Republic of Iraq
Ministry of Higher Education and Scientific Research
University of Basrah
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Department of Mathematics
Second Class

Advanced Calculus

Chapter Six
Partial Derivatives
First Lecture

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1: Functions of Several Variables:

Definition (1):

Suppose D is a set of $n - tuples\ of\ real\ numbers$ $(x_1, x_2, \cdots, x_n)$. A real valued function $f\ on\ D$ is a rule that assigns a unique (single) real number $w = f(x_1, x_2, \cdots, x_n)$ $to\ each$ element in D. The set D is the function's domain. The set of $w - values$ taken on by f is the function's range. The symbol w is the dependent $variable\ of\ f$, and f is said to be a function of the $n\ independent\ variables$ $x_1, x_2, \cdots, x_n$. We also call the x_j's the function's input variables and call w the function's output variable.

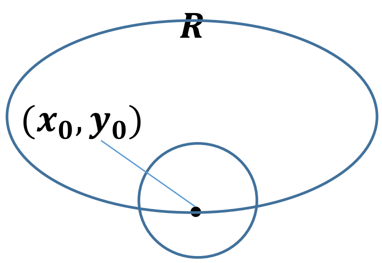
Example (1):

Find the domain and the range for the following functions:

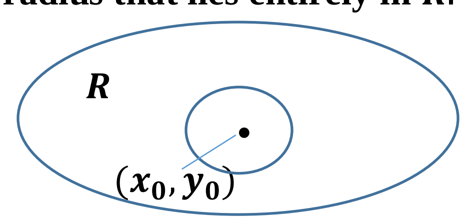
No.	Function	Domain	Range
1	$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
2	$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
3	$z = \sin xy$	Entire plane	$[-1, 1]$
4	$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
5	$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
6	$w = xy \ln z$	Half- space $z > 0$	$(-\infty, \infty)$

Definition (2): (Interior Point)

A point (x_0, y_0) in a region R in the $xy - plane$ is an interior point of R if it is a center of a disk of positive radius that lies entirely in R.



(x_0, y_0) ***boundary point***



(x_0, y_0) ***interior point***

Definition (3): (Boundary Point)

A point (x_0, y_0) is a boundary point of the region R if every disk centered at (x_0, y_0) contains points that lies outside of R as well as points that lies in R.

Definition (4): (Open Region)

A region is open if it consists entirely of interior points.

Definition (5): (Closed Region)

A region is closed if it contains all its boundary points.

For examples:

1: $\{ (x,y): x^2 + y^2 < 1 \} \rightarrow$ *open unit disk* (*every point is an interior*).

2: $\{ (x,y): x^2 + y^2 = 1\} \rightarrow$ *boundary of unit disk* (*unit circle*).

3: $\{(x,y): x^2 + y^2 \leq 1\} \rightarrow$ *closed unit disk* (*contains all boundary points*).

Note:

The empty set has no interior points and no boundary points. This implies that the empty set is open (because it does not contain points that are *not* interior points), and at the same time is closed (because there are no boundary points that it fails to contain).

Note:

The entire $xy - plane$ is also both open and closed. Open because every point in the plane is an interior point. Closed because it has no boundary points.

Definition (6): (Bounded and Unbounded Regions)

A region in the plane is bounded if it lies inside a disk of finite radius.

A region is unbounded if it is not bounded.

For example:

1: Bounded Regions: **line segments, triangles, rectangles, circles and disks.**

2: Unbounded Regions: **lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes and the plane itself.**

