituting 0 for x, you find that cos x approaches 1 and sin x – 3 approaches –3; hence,

$$\lim_{x \to 0} \frac{\cos x}{\sin x - 3} = -\frac{1}{3}$$

Example 2-18: Evaluate $\lim_{x \to 0^+} \cot x$.

Because cot x = cos x/sin x, you find $\lim_{x \to 0^+} \cos x/\sin x$. The numerator approaches 1 and the denominator approaches 0 through positive values because we are approaching 0 in the first quadrant; hence, the function increases without bound and $\lim_{x \to 0^+} \cot x = +\infty$, and the function has a vertical asymptote at $x = 0$.

Example 2-19: Evaluate $\lim_{x \to 0} \frac{\sin 4x}{x}$.

Multiplying the numerator and the denominator by 4 produces

$$\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} \frac{4 \sin 4x}{4x}$$
$$= \left(\lim_{x \to 0} 4\right) \cdot \lim_{x \to 0} \frac{\sin 4x}{4x}$$
$$= 4 \cdot 1$$
$$\lim_{x \to 0} \frac{\sin 4x}{x} = 4$$

Example 2-20: Evaluate $\lim_{x \to 0} \frac{\sec x - 1}{x}$.

Because $\sec x = 1/\cos x$, you find that

$$\lim_{x \to 0} \frac{\sec x - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{\cos x} - 1}{x}$$
$$= \lim_{x \to 0} \frac{1 - \cos x}{x \cos x}$$
$$= \lim_{x \to 0} \left(\frac{1}{\cos x}\right) \cdot \left(\frac{1 - \cos x}{x}\right)$$
$$= \left[\lim_{x \to 0} \frac{1}{\cos x}\right] \cdot \left[\lim_{x \to 0} \frac{1 - \cos x}{x}\right]$$
$$= 1 \cdot 0$$
$$\lim_{x \to 0} \frac{\sec x - 1}{x} = 0$$

Continuity

A function $f(x)$ is said to be **continuous** at a point $(c, f(c))$ if each of the following conditions is satisfied:

(1) $f(c)$ is defined (c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) $\lim_{x \to c} f(x) = f(c)$.

Geometrically, this means that there is no gap, split, or missing point for $f(x)$ at c and that a pencil could be moved along the graph of $f(x)$ through $(c, f(c))$ without lifting it off the graph. A function is said to be continuous at $(c, f(c))$ from the right if $\lim_{x \to c^+} f(x) = f(c)$ and continuous at $(c, f(c))$ from the left if $\lim_{x \to c^-} f(x) = f(c)$. Many of our familiar functions such as linear, quadratic and other polynomial functions, rational functions, and the trigonometric functions are continuous at each point in their domain.

A special function that is often used to illustrate one-sided limits is the greatest integer function. The *greatest integer function,* $[x]$, is defined to be the largest integer less than or equal to x (see Figure 2-6).

Some values of $[x]$ for specific x values are

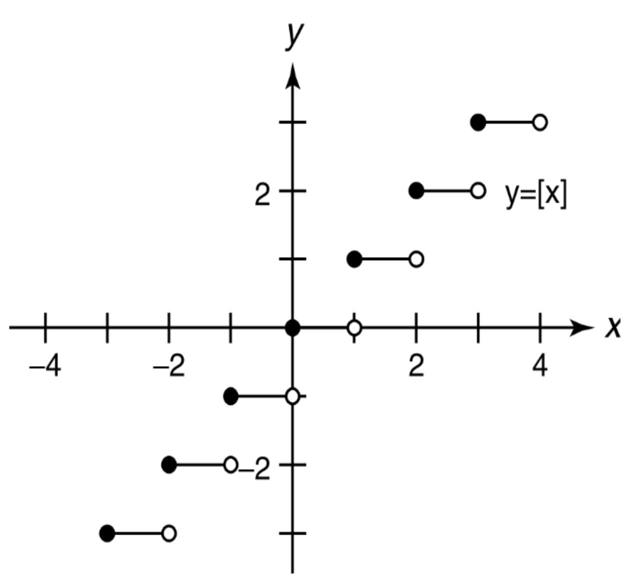
$$[2] = 2$$

$$[5.8] = 5$$

$$\left[-3\frac{1}{3}\right] = -4$$

$$[.46] = 0$$

Figure 2-6 The graph of the greatest integer function $y = [x]$.



The greatest integer function is continuous at any integer n from the right only because

$$f(n) = [n] = n$$

$$\text{and } \lim_{x \to n^+} f(x) = n$$

$$\text{but } \lim_{x \to n^-} f(x) = n - 1$$

hence, $\lim_{x \to n^-} f(x) \neq f(n)$ and $f(x)$ is not continuous at n from the left. Note that the greatest integer function is continuous from the right and from the left at any noninteger value of x.

Example 2-21: Discuss the continuity of $f(x) = 2x + 3$ at $x = -4$.

When the definition of continuity is applied to $f(x)$ at $x = -4$, you find that

$$(1)\ f(-4) = -5$$

$$(2)\ \lim_{x \to -4} f(x) = \lim_{x \to -4} (2x + 3) = -5$$

$$(3)\ \lim_{x \to -4} f(x) = f(-4)$$

hence, f is continous at $x = -4$.

