Optimization

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Chapter One

Basic Concepts

Lecture 2

7: Some Mathematical Foundations

We shall review a number of results from linear algebra and mathematical analysis which are useful in optimization theory and methods.

Definition (2): (Linear Space)

Let $X \,\&\, Y\ in\ R^n\ and\ \alpha \in R$, we define

$$X + Y = [x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (7)$$

And

$$\alpha X = [\alpha x_1, \alpha x_2, \cdots, \alpha x_n] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (8)$$

Then R^n together with the operations defined by (7)& (8) constitute a linear space.

We shall subsequently refer to the linear space as R^n.

Note (2):

A real $m \times n$ matrix $A = (a_{ij})$ defines a linear mapping from R^n to R^m and will be written as $A \epsilon\, R^{m \times n}$ or $A \epsilon\, L(R^n, R^m)$ to denote either the matrix or the linear operator.

Definition (3): (Vector Norm)

A function $\|.\|$: $R^n \rightarrow R$ is called **a vector norm** on R^n if and only if satisfies the following properties:

1: $\|X\| \geq 0 \; for \; all \; X \epsilon R^n$.

2: $\|X\| = 0 \; if \; and \; only \; if \; X = 0$.

3: $\|\alpha X\| = |\alpha| \|X\| \; for \; all \; X \epsilon R^n \; and \; \alpha \, \epsilon \, R$.

4: $\|X + Y\| \leq \|X\| + \|Y\| \; for \; all \; X, Y \epsilon \, R^n$.

Definition (4): (Normed Linear Space)

The linear space R^n together with a vector norm $\|.\|$ on R^n is called a normed linear space, we denote it by $(R^n, \|.\|)$.

Examples of vector norms as follows:

1: $L_\infty - Norm$: $\|X\|_\infty = \max_{1\le i\le n} |x_i|$.

2: $L_1 - Norm$: $\|X\|_1 = \sum_{i=1}^{n} |x_i|$.

3: $L_2 - Norm$: $\|X\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$.

4: $L_p - Norm$: $\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$.

Definition (5): (Matrix Norm)

Let $A, B \in R^{m\times n}$. A function $\|.\|: R^{m\times n} \to R$ is said to be **a matrix norm** if it satisfies the following properties:

1: $\|A\| \geq 0 \; for \; all \; A \in R^{m\times n}$.

2: $\|A\| = 0 \; if \; and \; only \; if \; A = 0$.

3: $\|\alpha A\| = |\alpha|\|A\| \; for \; all \; A \in R^{m\times n} \; and \; \alpha \in R$.

4: $\|A + B\| \leq \|A\| + \|B\| \; for \; all \; A, B \in R^{m\times n}$.

Examples of a matrix norm:

1: Maximum Column Norm: $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$.

2: Maximum Row Norm: $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$.

3: *Frobenius Norm*:

$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} = [tr(A^T A)]^{\frac{1}{2}}$, where $tr(\,.\,)$ denotes the trace of a square matrix $with \; tr(A) = \sum_{i=1}^{n} a_{ii}$.

Note (4):

The trace satisfies the following properties:

1: $tr(\alpha A + \beta B) = \alpha\, tr(A) + \beta\, tr(B)$.

2: $tr(A^T) = tr(A)$.

3: $tr(AB) = tr(BA)$.

4: $tr(A) = \sum_{i=1}^{n} \lambda_i$ *if the eigen values of A are denoted by* $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.

Note (5):

The norm of the identity matrix equals 1.

Definition (6): (Convergent)

A vector sequence $\{X_k\}$ is said to be convergent to X^* if $\lim_{k \to \infty} \|X_k - X^*\| = 0$.

A matrix sequence $\{A_k\}$ is said to be convergent to A if $\lim_{k \to \infty} \|A_k - A\| = 0$.

Note (6):

Choice of norms is irrelevant since all norms in finite dimensional space are equivalent.

8: Eigen Values Problem

Eigen values problem of a matrix A is that $AX = \lambda X$, $A \epsilon R^{n\times n}$, $X \neq 0$, $X \epsilon R^{n}$, $where\ \lambda\ is\ called\ an\ eigen\ value\ of\ A, X\ an\ eigen\ vector\ of\ A$ corresponding to λ , $and\ (\lambda, X)\ an\ eigen\ pair\ of\ A$.

Note (7):

If $A \epsilon R^{n\times n}$ $is\ symmetric\ with\ eigen\ values\ \lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, then

$$\|A\|_2 = \max_{1\leq i\leq n}|\lambda_i|.$$

9: Positive Definite Matrices

Definition (7):

If $A \epsilon R^{n\times n}$ be symmetric. $A\ is\ said\ to\ be\ positive\ definite\ if\ X^T AX > 0\ for\ all$ $X \epsilon R^n$, $X \neq 0$.

$A\ is\ said\ to\ be\ positive\ semi-definite\ if\ X^T AX \geq 0\ for\ all\ \ X \epsilon R^n$.

$A\ is\ said\ to\ be\ negative\ definite\ or\ negative\ semi-definite\ if\ {-A}\ is$ positive definite or positive semi – definite.

$A\ is\ said\ to\ be$ indefinite if it is neither positive semi – definite nor negative semi – definite.

For example, the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ positive definite because for all $X\ in\ R^2$

$$X^T AX = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 & 2x_1 + 4x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 + 4x_1x_2 + 4x_2^2 = (x_1 + 2x_2)^2 > 0\ for\ all\ X \neq 0.$$

While the matrix $-A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$ is negative definite.

Notes (8):

Let $A \epsilon R^{n\times n}$ be symmetric.

1: A is positive definite if and only if all its $eigen$ values are positive.

2: A is positive semi - definite if and only if all its $eigen$ values are nonnegative.

3: A is negative definite or negative semi – definite if and only if all its $eigen$ values are negative or non-positive.

4: A is indefinite if and only if it has both positive and negative $eigen$ values.