Optimization

Fourth Class

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Chapter One

Basic Concepts

Lecture 5

12: Optimality Conditions for Unconstrained Optimization

Definition (12): (Local Minimizer)

Let $f: R^n \to R$ be a given function and D be a given set in R^n. The point $X^* \epsilon\, D$ is called a local minimizer of f over D if and only if there exists $\varepsilon > 0$ such that $B(X^*, \varepsilon) \subset D$ *and* $f(X) \geq f(X^*)$ *for all* $X \epsilon\, B(X^*, \varepsilon)$, where $B(X^*, \varepsilon)$ is the open ball with center X^* and radius ϵ defined by

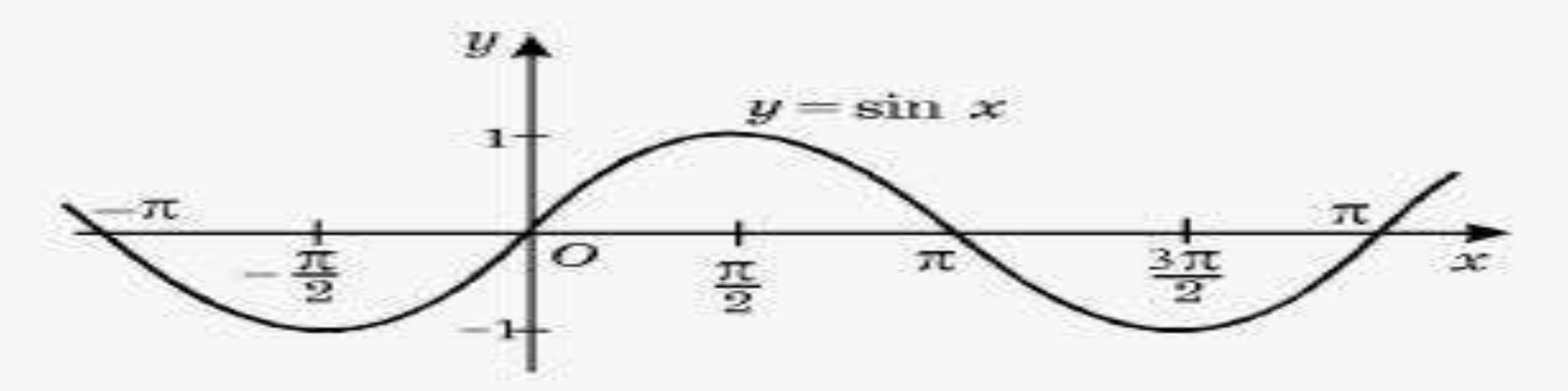
$B(X^*, \varepsilon) = \{X \epsilon\, R^n : \|X - X^*\| < \epsilon\}$ *and* $\|X - X^*\|$ *means the distance between* X *and* X^* in the normed linear space $(R^n, \| . \|)$.

A point $X^* \epsilon\, D$ is called a strict local minimizer of f over D if and only if there exists $\varepsilon > 0$ such that $B(X^*, \varepsilon) \subset D$ *and* $f(X) > f(X^*)$ *for all* $X \epsilon\, B(X^*, \varepsilon), X \neq X^*$.

Example (6):

Let $f: R \to R$ be defined by $f(x) = \sin x \ , x \in R$ in the normed linear space $(R, |.|)$ and $D = [0, 4\pi] \subset R$. Show that $X^* = \frac{3\pi}{2}$ is a strict local minimizer of $f \ over \ D$.

Solution:



Let $\varepsilon > 0 \ such \ that$

$B(X^*, \varepsilon) = \{X \in R : \ |X - X^*| < \varepsilon\} = (X^* - \varepsilon, X^* + \varepsilon) \subset D.$

$Take \ \ 0 < \varepsilon < \frac{3\pi}{2}$. Then $f(X) > f\left(\frac{3\pi}{2}\right) for \ all \ X \ in \ (X^* - \varepsilon, X^* + \varepsilon), \ X \neq \frac{3\pi}{2}$.

Hence $X^* = \frac{3\pi}{2}$ is a strict local minimizer of f over D.

Example (7):

Let $f: R \to R$ be defined by $f(x) = \begin{cases} 1 & , x \le \frac{1}{2} \\ 0 , & \frac{1}{2} < x < \frac{3}{2} \\ 1 , & \frac{3}{2} \le x \le \frac{5}{2} \\ 0, & x > \frac{5}{2} \end{cases}$. $Let\ D = (\frac{1}{2}, \frac{3}{2}) \subset R$.

Show that any point X^* in $(\frac{1}{2}, \frac{3}{2})$ is a local minimizer of f over $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Solution:

Let $\varepsilon > 0$ $such\ that$

$B(X^*, \varepsilon) = \{X \epsilon R : |X - X^*| < \varepsilon\} = (X^* - \varepsilon, X^* + \varepsilon) \subset \left(\frac{1}{2}, \frac{3}{2}\right)$.

Then $f(X) = f(X^*)$ for all values of X $in = (X^* - \varepsilon, X^* + \varepsilon)$.

Hence X^* is a local minimizer of f over $\left(\frac{1}{2}, \frac{3}{2}\right)$.

<u>Definition (13): (Global Minimizer)</u>

Let $f\colon R^n \to R$ be a given function and D be a given $set\ in\ R^n$. The point $X^* \epsilon\ D$ is called a global minimizer of f over D if and only if $f(X) \geq f(X^*)$ for all $X\ \epsilon\ D$.