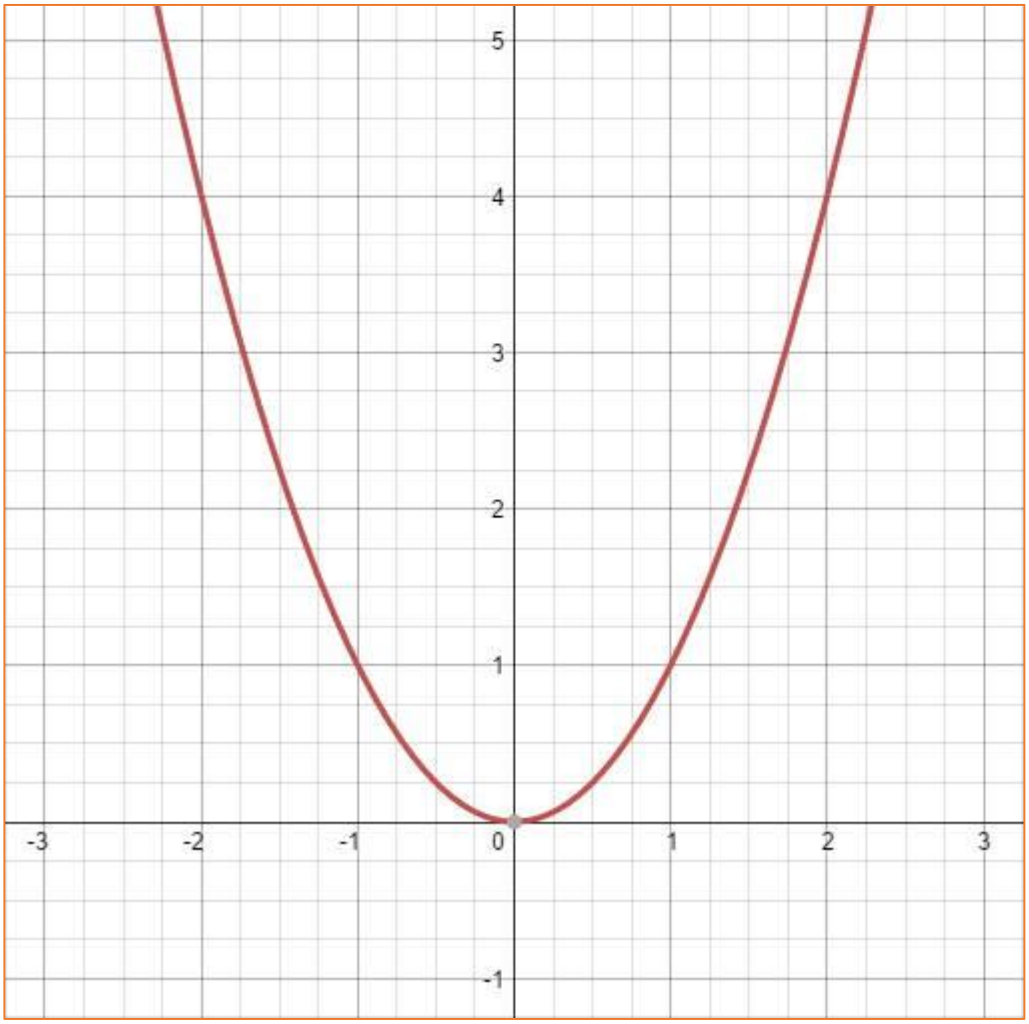
Example (2):

Describe the domain of the function $f(x,y) = \sqrt{y - x^2}$.

Solution:

Since f is defined only where $y - x^2 \geq 0$, the domain is closed, unbounded region. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the *domain's* interior.

Interior points where $y - x^2 > 0$



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Definition (7): Level Curve)

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve of f.

Definition (8): (Graph)

The set of all points $(x, y, f(x, y))$ in space for (x, y) in the domain of f is called the *graph of f*. The graph of f is also called the surface $z = f(x, y)$.

Example (3):

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0, f(x, y) = 51$ & $f(x, y) = 75$ *in the domain of f in the plane.*

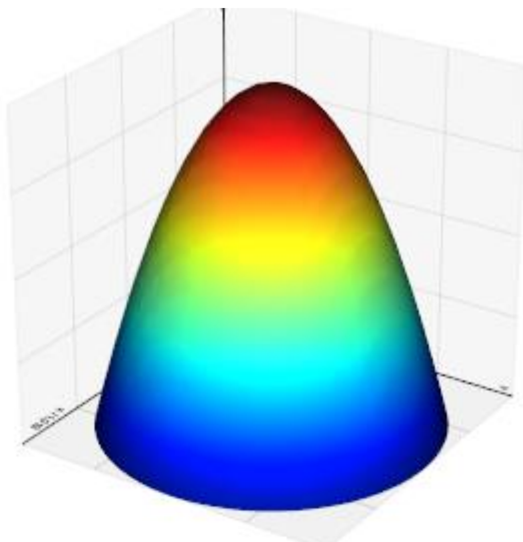
Solution:

The domain of f *is the entire xy − plane*, and the range of f is the set of real numbers less than or equal 100. *The graph is the paraboloid* $z = 100 - x^2 - y^2$.

