Optimization

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Chapter One

Basic Concepts

Lecture 9
Solved Problems

Problem (5):

Let the function $f: R^2 \rightarrow R \; is \; defined \; by$

$f(X) = 3x_1^2 - x_2^2 + x_1^3$. Show that:

A: The point $X^* = [-2, 0]^T$ is a local maximizer point for the function f.

B: The point $X^* = [0, 0]^T$ is a saddle point for the function f .

C: The function f has no local minimizers.

Solution:

A: We must prove the point $X^* = [-2, 0]^T$ is a local minimizer for the function $-f, say \; h = -f$.

$\therefore h(X) = -f(X) = -3x_1^2 + x_2^2 - x_1^3$.

1: We find the gradient of $h(X)$ as follows:

$$g(X) = \left[\frac{\partial h}{\partial x_1} \quad , \frac{\partial h}{\partial x_2}\right]^T = [-6x_1 - 3x_1^2 \quad , 2x_2]^T$$

2: We find the critical points as follows:

$$-6x_1 - 3x_1^2 = 0 \quad \dots \dots \dots \dots \dots \dots \dots \dots . (1)$$

$$2x_2 = 0 \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (2)$$

From (1), we have

$$-3x_1(2 + x_1) = 0 \ \rightarrow x_1 = 0 \ or \ x_1 = -2.$$

From (2), we have $x_2 = 0$.

∴The critical points are: $[0, 0]^T$ and $[-2, 0]^T$.

3: We find the Hessian matrix of $h(X)$ as follows:

$$\frac{\partial^2 h}{\partial x_1^2} = -6 - 6x_1 \,, \quad \frac{\partial^2 h}{\partial x_1 \partial x_2} = 0 = \frac{\partial^2 h}{\partial x_2 \partial x_1} \,, \quad \frac{\partial^2 h}{\partial x_2^2} = 2.$$

$\therefore$**The Hessian matrix of $h(X)$ is given by**

$$H(X) = \begin{bmatrix} -6 - 6x_1 & 0 \\ 0 & 2 \end{bmatrix}$$

If $X^* = [-2, 0]^T \rightarrow H(X^*) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}.$

4: We must prove $H(X^*)$ is positive definite matrix.

Let $Y \neq 0 \; in \; R^2$. We want to prove $Y^T H(X^*) Y > 0$.

$$\therefore Y^T H(X^*) Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6y_1 & 2y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 6y_1^2 + 2y_2^2 > 0 \; for \; all \; Y \neq 0.$$

$\therefore$ **$H(X^*)$ is positive definite matrix.**

$\therefore$The point $X^* = [-2, 0]^T$ is a local minimizer for the function $h(X) = -f(X)$.

$\therefore$ The point $X^* = [-2, 0]^T$ is a local maximizer for the function $f(X)$.

B:

1: We find the gradient vector for $f(X)$ as follows:

$$g(X) = \left[\frac{\partial f}{\partial x_1} \quad , \frac{\partial f}{\partial x_2}\right]^T = [6x_1 + 3x_1^2 \quad , -2x_2]^T$$

2: We find the Hessian matrix for $f(X)$ as follows:

$$\frac{\partial^2 f}{\partial x_1^2} = 6 + 6x_1 \, , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 = \frac{\partial^2 f}{\partial x_2 \partial x_1} , \quad \frac{\partial^2 f}{\partial x_2^2} = -2.$$

$\therefore$ **The Hessian matrix of $f(X)$ is given by**

$$G(X) = \begin{bmatrix} 6+6x_1 & 0 \\ 0 & -2 \end{bmatrix}.$$

If $X^* = [0,0]^T$, **then** $G(X^*) = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}.$

3: We determine whether $G(X^*)$ positive definite or not.

Let $Y \neq 0$.

$$\therefore Y^T G(X^*) Y = [y_1 \quad , y_2] \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [6y_1 \quad , -2y_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 6y_1^2 - 2y_2^2.$$

$\therefore$ **The last expression is not positive every where, for example if we take $Y = [1,0]^T$, then $Y^T G(X^*) Y = 6 > 0$ *while if we take* $Y = [0,1]^T$, then $Y^T G(X^*) Y = -2 < 0$.**

$\therefore X^* = [0,0]^T$ **is not local minimizer for** $f(X)$.

4: We find the Hessian matrix for the function $\mathbf{h(X)} = -f(X)$.

Since the Hessian matrix for the function $\mathbf{h(X)}$ **is given by**

$$H(X) = \begin{bmatrix} -6-6x_1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$\therefore H(X^*) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}, \text{ where } X^* = [0,0]^T.$$

5: We determine whether $H(X^*)$ **is positive definite or not.**

Let $V \neq 0$.

$$\therefore V^T H(X^*)\, V = [v_1 \quad v_2] \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= [-6v_1 \quad 2v_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -6v_1^2 + 2v_2^2 .$$

$\therefore$**The last expression is not positive every where, for example**

if we take $V = [1 \quad 0]^T$, then $V^T H(X^*)\, V = -6 < 0$,
$while\ if\ we\ take$ $V = [0 \quad 1]^T$, then $V^T H(X^*)\, V = 2 > 0$.
$\therefore X^* = [0, 0]^T$ is not a local minimizer for the function $-f(X)$.
$\therefore X^* = [0, 0]^T$ is not a local maximizer for the function $f(X)$.
$\therefore X^* = [0, 0]^T$ is not a local maximizer and not a local minimizer for the function $f(X)$.
$\therefore X^* = [0, 0]^T$ is a saddle point for the function $f(X)$.

C: Since the function $f(X)$ has only two critical points $[-2, 0]^T$ and $[0, 0]^T$. The point $[-2, 0]^T$ is a local maximum point and the point is a saddle point.
Then we conclude that the function $f(X)$ has no minimizers.

Problem (6):

Let the function $f\colon R^n \to R \text{ is defined by}$

$f(X) = \frac{1}{2} X^T A X - b^T X + c$, where A is a given $n \times n$ matrix, b is a given $n \times 1$ vector and c is a given real number. Let X^* be the solution of the system of linear equations $AX = b$. Show that if A is positive definite then X^* is a strong local minimizer of $f(X)$.

Solution:

By Theorem (7):

If X^* is a critical point and $G(X^*)$ is positive definite matrix, where $G(X^*)$ is the Hessian matrix of $f(X)$ at X^*, then X^* is a strong local minimizer of $f(X)$.

Since the gradient of $f(X)$ is given by: $g(X) = AX - b$ and

X^* **is the solution of the system** $AX = b, that\ is\ AX^* = b\ or$
$AX^* - b = 0\ .\ This\ means\ g(X^*) = AX^* - b = 0.$

$\therefore$ **X^* is the critical point for $f(X)$.**

The Hessian matrix of $f(X)$ is given by: $G(X) = A \rightarrow G(X^*) = A$ **.**

Since A is positive definite matrix.

$\therefore$**By Theorem (7), we conclude that the point X^* is a strong local minimizer for $f(X)$.**