Optimization

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Chapter Two

Line Search

Lecture 3

<u>3: The Golden Section Method</u>

Let $\Phi(\alpha) = f(X + \alpha d)$ be a unimodal function on the $interval$ $[a, b]$.

At the iteration k of the golden section method, let $the\ interval\ \ of\ uncertainly\ be$ $[a_k\ , b_k]$.

$Take\ two\ observations\ \lambda_k\ and\ \mu_k\ \epsilon\ [a_k\ , b_k]\ such\ that$ $\lambda_k\ < \mu_k$. Evaluate $\Phi(\lambda_k)\ and\ \Phi(\mu_k)$.

Then we have two cases:

<u>Case 1:</u> if $\Phi(\lambda_k) < \Phi(\mu_k)$, then set $a_{k+1} = a_k\ , b_{k+1} = \mu_k$.

<u>Case 2:</u> if $\Phi(\lambda_k) > \Phi(\mu_k)$, then set $a_{k+1} = \lambda_k\ , b_{k+1} = b_k$.

How to choose the observations λ_k *and* μ_k ?

We require that λ_k *and* μ_k satisfy the following conditions:

1: The distance from λ_k *and* μ_k to the end points of the *interval* $[a_k , b_k]$ are *equivalent, that is*:

$$b_k - \lambda_k = \mu_k - a_k.$$

2: *The reduction rate of the intervals of uncertainly for each iteration is the same, that is*:

$$b_{k+1} - a_{k+1} = \tau(b_k - a_k), \tau \epsilon (0, 1).$$

3: Only one extra observation is needed for each new iteration.

Now, we consider case 1:

Since

$$a_{k+1} = a_k \,, b_{k+1} = \mu_k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

And

$$b_{k+1} - a_{k+1} = \tau(b_k - a_k) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (2)$$

From (1) and (2) we have

$$\mu_k - a_k = \tau(b_k - a_k) \dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (3)$$

$$\therefore \quad \mu_k = \tau(b_k - a_k) + a_k \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (4)$$

Since

$$b_k - \lambda_k = \mu_k - a_k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (5)$$

∴From (4) and (5), we have:

$$\lambda_k = b_k - \mu_k + a_k \quad = b_k - (\tau(b_k - a_k) + a_k) + a_k$$

$$\lambda_k \;= (b_k - a_k \;) - \tau(b_k - a_k) + a_k$$

$$\therefore \quad \lambda_k = (1 - \tau)(b_k - a_k \;) + a_k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (6)$$

Note that, in this case the new interval is $[a_{k+1}, b_{k+1}] = [a_k, \mu_k]$.
For further reducing the interval of uncertainly, the observations λ_{k+1} and μ_{k+1} are selected.

Since $\mu_k = a_k + \tau(b_k - a_k)$

$$\therefore \mu_{k+1} = a_{k+1} + \tau(b_{k+1} - a_{k+1}) \,.$$

∴From case 1 and (4), we get

$$\mu_{k+1} = a_k + \tau(\mu_k - a_k) = a_k + \tau[a_k + \tau(b_k - a_k) - a_k]$$

$$\therefore \mu_{k+1} = a_k + \tau^2(b_k - a_k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

If we set

$$\tau^2 = 1 - \tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Then

$$\mu_{k+1} = a_k + (1 - \tau)(b_k - a_k) = \lambda_k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

It means that the new observation μ_{k+1} does not need to compute because coincides with λ_k.

Similarly, if we consider Case 2, the new observation λ_{k+1} coincides with μ_k.

Therefore, for each new iteration, only one extra observation is needed, which is just required by third condition.

Now, what is the reduction rate of the interval of uncertainly for each iteration?

By solving the equation $\tau^2 = 1 - \tau$ **, we immediately obtain**

$$\tau^2 + \tau - 1 = 0 \;\rightarrow\; \tau = \frac{-1 \mp \sqrt{1+4}}{2} = \frac{-1 \mp \sqrt{5}}{2}.$$

Since $\tau > 0$ **, then take**

$$\tau = \frac{b_{k+1} - a_{k+1}}{b_k - a_k} = \frac{-1 + \sqrt{5}}{2} \cong 0.618 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Then the formulas (6) and (4) can be written as

$$\lambda_k = a_k + 0.382(b_k - a_k) \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

$$\mu_k = a_k + 0.618(b_k - a_k) \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

Therefore, the golden section method is also called the 0.618 **method.**

Algorithm 2: (The Golden Section Method)

Step 1: (Initial Step)

Determine the initial interval $[a_1, b_1]$ and give the precision $\delta > 0$.

Compute initial observations λ_1 *and* μ_1 as

$$\lambda_1 = a_1 + 0.382(b_1 - a_1) \text{ and } \mu_1 = a_1 + 0.618(b_1 - a_1).$$

***Evaluate* $\Phi(\lambda_1)$ *and* $\Phi(\mu_1)$. Set $k = 1$.**

Step 2: (Compare the function values)

If $\Phi(\lambda_k) > \Phi(\mu_k)$, go to step 3.

If $\Phi(\lambda_k) \leq \Phi(\mu_k)$, go to step 4.

Step 3: (Case 2)

If $b_k - \lambda_k \leq \delta$, *stop and output* μ_k ; otherwise set

$a_{k+1} = \lambda_k$, $b_{k+1} = b_k$, $\lambda_{k+1} = \mu_k$, $\Phi(\lambda_{k+1}) = \Phi(\mu_k)$,

$\mu_{k+1} = a_{k+1} + 0.618\,(b_{k+1} - a_{k+1})$.

***Evaluate* $\Phi(\mu_{k+1})$ *and go to step* 5.**

Step 4: (Case 1)

If $\mu_k - a_k \leq \delta$ *, stop and output* λ_k *; otherwise set*

$$a_{k+1} = a_k \, , b_{k+1} = \mu_k \, , \mu_{k+1} = \lambda_k, \Phi(\mu_{k+1}) = \Phi(\lambda_k),$$

$$\lambda_{k+1} = a_{k+1} + 0.382 \, (b_{k+1} - a_{k+1}) \, .$$

Evaluate $\Phi(\lambda_{k+1})$ *and go to step* 5.

Step 5:

Set $k = k + 1$ *and go to step* 2.

H.W:

Use golden section method to find the location of the minimizer for $f(x) = x(1.5 - x)$.

Take the initial interval $(0, 1)$ *and* $\delta = 0.05$.