The level curve $f(x, y) = 0$ is the set of points in the $xy - plane$ at which $f(x, y) = 100 - x^2 - y^2 = 0$ *or* $x^2 + y^2 = 100$, which is the circle of radius 10 centered at the origin.

Similarly the level curves $f(x, y) = 51$ and $f(x, y) = 75$ are the circles $x^2 + y^2 = 49$ *and* $x^2 + y^2 = 25$.

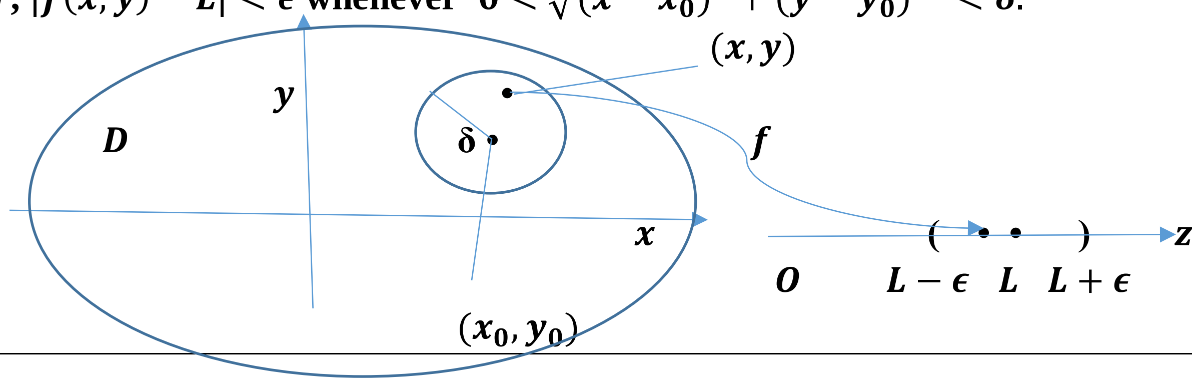
The level curve $f(x, y) = 100$ consists of the origin alone.



2: Limits and Continuity in Higher Dimensions

Definition (9): (Limits for a Function of Two Variables)

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) and write $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f, $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.



Notes:

1: The definition of the limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance between (x, y) and (x_0, y_0) is made sufficiently small (but not zero).

2: The definition applies to interior points (x_0, y_0) as well as boundary points of the domain of f, although a boundary point need not lie within the domain.

3: The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f.

Theorem (1): (Properties of Limits of Functions of Two Variables)

The following rules hold if L, M & k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \& \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \mp g(x,y) = L \mp M$.

2: $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$.

3: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = LM$.

4: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$.

5: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n, n\ is\ positive\ integer$.

6: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$, $n\ is\ a\ positive\ integer$ & $if\ n\ is\ even\ we\ assume\ L > 0$.

Example (4):

Find the following limits:

1: $\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2\,y+5xy-y^3} = \frac{3}{-1} = -3$.

2: $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25} = 5$.

Example (5):

Find $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$.

Solution:

Since the denominator $\sqrt{x}-\sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$. We $can\ not$ use the Quotient Rule from Theorem (1). If we multiply numerator and denominator by $\sqrt{x}+\sqrt{y}$, however, we produce an equivalent fraction whose limit we can find.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \left(\frac{x^2-xy}{\sqrt{x}-\sqrt{y}}\right)\left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) = \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{x-y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x-y} = \lim_{(x,y)\to(0,0)} x\left(\sqrt{x}+\sqrt{y}\right) = 0 .$$

Note:

We can cancel the factor $(x-y)$ because the path $y = x$ is not in the domain of the function.

Example (6):

Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ **if it exists.**

Solution:

We first observe that along the line $x = 0$.

$\therefore f(x,y) = 0 \rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Now, we observe that along the line $y = 0$.

