أمثلة ومسائل-1

(1-1) أمثلـــــة

مثال (1): أثبت أن المتجهين $\vec{A} = \hat{\imath} + 4\hat{\jmath} + 3\hat{\jmath}$ & $\vec{B} = 4\hat{\imath} + 2\hat{\jmath} - 4\hat{\jmath}$ متعامدان.

الحل: اذا كانت نتيجة $\vec{A}.\vec{B} = 0$ فأن المتجهين متعامدين.

ولايجاد $\vec{A}.\vec{B}$ نستخدم المعادلة التالية:

$$\vec{A}.\vec{B} = (\hat{\imath} + 4\hat{\jmath} + 3\hat{k}).(4\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) = 4 + 8 - 12 = 0$$

بما ان :

$$\vec{A}.\vec{B} = ABcos\theta$$

$$cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{zero}{AB}$$

$$\therefore \theta = 90 \Rightarrow \vec{A} \perp \vec{B}$$

مثال (2): أثبت أن المتجهين $\nabla|\vec{r}| = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$ حيث ان:

$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ و $|\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ و $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

الحل:

$$\vec{\nabla}|\vec{r}| = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$\vec{\nabla}|\vec{r}| = \hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{1}{2}} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{\frac{1}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

$$= \hat{\imath}\left[\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2x\right] + \hat{\jmath}\left[\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2y\right] + \hat{k}\left[\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}.2z\right]$$

$$= \frac{\hat{\imath}x}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\hat{\jmath}y}{(x^2 + y^2 + z^2)^{1/2}} + \frac{\hat{k}z}{(x^2 + y^2 + z^2)^{1/2}} = \frac{(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

مثال (3): أثبت أن المتجهين $\nabla \frac{1}{|\vec{r}|} = -\frac{\hat{r}}{|\vec{r}|^2}$ حيث ان:

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z \text{ و } |\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \text{ و } \vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

الحل:

$$\nabla \frac{1}{|\vec{r}|} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^2 + y^2 + z^2)^{\frac{-1}{2}}$$

$$= \hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{-1}{2}} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{\frac{-1}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{\frac{-1}{2}}$$

$$= \hat{\imath}\left[-\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}.2x\right] + \hat{\jmath}\left[-\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}.2y\right] + \hat{k}\left[-\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}.2z\right]$$

$$= \frac{-\hat{\imath}x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{\hat{\jmath}y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{\hat{k}z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \frac{-(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)}{\left[(x^2 + y^2 + z^2)^{\frac{1}{2}}\right]^3} = -\frac{\vec{r}}{|\vec{r}|^3} = -\frac{\vec{r}}{|\vec{r}|^2.|\vec{r}|}$$

وبما ان:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \Longrightarrow \vec{\nabla}\frac{1}{|\vec{r}|} = -\frac{\hat{r}}{|\vec{r}|^2}$$

مثال (4): أثبت أن المتجهين $\nabla \frac{1}{|\vec{r}|^2} = -\frac{2\vec{r}}{|\vec{r}|^3} = -\frac{2\vec{r}}{|\vec{r}|^4}$ حيث ان:

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z \text{ و } |\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} \text{ و } \vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

الحل:

$$\nabla \frac{1}{|\vec{r}|^2} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^2 + y^2 + z^2)^{-1}$$

$$= \hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-1}$$

$$= \hat{\imath}\left[-1(x^2 + y^2 + z^2)^{-2}.2x\right] + \hat{\jmath}\left[-1(x^2 + y^2 + z^2)^{-2}.2y\right] + \hat{k}\left[-1(x^2 + y^2 + z^2)^{-2}.2z\right]$$

$$= \frac{-2\hat{\imath}x}{(x^2 + y^2 + z^2)^2} - \frac{2\hat{\jmath}y}{(x^2 + y^2 + z^2)^2} - \frac{2\hat{k}z}{(x^2 + y^2 + z^2)^2} = \frac{-2(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)}{\left[(x^2 + y^2 + z^2)^{\frac{1}{2}}\right]^4} = -\frac{2\vec{r}}{|\vec{r}|^4} = -\frac{2}{|\vec{r}|^3}\hat{r}$$

مثال (5): أثبت أن المتجهين $-\frac{3\vec{r}}{|\vec{r}|^5} = -\frac{3\hat{r}}{|\vec{r}|^4} = \nabla\frac{1}{|\vec{r}|^3}$ حيث ان:

$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ و $|\vec{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$ و $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

الحل:

$$\nabla\frac{1}{|\vec{r}|^3} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$= \hat{\imath}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{3}{2}} + \hat{\jmath}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{3}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= \hat{\imath}\left|-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}.2x\right| + \hat{\jmath}\left|-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}.2y\right| + \hat{k}\left|-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}.2z\right|$$

$$= \frac{-3\hat{\imath}x}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{3\hat{\jmath}y}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{3\hat{k}z}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{-3(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^5} = -\frac{3\vec{r}}{|\vec{r}|^5} = -\frac{3}{|\vec{r}|^4}\hat{r}$$

مثال (6): أذا كان المتجه $\vec{r}$ يعرف بالمعادلة التالية: $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

أثبت ما يلي :

-1 $(\nabla\cdot\vec{r} = 3)$

-2 $(\nabla\times\vec{r} = 0)$

-3 $(\vec{u}\cdot\nabla)\vec{r} = \vec{u}$ حيث ان $\vec{u} = \hat{\imath}u_x + \hat{\jmath}u_y + \hat{\jmath}u_z$

الحل:

1- $$\vec{\nabla}.\vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{\imath}x + \hat{\jmath}y + \hat{k}z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

2- $$\vec{\nabla}x\vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

3- $$(\vec{u}\cdot\nabla)\vec{r} = \left[(\hat{\imath}u_x + \hat{\jmath}u_y + \hat{k}u_z).(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\right](\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$$

$$= \left[u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right](\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$$

$$= u_x\frac{\partial}{\partial x}(\hat{\imath}x + \hat{\jmath}y + \hat{k}z) + u_y\frac{\partial}{\partial y}(\hat{\imath}x + \hat{\jmath}y + \hat{k}z) + u_z\frac{\partial}{\partial z}(\hat{\imath}x + \hat{\jmath}y + \hat{k}z) = \hat{\imath}u_x + \hat{\jmath}u_y + \hat{k}u_z = \vec{u}$$
