Optimization

Fourth Class

2020 - 2021

By

Dr. Jawad Mahmoud Jassim

Dept. of Math.

Education College for Pure Sciences

University of Basrah

Iraq

Chapter Two

Line Search

Lecture 4

4: Fibonacci Method:

Definition (14): (Fibonacci - Sequence)

The Fibonacci – sequence $\{F_k\}$ is defined as follows:

$$F_0 = F_1 = 1 , \quad F_{k+1} = F_k + F_{k-1} , k = 1, 2, 3 , \cdots \qquad \dots\dots\dots \quad (13)$$

$Thus\ the\ Fibomacci\ numbers\ are\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$

If we use $\frac{F_{n-k}}{F_{n-k+1}}$ instead of τ in (6) and (4) in golden section method, we immediately obtain the formula

$$\lambda_k = a_k + \left[1 - \frac{F_{n-k}}{F_{n-k+1}}\right](b_k - a_k)$$

$$\lambda_k = a_k + \left[\frac{F_{n-k+1} - F_{n-k}}{F_{n-k+1}}\right](b_k - a_k)$$

$$\lambda_k = a_k + \frac{F_{n-k-1}}{F_{n-k+1}}(b_k - a_k) , k = 1, 2, \cdots , n - 1 \dots\dots\dots\dots\dots\dots\dots \quad (14)$$

$$\mu_k = a_k + \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k) \ , k = 1, 2, \cdots , n - 1 \dots\dots\dots\dots\dots\dots \quad (15)$$

Which is called the Fibonacci Formula.

As stated in the last section (Golden Search Method), in Case 1, if $\Phi(\lambda_k) \leq \Phi(\mu_k)$, the new interval of uncertainly is $[a_{k+1}, b_{k+1}] = [a_k, \mu_k]$, so $b_{k+1} - a_{k+1} = \mu_k - a_k$. From (15), we have

$$\mu_k - a_k = \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k)$$. Hence

$$b_{k+1} - a_{k+1} = \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Which give a reduction in each iteration. This is also true for Case 2.

Assume that we ask for the length of finial interval no more than δ, *i.e.* $b_n - a_n \leq \delta$.

جامعة البصرة * كلية التربية
الدكتور
جواد
محمود
جاسم

If we put $k + 1 = n$ in (16), we have

$$b_n - a_n = \frac{F_1}{F_2}(b_{n-1} - a_{n-1})$$
$$= \frac{F_1}{F_2}\frac{F_2}{F_3}\frac{F_3}{F_4}\cdots\frac{F_{n-1}}{F_n}(b_1 - a_1) = \frac{F_1}{F_n}(b_1 - a_1) = \frac{1}{F_n}(b_1 - a_1)$$

$$\therefore \quad b_n - a_n = \frac{1}{F_n}(b_1 - a_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (17)$$

Since $b_n - a_n \leq \delta$

$$\therefore \quad \frac{1}{F_n}(b_1 - a_1) \leq \delta$$

$$\therefore F_n \geq \frac{b_1 - a_1}{\delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

The procedure of the Fibonacci method is similar to the Algorithm of Golden Section Method (Algorithm 2).

الدكتور
جواد
محمود
جاسم

Let $F_k = r^k$. Since $F_{k+1} = F_k + F_{k-1}$, $k = 1, 2, \cdots$

$$\therefore r^{k+1} = r^k + r^{k-1} \;\rightarrow\; r^k r = r^k + r^k r^{-1} \;\rightarrow r^k r - r^k - r^k r^{-1} = 0 \;\rightarrow$$

$$r^k\left(r - 1 - r^{-1}\right) = 0.$$

Since $r^k \neq 0$ (*properties of the exponential function*)

$$\therefore r - 1 - r^{-1} = 0 \;\rightarrow\; r^2 - r - 1 = 0 \;\dots\dots\dots\dots\dots\dots\dots\dots\dots.\; (19)$$

$$\therefore r = \frac{1 \mp \sqrt{5}}{2} \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \;(20)$$

Then the general solution of the difference equation $F_{k+1} = F_k + F_{k-1}$ is

$$F_k = A r_1^k + B r_2^k \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \;(21)$$

Using the initial conditions $F_0 = F_1 = 1$, we get

$$A + B = 1 \dots\dots\dots\dots\dots\dots.. \;(*)$$

$$A r_1 + B r_2 = 1 \dots\dots\dots\dots.. \;(**)$$

Now, we solve the above two equations to find A *and* B.

