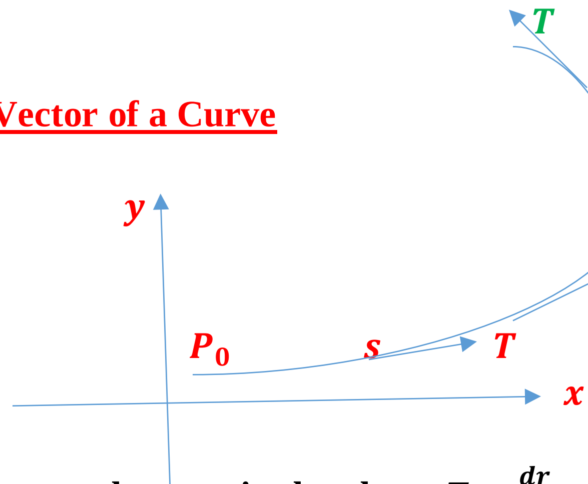
Note:

$$\frac{dr}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds} = v\frac{1}{|v|} = \frac{v}{|v|} = T \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (5)$$

This equation says that $\frac{dr}{ds}$ is the unit tangent vector in the direction of the velocity vector.

4: Curvature and Normal Vector of a Curve



As a particle moves along a smooth curve in the plane, $T = \frac{dr}{ds}$ turns as the curve bends. Since T is a unit vector, its length remains constant and only its direction changes as particle moves along the curve. The rate at which T turns per unit of length along the curve is called the curvature.

Definition (9):

If T is a unit vector of a smooth curve, the curvature function of the curve is $\kappa = \left|\frac{dT}{ds}\right|$.

Formula for Calculating Curvature

If a smooth curve $r(t)$ is already given in terms of some parameter t other than the arc $length\ parameter\ \ s$, we can calculate the curvature as:

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\frac{dt}{ds}\right| = \left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right| = \frac{1}{\left|\frac{ds}{dt}\right|}\left|\frac{dT}{dt}\right| = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$$

Note:

The curvature of the straight line is zero.

Example (10):

Find the curvature of the circle.

Solution:

Let $r(t)=(a\cos t)i+(a\sin t)j$ *the parameterization of a circle of radius a.*

$\therefore v=\frac{dr}{dt}=(-a\sin t)i+(a\cos t)j \rightarrow |v|=\sqrt{a^2 sin^2 t+a^2 cos^2 t}=a$.

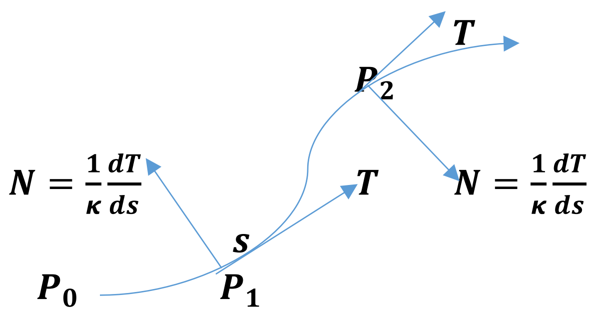
$\therefore T=\frac{v}{|v|}=(-\sin t)i+(\cos t)j \rightarrow \frac{dT}{dt}=(-\cos t)i-(\sin t)j$.

$\therefore \left|\frac{dT}{dt}\right|=\sqrt{cos^2 t+sin^2 t}=1$.

$\therefore \kappa=\frac{1}{|v|}\left|\frac{dT}{dt}\right|=\frac{1}{|v|}=\frac{1}{a}=\frac{1}{radius}$.

Normal Vector of a Curve

Since T **has a constant length, the derivative** $\frac{dT}{ds}$ **is orthogonal to** T**. Therefore, if we divide** $\frac{dT}{ds}$ **by its length** κ **(curvature), we obtain unit vector** N **orthogonal to** T.



Definition (10):

At a point where $\kappa \neq 0$**, the priniciple unit normal vector for a smooth curve in the plane is** $N=\frac{1}{\kappa}\frac{dT}{ds}$.

Note:

The principle normal vector N **will point toward the concave side of the curve.**

Formula for Calculating Normal Unit Vector

If $r(t)$ **is a smooth curve, then the principle unit normal vector is given by**

$$N = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|} \text{ ....} \quad (2)$$

Example (11):

Find T & N for the circular motion $r(t) = (\cos 2t)i + (\sin 2t)j$.

Solution:

$T = \frac{v}{|v|}$, $v = r'(t) = (-2\sin 2t)i + (2\cos 2t)j$ & $|v| = \sqrt{4sin^2 2t + 4cos^2 2t}$

$\therefore |v| = 2$.

$\therefore T = (-\sin 2t)i + (\cos 2t)j$.

