Optimization

Fourth Class

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Chapter One

Basic Concepts

Lecture 4

Definition (11): (Convex Function)

Let $D \subset R^n$ be a nonempty convex set. Let $f: D \subset R^n \rightarrow R$. If, for any $X, Y \in D$ and all $\alpha \in (0,1)$, we have

$$f(\alpha X + (1-\alpha)Y) \leq \alpha f(X) + (1-\alpha) f(Y).$$

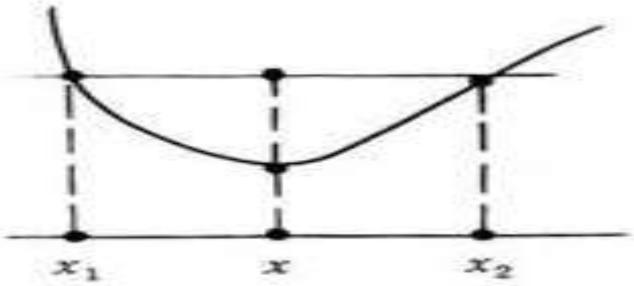
Then f is said to be convex function on D.

If the above inequality is true as a strict inequality for all $X \neq Y$, i.e.

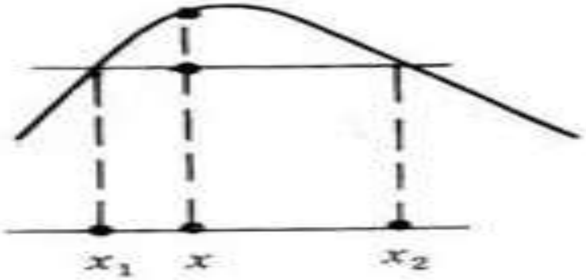
$$f(\alpha X + (1-\alpha)Y) < \alpha f(X) + (1-\alpha) f(Y).$$

Then f is called strictly convex function on D.

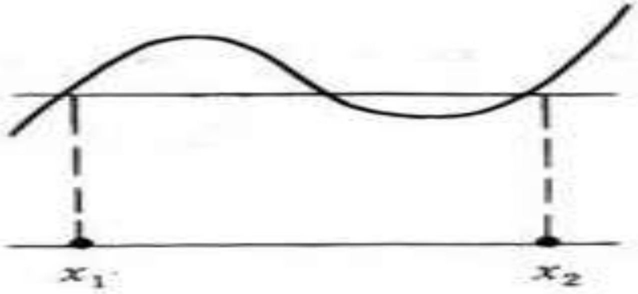
If $-f$ is a convex (strictly convex) function on D, then f is said to be a concave (strictly concave) function on D.



$x = \alpha x_1 + (1-\alpha) x_2$
Convex Function



Concave Function



Nonconvex and Nonconcave Function

<u>Note (13):</u>

The geometrical interpolation of a convex function says that the function values are below the corresponding chord, that is, the values of a convex function at points on the line segment $\alpha x_1 + (1-\alpha)x_2$ are less than or equal to the height of the chord joining the points $\left(X, f(X)\right)$ *and* $\left(Y, f(Y)\right)$.

<u>Example (4):</u>

Show that the function $f: R \to R$ defined by $f(x) = (1-x)^2$, $x \epsilon R$ is strictly convex on any interval $[a, b] \subset R$.

Solution:

$Let\ x, y \in R\ and\ \alpha \in (0,1).$

We want to prove that $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$.