The point $X^* \epsilon\ D$ is called a strict global minimizer of f over D if and only if $f(X) \geq f(X^*)$ for all $X\ \epsilon\ D\ , X \neq X^*$.

<u>Note (13):</u>

The norm $\|X\|_E$ is called Euclidean norm and defined as:

$$\|X\|_E = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, X = [x_1\ , x_2\ , x_3\ , \cdots, x_n\ ]^T.$$

<u>Example (8):</u>

Let $f\colon R^2 \to R\ defined\ by\ f(X) = x_1^2 + x_2^2\ , X\ \epsilon\ R^2$.

Let $D = B(0,1) = \{X\ \epsilon\ R^2\colon\ \|X\|_E < 1\}$. Show that $X^* = 0$ is a strict global minimizer of f over D.

<u>Solution:</u>

Notice that, geometrically, the set D consists of all points X in the interior of the circular disk with $center$ 0 $and\ radius$ 1. Clearly $f(X) > f(0)\ for\ all\ X\ \epsilon\ D$, $X \neq 0$. So, $X^* = 0$ is a strict global minimizer of f over D.

Example (9):

Let f: $R^2 \to R$ *defined by* $f(X) = (x_2 - x_1^2)^2 + (1 - x_1)^2$. **Show that** $X^* = [1, 1]^T$ *is a strict global minimizer of f over* R^2.

Solution:

$Let\ \delta = [\delta_1\ , \delta_2]^T \subset R^2$.

$$\therefore f(X^* + \delta) = f(1 + \delta_1, 1 + \delta_2) = [1 + \delta_2 - (1 + \delta_1)^2]^2 + [1 - (1 + \delta_1)]^2$$

$$= [1 + \delta_2 - \left(1 + 2\,\delta_1 + \delta_1^2\right)]^2 + [1 - 1 - \delta_1]^2$$

$$= [1 + \delta_2 - 1 - 2\delta_1 - \delta_1^2]^2 + \delta_1^2 = [\delta_2 - 2\delta_1 - \delta_1^2]^2 + \delta_1^2.$$

Since $f(X^*) = f(1, 1) = 0$.

$$\therefore f(X^* + \delta) = 0 + [\delta_2 - 2\delta_1 - \delta_1^2]^2 + \delta_1^2 = f(X^*) + [\delta_2 - 2\delta_1 - \delta_1^2]^2 + \delta_1^2\ .$$

Since $[\delta_2 - 2\delta_1 - \delta_1^2]^2 > 0\ \ and\ \delta_1^2 > 0\ , for\ all\ \delta \neq 0$.

$$\therefore f(X^* + \delta) > \ f(X^*), for\ all\ \ \delta \neq 0\ .$$

Hence X^* **is a strong global minimizer** $of\ f\ over\ R^2$.

Example (10):

Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} (x-1)^2 \,, x \;\; not\ in\ the\ interval\ \ [0,2] \\ \quad 1 \qquad\qquad\quad , x\ in\ the\ interval\ \ [0,2] \end{cases}.$$

Show that any point in $[0,2]$ is a global minimizer $of\ f\ over\ R^2$.

Solution:

Let x^* be any point $in\ the\ interval\ [0,2]$, then $f(x^*) = 1$
and $f(x^*) > 1\ for\ all\ x^* not\ in\ the\ interval\ [0,2]$.

Hence x^* is a global minimizer $of\ f\ over\ R^2$.

Definition (14): (Maximizer)

Let $f: R^n \to R$ be a given function and D be a given $set\ in\ R^n$. The point $X^* \epsilon\ D$ is called a maximizer of f over D if and only if X^* is a minimizer of $-f\ over\ D$.

Definition (15): (Critical Point)

Let $f: R^n \to R\ have\ first\ partial\ derivatives\ in\ D \subset R^n$. The point $X^* \epsilon\ D$ is called a critical point $f\ in\ D$ if and only if $g(X^*) = 0\ , where\ g$ is the gradient vector of f.

Example (11):

Let $f: R^2 \rightarrow R$ *be defined by* $f(X) = x_1^3 - 2x_1^2x_2 + x_2^2, X \epsilon R^2$. Find the critical points of f.

Solution:

First, we find the gradient of f *which is* $g(X) = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}]^T$.

$\frac{\partial f}{\partial x_1} = 3x_1^2 - 4x_1x_2$, $\frac{\partial f}{\partial x_2} = -2x_1^2 + 2x_2$.

$\therefore g(X) = [3x_1^2 - 4x_1x_2, -2x_1^2 + 2x_2]^T$.

Second, put $g(X) = 0$.

$\therefore \qquad 3x_1^2 - 4x_1x_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

and $\quad -2x_1^2 + 2x_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

From (2) we get $x_2 = x_1^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3) and (1) we get $3x_1^2 - 4x_1^3 = 0 \rightarrow x_1^2(3 - 4x_1) = 0 \rightarrow$

$x_1 = 0$ *or* $x_1 = \frac{3}{4}$. From (3) we *get* $x_2 = 0$ *or* $x_2 = \frac{9}{16}$.

$\therefore$*The critical points are* $[0, 0]^T$ *and* $[\frac{3}{4}, \frac{9}{16}]^T$.