$N = \frac{\frac{dT}{dt}}{|\frac{dT}{dt}|}$, $\frac{dT}{dt} = (-2\cos 2t)i - (2\sin 2t)j$, $|\frac{dT}{dt}| = 2$.

$\therefore N = (-\cos 2t)i - (\sin 2t)j$.

Notice that $T \cdot N = 0$ (N is orthogonal to T).

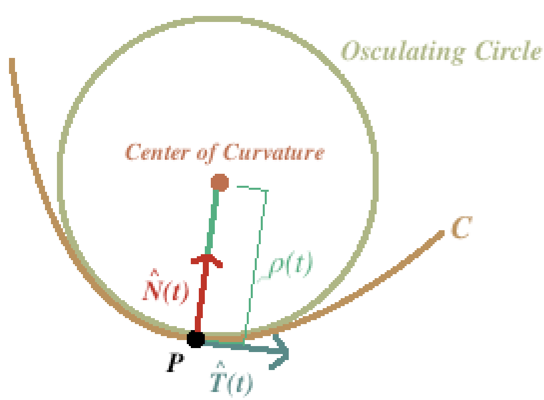
Circle of Curvature for Plane Curves

The circle of the curvature or osculating circle at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1: is tangent to the curve at P (has the same tangent line the curve has).

2: *has the same curvature the curve has at P*.

3: has center that lies toward the concave or inner side of the curve.



The radius of curvature of the curve at P is the radius of the circle of curvature which *is* $\rho = \frac{1}{\kappa}$.

Example (12):

Find the osculating circle (circle of curvature) of the parabola $y = x^2$ at the origin.

Solution:

We parameterize the parabola using the parameter $t = x, then\ y = t^2$.

$\therefore r(t) = ti + t^2 j$.

Since $\kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|, where\ T = \frac{v}{|v|}$.

$\therefore v = r'(t) = i + 2tj \ \rightarrow |v| = \sqrt{1 + 4t^2}$.

So that

$$T = \frac{1}{\sqrt{1+4t^2}}(i + 2tj) \ \rightarrow \frac{dT}{dt} = \frac{1}{\sqrt{1+4t^2}}(2j) - \frac{4t}{\sqrt{(1+4t^2)^3}}(i + 2tj).$$

So that, at the origin $t = 0 \ \rightarrow v(0) = i$ & $\frac{dT}{dt}(0) = 2j$.

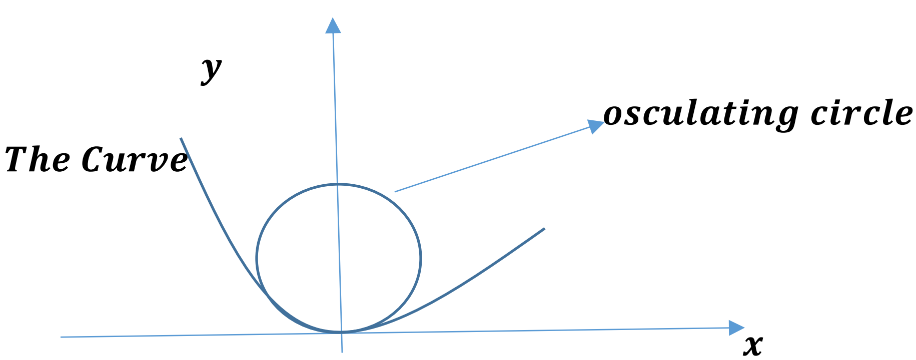
$\therefore |v(0)| = 1$ & $\left|\frac{dT}{dt}(0)\right| = 2$.

$\therefore \kappa(0) = 2$.

$\therefore$*The radius of curvature is* $\rho = \frac{1}{\kappa} = \frac{1}{2}$.

The center of the circle is $(0, \frac{1}{2})$.

$\therefore$*The equation of the osculating circle is* $x^2 - \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$.



Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $r(t)$ as a function of $some\ parameter\ t$, and if s is the arc length of the curve, then the unit tangent vector $T = \frac{dr}{ds} = \frac{v}{|v|}$.

The curvature in space is then defined to be

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{|v|}\left|\frac{dT}{dt}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (3)$$

The vector $\frac{dT}{ds}$ $is\ orthogonal\ to\ T$, and we define the principle unit normal to be

$$N = \frac{1}{\kappa}\frac{dT}{ds} = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (4)$$

Example (13):

Find the curvature for the helix $r(t) = (a\cos t)i + (a\sin t)j + btk$, $a, b \geq 0$ & $a^2 + b^2 \neq 0$.

Solution:

$The\ curvature\ is\ given\ by\ \kappa = \frac{1}{|v|}\left|\frac{dT}{dt}\right|, T = \frac{v}{|v|}$.

$\therefore v = r'(t) = (-a\sin t)i + (a\cos t)j + bk \rightarrow |v| = \sqrt{a^2 + b^2}$.