$\therefore f(\alpha x+(1-\alpha)y) = [1-\{\alpha x+(1-\alpha)y\}]^2$

$= 1-2\{\alpha x+(1-\alpha)y\}+\{\alpha x+(1-\alpha)y\}^2$

$= 1-2\alpha x-2(1-\alpha)y+\alpha^2x^2+2\alpha(1-\alpha)xy+(1-\alpha)^2y^2$

$= 1-2\alpha x-2(1-\alpha)y+\alpha^2x^2+2\alpha(1-\alpha)xy+(1-\alpha)^2y^2$

$+\alpha-\alpha+\alpha x^2-\alpha x^2+(1-\alpha)y^2-(1-\alpha)y^2$

$= [\alpha-2\alpha x+\alpha x^2]+[1-\alpha-2(1-\alpha)y+(1-\alpha)^2y^2]+$

$[\alpha^2x^2-\alpha x^2]$

$+[2\alpha(1-\alpha)xy]+[(1-\alpha)^2y^2-(1-\alpha)y^2]$

$= \alpha[1-2x+x^2]+(1-\alpha)[1-2y+y^2]+[\alpha^2-\alpha]x^2+$

$2\alpha(1-\alpha)xy+[(1-\alpha)^2-(1-\alpha)]y^2$

$$= \alpha(1-x)^2+(1-\alpha)(1-y)^2 + \alpha(\alpha-1)x^2 + 2\alpha(1-\alpha)xy$$
$$+[(1-\alpha)(1-\alpha-1)y^2$$
$$= \alpha(1-x)^2+(1-\alpha)(1-y)^2 - \alpha(1-\alpha)x^2 + 2\alpha(1-\alpha)xy - \alpha(1-\alpha)y^2$$
$$= \alpha(1-x)^2+(1-\alpha)(1-y)^2 - \alpha(1-\alpha)[x^2-2xy+y^2]$$
$$= \alpha(1-x)^2+(1-\alpha)(1-y)^2 - \alpha(1-\alpha)(x-y)^2$$
$$= \alpha f(x) + (1-\alpha)f(y) - \alpha(1-\alpha)(x-y)^2$$
$$\therefore f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y) - \alpha(1-\alpha)(x-y)^2$$
$$\therefore f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y).$$

$\therefore f$ **is strictly convex function.**

Example (5):

Let $f: R^n \to R$ $defined\ by\ f(X) = \frac{1}{2} X^T AX + b^T X + c\ , X \epsilon R^n$, where $A\ is\ an\ n \times n$ matrix, $b\ is\ an\ n \times 1$ vector and c is a real number.

Show that f is convex if A is positive semi – definite and strictly convex if A is positive definite.

Solution: (H.W.)

Theorem (2):

1: Let f be a convex function on a convex set $D \subset R^n$ and $real\ number$ $\alpha \geq 0$. Then αf is also a convex function on D.

2: Let f_1, f_2 be convex functions on $a\ convex\ set\ D \subset R^n$. Then $f_1 + f_2$ is also a convex function on D.

3: Let $f_1, f_2, f_3, \cdots, f_m$ be convex functions on a convex set $D \subset R^n$ and real $numbers\ \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_m \geq 0$. Then $\sum_{i=1}^{m} \alpha_i f_i$ is also a convex function on D.

Proof: (H.W.)

Theorem (3):

Let $S \subset D$ be an open convex set. Let $f: D \subseteq R^n \rightarrow R$ be a convex function.
Then f is continuous function on S.

Theorem (4):

Let $f: R^n \rightarrow R$ have continuous first partial derivatives in a convex set $D \subset R^n$. Then:

1: $f \; is \; convex \; on \; D \; if \; and \; only \; if \; for \; all \; X, Y \epsilon D,$
$f(Y) \geq f(X) + g(X)^T (Y - X), where \; g \; is \; the \; gradient \; vector$
$of \; f \; at \; X.$

2: f is strictly convex on D if and only if $for \; all \; X, Y \epsilon D, X \neq Y,$

$f(Y) > f(X) + g(X)^T (Y - X), where \; g \; is \; the \; gradient \; vector \; of \; f \; at \; X.$

<u>Theorem (5):</u>

Let $f: R^n \to R$ have continuous second partial derivatives in a convex set $D \subset R^n$. Then:

1: $f\ is\ convex\ on\ D$ if and only if it its Hessian matrix is $positive\ semi-\ definite$ at each point in D.

2: $f\ is\ strictly\ convex\ on\ D$ if and only if it its Hessian matrix is $positive\ definite$ at each point in D.