Multiply the equation $(*)$ *by* r_1, we get

$$A r_1 + B r_1 = r_1 \dots\dots\dots\dots. \;(***)$$

الدكتور
جواد
محمود
جاسم

Now, the difference between the equations $(***)$ *and* $(**)$, *we get*

$B = \frac{r_1 - 1}{r_1 - r_2}$. Since $r_1 = \frac{1+\sqrt{5}}{2}$ *and* $r_2 = \frac{1-\sqrt{5}}{2} \rightarrow r_1 - r_2 = \sqrt{5}$ *and*

$$r_1 - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{1+\sqrt{5}-2}{2} = \frac{-1+\sqrt{5}}{2} = -\frac{1-\sqrt{5}}{2} = -r_2 .$$

$\therefore \; B = -\frac{r_2}{\sqrt{5}}$. From the equation $(*)$, *we have*

$$A = 1 - B = 1 - \left(-\frac{r_2}{\sqrt{5}}\right) = 1 + \frac{r_2}{\sqrt{5}} = \frac{\sqrt{5} + r_2}{\sqrt{5}}$$

$$\therefore A = \frac{\sqrt{5} + \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2}}{\sqrt{5}} = \frac{r_1}{\sqrt{5}} .$$

Substituting the values of A *and* B *in the equation* (21), *we get*

$$F_k = Ar_1^k + Br_2^k = \frac{1}{\sqrt{5}}\left[r_1^{k+1} - r_2^{k+1}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \quad (22)$$

الدكتور
جواد
محمود
جاسم

Theorem (6):

$$\lim_{k\to\infty}\frac{F_{k-1}}{F_k}=\frac{\sqrt{5}-1}{2}=\tau, \textit{where } \tau \textit{ is given in Golden Section Method}.$$

Proof:

Since $F_{k+1}=F_k+F_{k-1}\ , k=1,2,\cdots$

$$\therefore \quad F_{k-1}=F_{k+1}-F_k \ \rightarrow\ \frac{F_{k-1}}{F_k}=\frac{F_{k+1}-F_k}{F_k}=\frac{F_{k+1}}{F_k}-1\ .$$

Let $\lim\limits_{k\to\infty}\frac{F_{k-1}}{F_k}=L\ >0\ .$

$$\therefore \frac{1}{\frac{F_{k-1}}{F_k}}=\frac{F_k}{F_{k-1}}=\frac{F_{k+1}}{F_k}.$$

$\therefore$We can write $\lim\limits_{k\to\infty}\frac{F_{k+1}}{F_k}=\frac{1}{L}$

$$\therefore \lim_{k\to\infty}\frac{F_{k-1}}{F_k}=\lim_{k\to\infty}\frac{F_{k+1}}{F_k}-1\ .$$

$$\therefore L=\frac{1}{L}-1\ \rightarrow\ L^2+L-1=0\ \rightarrow L=\frac{-1\mp\sqrt{2}}{2}\ .$$

$$\therefore \lim_{k\to\infty}\frac{F_{k-1}}{F_k}=\frac{\sqrt{5}-1}{2}=\tau\left((\textit{Since } L>0)\right).$$

الدكتور
جواد
محمود
جاسم

Note (14):

Theorem (6) shows that when $k \to \infty$ the Fibonacci method and the golden section method have the same reduction rate of the interval of uncertainly. Therefore the Fibonacci method converges with converges ratio τ .

Algorithm (3): (Fibonacci Method)

Step 1:

Let initial interval $[a_1, b_1]$ be given and the number of function evaluations $N-1, (N \geq 3)$ be preset.

Step 2:

For $k = 1, 2, \cdots, N-2$, compute

$x_1^k = \frac{F_{N-k-1}}{F_{N-k+1}}[b_k - a_k] + a_k$ *and* $x_2^k = \frac{F_{N-k}}{F_{N-k+1}}[b_k - a_k] + a_k$. **Then calculate** $f(x_1^k)$ *and* $f(x_2^k)$.

Step 3:

If $f(x_2^k) > f(x_1^k)$, *set* $a_{k+1} = a_k$, $b_{k+1} = x_2^k$, and go to step 5. Otherwise go to step 4.

Step 4:

If $f(x_2^k) \leq f(x_1^k)$, *set* $a_{k+1} = x_1^k$, $b_{k+1} = b_k$, and go to step 5.

الدكتور
جواد
محمود
جاسم

Step 5:

If $k = N-2$, stop and determine the required interval is $[a_{k+1}, b_{k+1}]$. Otherwise go to step 2.

<u>H.W.</u>

Use the Fibonacci method to find the location of the minimizer for the function

$$f(x) = \begin{cases} \frac{x}{2} & for\ x \le 2 \\ -x + 3 & for\ x > 2 \end{cases}.$$

Use the initial interval $[0, 3]$ $and\ N = 6$.

الدكتور
جواد
محمود
جاسم