$\therefore T = \frac{1}{\sqrt{a^2+b^2}}((-a\sin t)i + (a\cos t)j + bk)$.

$\therefore \frac{dT}{dt} = \frac{1}{\sqrt{a^2+b^2}}((-a\cos t)i - (a\sin t)j) \rightarrow \left|\frac{dT}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$.

$\therefore \kappa = \frac{a}{a^2+b^2}$.

Example (14):

Find N for the helix in Example (13).

Solution:

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = (-\cos t)i - (\sin t)j.$$

Note:

If $y = f(x)$ is a twice differentiable $function\ of\ x$, then the curvature of the graph of a function in the $xy - plane$ is given by

$$\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}.$$

Example (15):

Find the curvature of the curve $y = \ln(\cos x)\,,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution:

$y' = -\tan x \ \ \rightarrow y'' = -sec^2 x$.

$\therefore \kappa(x) = \frac{|f''(x)|}{[1+(f'(x)^2]^{\frac{3}{2}}} = \frac{|-sec^2 x|}{[1+tan^2 x]^{\frac{3}{2}}} = \frac{sec^2 x}{[sec^2 x]^{\frac{3}{2}}} = \frac{sec^2 x}{sec^3 x} = \frac{1}{secx} = \cos x$.

Note:

If $x = g(y)$ **is a twice differentiable function of** y**, then the curvature is given by** $\kappa(y) = \frac{|g''(y)|}{[1+(g'(y)^2]^{\frac{3}{2}}}$.

Note:

The curvature of a smooth curve defined by twice differentiable functions $x = f(t)$ & $y = g(t)$ *is given* **by the formula**

$\kappa(t) = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2\,]^{\frac{3}{2}}}$.

The dots in the formula denote differentiation with respect to t, one derivative for each dot.

Example (16):

Find the curvature of the curve $r(t) = ti + (\ln \sin t)j\,,\ \ 0 < t < \pi$.

Solution:

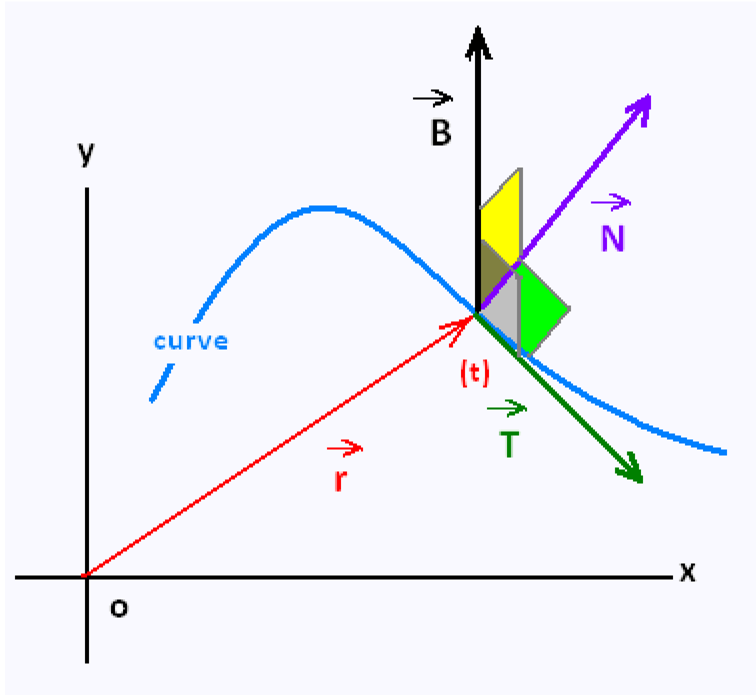
The parametric equations for the curve are $x = t$ & $y = \ln \sin t$.

$\therefore\ \ \dot{x} = 1\,, \dot{y} = \cot t\,,\ \ \ddot{x} = 0\ \ \&\ \ddot{y} = -\ csc^2 t$

$\therefore \kappa(t) = \ \kappa(t) = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2\,]^{\frac{3}{2}}} = \frac{|-csc^2 t}{[1+cot^2 t]^{\frac{3}{2}}} = \frac{csc^2 t}{[csc^2 t]^{\frac{3}{2}}} = \frac{csc^2 t}{csc^3 t} = \frac{1}{csc\ t} = sin\ t$.

Definition (11):

The bi-normal vector of a curve in space is $B = T \times N$, *which* **is a unit vector that is orthogonal to both** T & N.



Definition (12):

Together $\boldsymbol{T}, \boldsymbol{N}$ & $\boldsymbol{B}$ define a moving right handed vector frame that plays a significant role in calculating the paths of particle's moving through space. It is called the $\boldsymbol{Frenet\ frame}$ or the $\boldsymbol{TNB\ frame}$.