$\therefore f(x,y) = 0 \rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

$\therefore \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0 .$

Example (7):

$If\ f(x,y) = \frac{y}{x}.\ Does \lim_{(x,y)\to(0,0)} f(x,y)\ exist?$

Solution:

We first observe that along the line $y = 0$.

$\therefore f(x,y) = 0 \rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Second, we observe that along the line $y = x$.

$\therefore f(x,y) = f(x,x) = \frac{x}{x} = 1 \rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 1 .$

Since the limits of f are different along different paths.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)\ does\ not\ exist.$

Definition (10): (Continuity)

A function $f(x,y)$ is continuous at the point (x_0, y_0) if:

1: $f\ is\ defined\ at\ (x_0, y_0).$

2: $\lim_{(x,y)\to(x_0,y_0)} f(x,y)\ exists.$

3: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

A function is continuous if it is continuous at every point of its domain.

Example (8):

Show that the function $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$ **is continuous at every point except the origin.**

Solution:

The function f is continuous at every point (x,y) except $(0,0)$ because its values at points other than $(0,0)$ are given by a rational function of x *and* y and there for at those points the limiting value is simply obtained by substituting the values of x *and* y in to that rational expression.

At $(0,0)$, the value of f is defined but f has *no limit as* $(x,y)\to(0,0)$. The reason is that different paths of approach to the origin can lead to different results.

For every value of m the function f has a constant value on the line $y = mx$, $x \neq 0$ *because*

$$f(x,y) = f(x,mx) = \frac{2mx^2}{x^2\,(1+m^2\,)} = \frac{2m}{1+m^2}.$$

$\therefore \lim\limits_{(x,y)\to(0,,0)} f(x,y) = \frac{2m}{1+m^2}$.

This limit changes with each value of the slope m.

$\therefore$There is no single number we may call the limit of f *as* $(x,y)\to(0,,0)$. The limit fails to exist and the function is not continuous at the origin.

Note:

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) *approaches* $(x_0\,,y_0)$, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example (9):

Show that the function $f(x,y) = \frac{2x^2\,y}{x^4+y^2}$ has no limit as $(x,y)\to(0,\;0)$.

Solution:

The limit cannot be found by direct substitution, which gives the indeterminate form $\frac{0}{0}$. We examine the values of f along parabolic curves that end at $(0,\ 0)$. $Along$ the curve $y = kx^2, x \neq 0$, the function hasthe constant value.

$\therefore f(x, y) = f(x, kx^2) = \frac{(2x^2)(kx^2)}{x^4 + k^2 x^4} = \frac{2kx^4}{x^4 + k^2 x^4} = \frac{2k}{1+k^2}$.

$\therefore \lim_{(x,y) \to (0,,0)} f(x, y) = \frac{2k}{1+k^2}$.

This limit varies with the path of approach.

$\therefore$ **f has no limit as $(x, y) \to (0,,0)$.**

Note:

Having the same limit along all straight lines approaching $(x_0\ , y_0)$ does not imply that a limit exists at $(x_0\ , y_0)$.

For example, the $\lim_{(x,y) \to (0,,0)} \frac{2x^2\, y}{x^4+y^2} = 0$ along every straight $line\ path\ y = mx$.

But we show that in Example (9) $\lim_{(x,y) \to (0,,0)} \frac{2x^2\, y}{x^4+y^2}$ does not exist.

Note:

If f is continuous at $(x_0\ , y_0)$ and g is a simple-valued function continuous at $f(x_0\ , y_0)$, then the composition function $h = gof$ defined by $h(x, y) = g(f(x, y))$ *is continuous at* $(x_0\ , y_0)$.

Theorem (2): (Sandwich Theorem)

If $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) $and\ if\ g\ \&\ h\ have$ the same finite limit L as $(x, y) \to (x_0, y_0)$, then $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L$.

Example (10):

If $1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$. Find $\lim_{(x,y) \to (0,0)} \frac{\tan^{-1} xy}{xy}$.

Solution:

By Sandwich Theorem

Since $\lim_{(x,y) \to (0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\lim_{(x,y) \to (0,0)} 1 = 1$.

$\therefore \lim_{(x,y) \to (0,0)} \frac{\tan^{-1} xy}{xy} - 1.$