Example 2-22: Discuss the continuity of $f(x) = \frac{x^2 - 4}{x - 2}$ at $x=2$.

When the definition of continuity is applied to $f(x)$ at $x = 2$, you find that $f(2)$ is not defined; hence, f is not continuous (discontinuous) at $x = 2$.

Example 2-23: Discuss the continuity of $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

When the definition of continuity is applied to $f(x)$ at $x = 2$, you find that

$$(1)\ f(2) = 4$$

$$(2)\ \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$$

$$= \lim_{x \to 2} (x + 2) = 4$$

$$(3)\quad \lim_{x \to 2} f(x) = f(2)$$

hence, f is continous at $x = 2$.

Example 2-24: Discuss the continuity of $f(x) = \sqrt{x}$ at $x = 0$.

When the definition of continuity is applied to $f(x)$ at $x = 0$, you find that

(1) $f(0) = 0$

(2) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sqrt{x}$ DNE because $\lim_{x \to 0^+} \sqrt{x} = 0$,

but $\lim_{x \to 0^-} \sqrt{x}$ DNE

(3) $\lim_{x \to 0^+} f(x) = f(0)$

hence, f is continuous at $x = 0$ from the right only.

Example 2-25: Discuss the continuity of $f(x) = \begin{cases} 5 - 2x, & x < -3 \\ x^2 + 2, & x \geq -3 \end{cases}$ at $x = -3$.

When the definition of continuity is applied to $f(x)$ at $x = -3$, you find that

(1) $f(-3) = (-3)^2 + 2 = 11$

(2) $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (5 - 2x) = 11$

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (x^2 + 2) = 11$

hence, $\lim_{x \to 3} f(x) = 11$ because $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x)$

(3) $\lim_{x \to -3} f(x) = f(-3)$

hence, f is continuous at $x = -3$.

Many theorems in calculus require that functions be continuous on intervals of real numbers. A function $f(x)$ is said to be continuous on an open interval (a,b) if f is continuous at each point $c \in (a,b)$. A function $f(x)$ is said to be continuous on a closed interval $[a,b]$ if f is continuous at each point $c \in (a,b)$ and if f is continuous at a from the right and continuous at b from the left.

Example 2-26:

(a) $f(x) = 2x + 3$ is continuous on $(-\infty,+\infty)$ because f is continuous at every point $c \in (-\infty,+\infty)$.

(b) $f(x) = (x - 3)/(x + 4)$ is continuous on $(-\infty,-4)$ and $(-4,+\infty)$ because f is continuous at every point $c \in (-\infty,-4)$ and $c \in (-4,+\infty)$

(c) $f(x) = (x - 3)/(x + 4)$ is not continuous on $(-\infty,-4]$ or $[-4,+\infty)$ because f is not continuous on -4 from the left or from the right.

(d) $f(x) = \sqrt{x}$ is continuous on $[0, +\infty)$ because f is continuous at every point $c \in (0,+\infty)$ and is continuous at 0 from the right.

(e) $f(x) = \cos x$ is continuous on $(-\infty,+\infty)$ because f is continuous at every point $c \in (-\infty,+\infty)$.

(f) $f(x) = \tan x$ is continuous on $(0,\pi/2)$ because f is continuous at every point $c \in (0,\pi/2)$.

(g) $f(x) = \tan x$ is not continuous on $[0,\pi/2]$ because f is not continuous at $\pi/2$ from the left.

(h) $f(x) = \tan x$ is continuous on $[0,\pi/2)$ because f is continuous at every point $c \in (0,\pi/2)$ and is continuous at 0 from the right.

(i) $f(x) = 2x/(x^2 + 5)$ is continuous on $(-\infty,+\infty)$ because f is continuous at every point $c \in (-\infty,+\infty)$.

(j) $f(x) = |x - 2|/(x - 2)$ is continuous on $(-\infty,2)$ and $(2,+\infty)$ because f is continuous at every point $c \in (-\infty,2)$ and $c \in (2,+\infty)$.

(k) $f(x) = |x - 2|/(x - 2)$ is not continuous on $(-\infty,2]$ or $[2,+\infty)$ because f is not continuous at 2 from the left or from the right.

Questions

1. Evaluate the following

(a) $\lim\limits_{x \to 3^+} \dfrac{|x^2 - 9|}{x - 3}$

(b) $\lim\limits_{x \to 3^-} \dfrac{|x^2 - 9|}{x - 3}$

(c) $\lim\limits_{x \to 3} \dfrac{|x^2 - 9|}{x - 3}$

2. Evaluate $\lim\limits_{x \to 2^+} \dfrac{x + 2}{x^2 - 4}$

3. Evaluate $\lim\limits_{x \to +\infty} \dfrac{x^2}{x^3 - 1}$

4. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 5x}{3x}$

5. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 2}, & x \neq -1 \\ 2, & x = -1 \end{cases} \text{ at } x = -1.$$

Answers: 1. (a) 6 (b) –6 (c) DNE **2.** $+\infty$ **3.** 0 **4.** 5/3 **5.** f is not continuous at $x = -1$.

DNE: Does Not Exist