Republic of Iraq
Ministry of Higher Education and Scientific Research
University of Basrah
Collage of Education for Pure Sciences
Department of Mathematics
Second Class

Advanced Calculus

Chapter Six
Partial Derivatives
Lecter 2

By
Dr.Jawad Mahmoud Jassim

2019-2020

3: Partial Derivatives of a Function of Two Variables:

Definition (11):

The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is:

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}, \textit{ provided the limit exists}.$$

Notes:

1: The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is the same as the ordinary derivative of $f(x,y_0)$ *at the point* x_0. That is

$$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \frac{d}{dx}f(x,y)\Big|_{x_0}.$$

2: A variety notations are used to denote the partial derivative at a point (x_0,y_0) including $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)},\ f_x((x_0,y_0),\ and\ \frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}$.

3: When we do not specify a specific point (x_0,y_0) at which the partial derivative is being evaluated, then the partial derivative is being evaluated, then the partial derivative becomes a function whose domain is the points where the partial derivative exists. Notations for this function include $\frac{\partial f}{dx},\ f_x\ ,\ and\ \frac{\partial z}{\partial x}$.

4: The slope of the curve of the curve $z = f(x,y_0)$ at the point $(x_0,y_0,f(x_0,y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0,y_0).

5: The tangent line to the curve at $P(x_0,y_0,f(x_0,y_0))$ is the line in the plane $y = y_0$ *that passes* through $P(x_0,y_0,f(x_0,y_0))$ with slope $f_x((x_0,y_0)$.

6: The partial derivative $\frac{\partial f}{dx}$ at (x_0,y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0 .

Definition (12):

The partial derivative of $f(x,y)$ with respect to$y$ at the point (x_0,y_0) is:

$$\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0,y_{0+h})-f(x_0,y_0)}{h},\ \textit{provided the limit exists}.$$

Notes:

1: The slope of the curve $z = f(x_0,y)$ at the point $P(x_0,y_0,f(x_0,y_0))$ in the vertical plane $x = x_0$ is the partial derivative of f with respect to y *at* (x_0,y_0).

2: The tangent line to the curve at $P(x_0,y_0,f(x_0,y_0))$ is the line in the plane

$x = x_0$ *that passes through* $P(x_0, y_0, f(x_0, y_0))$ *with the slope* $\frac{\partial f}{\partial y}|_{(x_0,y_0)}$.

3: The partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0) gives the rate of change of f with respect to y when x is held fixed at the value x_0 .

4: The partial derivative with respect to y is denoted by

$$\frac{\partial f}{\partial y}|_{(x_0,y_0)} \quad , \quad f_y(x_0, y_0) \ , \ and \quad \frac{\partial z}{\partial y}|_{(x_0,y_0)} .$$

5: When we do not specify a specific point, the partial derivative becomes a function and denoted as $\frac{\partial f}{\partial x}$, f_y , $\frac{\partial z}{\partial x}$.

Example (11):

Find the values of $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution:

$$\frac{\partial f}{\partial x} = 2x + 3y \rightarrow \frac{\partial f}{\partial x}|_{(4,-5)} = 2(4) + 3(-5) = 8 - 18 = -7.$$

$$\frac{\partial f}{\partial y} = 3x + 1 \rightarrow \frac{\partial f}{\partial y}|_{(4,-5)} = 3(4) + 1 = 12 + 1 = 13.$$

Example (12):

Find $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$.

Solution:

$$\frac{\partial f}{\partial x} = y^2 \cos xy .$$

$$\frac{\partial f}{\partial y} = xy \cos xy + \sin xy .$$

Example (13):

Find f_x & f_y *if* $f(x, y) = \frac{2y}{y+\cos x}$.

Solution:

$$f_x = \frac{2y \sin x}{(y+\cos x)^2} \quad \& \quad f_y = \frac{2 \cos x}{(y+\cos x)^2} .$$

Example (14):

Find $\frac{\partial z}{\partial x}$ assuming the equation $yz - \ln z = x + y$ defines z as a function of two independent variables x *and* y and the partial derivative exists.

Solution:

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x + y) \rightarrow y\frac{\partial z}{\partial x} + z\frac{\partial y}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + \frac{\partial y}{\partial x} \rightarrow$$
$$y\frac{\partial z}{\partial x} + 0 - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \rightarrow y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 \rightarrow \frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1 \rightarrow$$
$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{zy - 1}.$$

Example (15):

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.

Solution:

The slope is

$$\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)} = \frac{\partial z}{\partial y}\Big|_{(1,2)} = 2y|_{(1,2)} = 4.$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two variables are similar to the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant. That is, if $w = f(x, y, z)$, *then*

$\frac{\partial f}{\partial x} = \frac{\partial w}{\partial x}$ ***is the partial derivative with respect to x.***

$\frac{\partial f}{\partial y} = \frac{\partial w}{\partial y}$ ***is the partial derivative with respect to y.***

$\frac{\partial f}{\partial z} = \frac{\partial w}{\partial z}$ ***is the partial derivative with respect to z.***

Example (16):

If x, y & z are independent variables and $f(x, y, z) = x\sin(y + 3z)$.

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ & $\frac{\partial f}{\partial z}$.

Solution:

$$\frac{\partial f}{\partial x} = \sin(y + 3z).$$
$$\frac{\partial f}{\partial y} = x\cos(y + 3z).$$
$$\frac{\partial f}{\partial z} = 3x\cos(y + 3z).$$